Determining the price of a basket option is not a trivial task, because there is no explicit analytical expression available for the distribution of the weighted sum of prices of the the assets in the basket. However, by using a conditioning variable, this price can be decomposed in two parts, one of which can be computed exactly. For the remaining part we first derive a lower and an upper bound based on comonotonicity, and another upper bound equal to that lower bound plus an error term. Secondly, we derive an approximation by applying some moment matching method.

Keywords: basket option; comonotonicity; analytical bounds; moment matching; Asian basket option; Black & Scholes model

AMS 2000 Subject Classification: Primary 91B28
Secondary 60E15; 60J65

JEL Classification: G13

† This research was carried out while the author was employed at the Ghent University
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1. Introduction

One of the more extensively sold exotic options is the basket option, an option whose payoff depends on the value of a portfolio or basket of assets. In this paper, we concentrate first upon the basket call option which pays off at maturity the greater of zero and the difference between the average of the prices of the \( n \) different assets in the basket and the exercise price. The theory can be developed in a similar way for puts, see Section 6.

The typical underlying of a basket option is a basket consisting of several stocks, that represents a certain economy sector, industry or region.

The main advantage of a basket option is that it is cheaper to use a basket option for portfolio insurance than to use the corresponding portfolio of plain vanilla options. Indeed, a basket option takes the imperfect correlation between the assets in the basket into account and moreover the transaction costs are minimized because an investor has to buy just one option instead of several ones.

For pricing simple options on one underlying, the celebrated Black & Scholes model leads to a closed form solution since the stock price at a fixed time follows a lognormal distribution. However, using the famous Black & Scholes model for a collection of underlying stocks, does not provide us with a closed form solution for the price of a basket option. The difficulty stems primarily from the lack of availability of the distribution of a weighted sum of non-independent lognormals, a feature that has hampered closed-form basket option pricing characterization. Indeed, the value of a portfolio is the weighted average of the underlying stocks at the exercise date.

One can use Monte Carlo simulation techniques (by assuming that the assets follow correlated geometric Brownian motion processes) to obtain a numerical estimate of the price. Other techniques consist of approximating the real distribution of the payoffs by another more tractable one. For instance, in industry it is common to use the lognormal distribution as an
Pricing of arithmetic basket options by conditioning

approximation for the sum of lognormals, although it is known that this methodology leads sometimes to poor results. An extensive discussion of different methods can be found in the theses of Arts (1999), Beißer (2001) and Van Diepen (2002).

Obviously, the payoff structure of a basket option resembles the payoff structure of an Asian option. But whereas the Asian option is a path-dependent option, that is, its payoff at maturity depends on the price process of the underlying asset, the basket option is a path-independent option whose terminal payoff is a function of several asset prices at the maturity date. Nevertheless, in literature, different authors tried out initial methods for Asian options to the case of basket options. In this respect, it seems natural to adapt the methods for valuing Asian options as developed first in Simon, Goovaerts and Dhaene (2000) and Dhaene et al. (2002a,b), and refined and extended in various ways in Vanmaele et al. (2002).

Combining both types of options one can consider an Asian option on a basket of assets instead of on one single asset. In this case we talk about an Asian basket option. Dahl and Benth (2001a,b) value such options by quasi-Monte Carlo techniques and singular value decomposition.

But as these approaches are rather time consuming, it would be vital to have accurate, analytical and easily computable bounds or approximations of this price. As the financial institutions dealing with baskets are perhaps even more concerned about the ability of controlling the risks involved, it is important to offer an interval of hedge parameters. Confronted with such issues, the objective of this paper is to obtain accurate analytical lower and upper bounds as well as approximations. To this end, we use on one hand the method of conditioning as in Curran (1994) and in Rogers and Shi (1995), and on the other hand results on a general technique based on comonotonic risks for deriving upper and lower bounds for stop-loss premiums of sums of dependent random variables (see Kaas, Dhaene and Goovaerts (2000)).

All lower and upper bounds can be expressed as an average of Black & Scholes option prices,
sometimes with a synthetic underlying asset. Therefore, hedging parameters can be obtained in a straightforward way.

A basket option is an option whose payoff depends on the value of a portfolio (or basket) of assets (stocks). Thus, an arithmetic basket call option with exercise date \(T\), \(n\) risky assets and exercise price \(K\) generates a payoff \(\left(\sum_{i=1}^{n} a_i S_i(T) - K\right)_+\) at \(T\), that is, if the sum \(S = \sum_{i=1}^{n} a_i S_i(T)\) of asset prices \(S_i\) weighted by positive constants \(a_i\) at date \(T\) is more than \(K\), the payoff equals the difference; if not, the payoff is zero. The price of the basket option at current time \(t = 0\) is given by

\[
BC(n, K, T) = e^{-rT} E^Q \left[ \left(\sum_{i=1}^{n} a_i S_i(T) - K\right)_+ \right]
\]  

under a martingale measure \(Q\) and with \(r\) the risk-neutral interest rate. Assuming a Black & Scholes setting, the random variables \(S_i(T)/S_i(0)\) are lognormally distributed under the unique risk-neutral measure \(Q\) with parameters \((r - \sigma_i^2/2)T\) and \(\sigma_i^2 T\), when \(\sigma_i\) is the volatility of the underlying risky asset \(S_i\). Therefore we do not have an explicit analytical expression for the distribution of the sum \(\sum_{i=1}^{n} a_i S_i(T)\) and determining the price of the basket option is not a trivial task. Since the problem of pricing arithmetic basket options turns out to be equivalent to calculating stop-loss premiums of a sum of dependent risks, we can apply the results on comonotonic upper and lower bounds for stop-loss premiums, which have been summarized in Section 2.

The paper is organized as follows. Section 2 recalls from Dhaene et al. (2002) and Kaas et al. (2000) procedures for obtaining the lower and upper bounds for prices by using the notion of comonotonicity. In Section 3 the price of the basket option in the Black & Scholes setting is decomposed in two parts, one of which is computed exactly. For the remaining part we derive bounds in Section 4, first by concentrating on the comonotonicity and then by applying the Rogers and Shi approach to carefully chosen conditioning variables. We discuss different conditioning variables in order to determine some superiority. In Section 5 the remaining
part in the decomposition of the basket option price is approximated using a moment matching method. Section 6 contains some general remarks. In Section 7, several sets of numerical results are given and the different bounds and approximations are discussed. Section 8 discusses the pricing of Asian basket options, which can be done by the same reasoning. Section 9 concludes the paper.

2. Some theoretical results

In this section, we recall from Dhaene et al. (2002a,b) and the references therein the procedures for obtaining the lower and upper bounds for stop-loss premiums of sums $S$ of dependent random variables by using the notion of comonotonicity. A random vector $(X_1^c, \ldots, X_n^c)$ is comonotonic if each two possible outcomes $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ of $(X_1^c, \ldots, X_n^c)$ are ordered componentwise.

In both financial and actuarial context one encounters quite often random variables of the type $S = \sum_{i=1}^n X_i$ where the terms $X_i$ are not mutually independent, but the multivariate distribution function of the random vector $X = (X_1, X_2, \ldots, X_n)$ is not completely specified because one only knows the marginal distribution functions of the random variables $X_i$. In such cases, to be able to make decisions it may be helpful to find the dependence structure for the random vector $(X_1, \ldots, X_n)$ producing the least favourable aggregate claims $S$ with given marginals. Therefore, given the marginal distributions of the terms in a random variable $S = \sum_{i=1}^n X_i$, we shall look for the joint distribution with a smaller resp. larger sum, in the convex order sense. In short, the sum $S$ is bounded below and above in convex order ($\preceq_{cx}$) by sums of comonotonic variables:

$$S^\ell \preceq_{cx} S \preceq_{cx} S^c,$$
where $X$ is said to precede $Y$ in convex order sense, notation $X \preceq_{cv} Y$, if and only if
\[
E[X] = E[Y] \\
E[(X - d)_+] \leq E[(Y - d)_+],
\]
for $-\infty < d < +\infty$. This definition of convex order implies that
\[
E[(S - d)_+] \leq E[(S - d)_+] \leq E[(S^c - d)_+]
\]
for all $d$ in $\mathbb{R}^+$, while $E[S^t] = E[S] = E[S^c]$ and $\text{var}[S^t] \leq \text{var}[S] \leq \text{var}[S^c]$.

### 2.1. Comonotonic upper bound

As proven in Dhaene et al. (2002), the convex-largest sum of the components of a random vector with given marginals is obtained by the comonotonic sum $S^c = X^c_1 + X^c_2 + \cdots + X^c_n$ with
\[
S^c = \frac{d}{\sum_{i=1}^{n} F_{X_i}^{-1}(U)}, \tag{2}
\]
where the usual inverse of a distribution function, which is the non-decreasing and left-continuous function $F_{X_i}^{-1}(p)$, is defined by
\[
F_{X_i}^{-1}(p) = \inf \{x \in \mathbb{R} \mid F_X(x) \geq p\}, \quad p \in [0, 1],
\]
with $\inf \emptyset = +\infty$ by convention.

Kaas et al. (2000) have proved that the inverse distribution function of a sum of comonotonic random variables is simply the sum of the inverse distribution functions of the marginal distributions. Moreover, in case of strictly increasing and continuous marginals, the cumulative distribution function (cdf) $F_{S^c}(x)$ is uniquely determined by
\[
F_{S^c}^{-1}(F_{S^c}(x)) = \sum_{i=1}^{n} F_{X_i}^{-1}(F_{S^c}(x)) = x, \quad F_{S^c}^{-1}(0) < x < F_{S^c}^{-1}(1). \tag{3}
\]
Hereafter we restrict ourselves to this case of strictly increasing and continuous marginals.
In the following theorem Dhaene et al. (2002) have proved that the stop-loss premiums of a sum of comonotonic random variables can easily be obtained from the stop-loss premiums of the terms.

**Theorem 1.** The stop-loss premiums of the sum $S^c$ of the components of the comonotonic random vector $(X_1^c, X_2^c, \ldots, X_n^c)$ are given by

$$E \left[ (S^c - d)_+ \right] = \sum_{i=1}^{n} E \left[ (X_i - F_{X_i}^{-1}(F_{S^c}(d)))_+ \right], \quad (F_{S^c}^{-1}(0) < d < F_{S^c}^{-1}(1)).$$

If the only information available concerning the multivariate distribution function of the random vector $(X_1, \ldots, X_n)$ are the marginal distribution functions of the $X_i$, then the distribution function of $S^c = F_{X_1}^{-1}(U) + F_{X_2}^{-1}(U) + \cdots + F_{X_n}^{-1}(U)$ is a prudent choice for approximating the unknown distribution function of $S = X_1 + \cdots + X_n$. It is a supremum in terms of convex order. It is the best upper bound that can be derived under the given conditions.

**2.2. Lower bound**

Let $X = (X_1, \ldots, X_n)$ be a random vector with given marginal cdfs $F_{X_1}, F_{X_2}, \ldots, F_{X_n}$. Let us now assume that we have some additional information available concerning the stochastic nature of $(X_1, \ldots, X_n)$. More precisely, we assume that there exists some random variable $\Lambda$ with a given distribution function, such that we know the conditional distribution, given $\Lambda = \lambda$, of the random variables $X_i$, for all possible values of $\lambda$. We recall from Kaas et al. (2000) that a lower bound, in the sense of convex order, for $S = X_1 + \cdots + X_n$ is

$$S^\ell = E [S \mid \Lambda].$$

This idea can also be found in Rogers and Shi (1995) for the continuous case.

Let us further assume that the random variable $\Lambda$ is such that all $E [X_i \mid \Lambda]$ are non-decreasing and continuous functions of $\Lambda$ and in addition assume that the cdfs of the random variables $E [X_i \mid \Lambda]$ are strictly increasing and continuous, then the cdf of $S^\ell$ is also strictly increasing.
and continuous, and we get for all \( x \in (F_{g^t}^{-1}(0), F_{g^t}^{-1}(1)) \),

\[
\sum_{i=1}^{n} F_{E[X_i|\Lambda]}^{-1}(F_{g^t}(x)) = x \iff \sum_{i=1}^{n} E \left[ X_i \mid \Lambda = F_{\Lambda}^{-1}(F_{g^t}(x)) \right] = x, \tag{6}
\]

which unambiguously determines the cdf of the convex order lower bound \( S^t \) for \( S \). Using Theorem 1, the stop-loss premiums of \( S^t \) can be computed as:

\[
E \left[ (S^t - d)^+ \right] = \sum_{i=1}^{n} E \left[ (E \left[ X_i \mid \Lambda \right] - E \left[ X_i \mid \Lambda = F_{\Lambda}^{-1}(F_{g^t}(d)) \right])^+ \right], \tag{7}
\]

which holds for all retentions \( d \in (F_{g^t}^{-1}(0), F_{g^t}^{-1}(1)) \).

So far, we considered the case that all \( E \left[ X_i \mid \Lambda \right] \) are non-decreasing functions of \( \Lambda \). The case where all \( E \left[ X_i \mid \Lambda \right] \) are non-increasing and continuous functions of \( \Lambda \) also leads to a comonotonic vector \( (E \left[ X_1 \mid \Lambda \right], E \left[ X_2 \mid \Lambda \right], \ldots, E \left[ X_n \mid \Lambda \right]) \), and can be treated in a similar way.

### 3. Basket options in a Black & Scholes setting: An exact part

We now shall concentrate on the pricing of basket options in the famous Black & Scholes model by noticing that there is a part which can be calculated in an exact way. The remaining part will be treated in subsequent paragraphs by lower and upper bounds and by approximations. Numerical computations show that the exact part contributes more than 90% to the (Monte Carlo) price of basket options.

Denoting by \( S_i(t) \) the price of the \( i \)-th asset in the basket at time \( t \), the basket is given by

\[
S(t) = \sum_{i=1}^{n} a_i S_i(t),
\]

where \( a_i \) are deterministic, positive and constant weights specified by the option contract. We assume that under the risk neutral measure \( Q \)

\[
dS_i(t) = rS_i dt + \sigma_i S_i dW_i(t),
\]
where \(\{W_i(t), t \geq 0\}\) is a standard Brownian motion associated with the price process of asset \(i\). Further, we assume the different asset prices to be instantaneously correlated according to

\[
\text{corr}(dW_i, dW_j) = \rho_{ij} dt.
\]  

(8)

Given the above dynamics, the \(i\)-th asset price at time \(t\) equals

\[
S_i(t) = S_i(0)e^{(r - \frac{1}{2}\sigma_i^2)t + \sigma_i W_i(t)}.
\]

We can rewrite the basket as a sum of lognormal variables

\[
S(t) = \sum_{i=1}^{n} X_i(t) = \sum_{i=1}^{n} \alpha_i(t)e^{Y_i(t)},
\]  

(9)

where \(\alpha_i(t) = a_i S_i(0)e^{(r - \frac{1}{2}\sigma_i^2)t} \geq 0\) and \(Y_i(t) = \sigma_i W_i(t) \sim N(0, \sigma_i^2 t)\) and thus \(X_i(t)\) is lognormally distributed: \(X_i(t) \sim LN(\ln(a_i S_i(0)) + (r - \frac{1}{2}\sigma_i^2)t, \sigma_i^2 t)\).

In this case the stop-loss premium with some retention \(d_i\), namely \(E^Q[(X_i - d_i)_+]\), is well-known since \(\ln(X_i(t)) \sim N(\mu_i(t), \sigma_{Y_i(t)}^2)\) with \(\mu_i(t) = \ln(\alpha_i(t))\) and \(\sigma_{Y_i(t)}^2 = \sigma_i \sqrt{t}\), and equals for \(d_i > 0\)

\[
E^Q[(X_i(t) - d_i)_+] = e^{\mu_i(t) + \frac{\sigma_{Y_i(t)}^2}{2}} \Phi(d_{i,1}(t)) - d_i \Phi(d_{i,2}(t)),
\]  

(10)

where \(d_{i,1}\) and \(d_{i,2}\) are determined by

\[
d_{i,1}(t) = \frac{\mu_i(t) + \sigma_{Y_i(t)}^2 - \ln(d_i)}{\sigma_{Y_i(t)}}, \quad d_{i,2}(t) = d_{i,1}(t) - \sigma_{Y_i(t)},
\]  

(11)

and where \(\Phi\) is the cdf of the \(N(0,1)\) distribution.

The case \(d_i < 0\) is trivial.

In what follows we only consider the basket at maturity date \(T\) and for the sake of notational simplicity, we shall drop the explicit dependence on \(T\) in \(X_i, \alpha_i\) and \(Y_i\).

For any normally distributed random variable \(\Lambda\), with cdf \(F_\Lambda(\cdot)\), for which there exists a \(d_\Lambda \in \mathbb{R}\), such that \(\Lambda \geq d_\Lambda\) implies \(S \geq K\), it follows that

\[
E^Q[(S - K)_+ | \Lambda] = E^Q[S - K | \Lambda] \equiv (S^\ell - K)_+.
\]  

(12)
For such \( \Lambda \), we can decompose the option price (1) in the following way

\[
e^{-rT}E^Q[(S - K)_+] = e^{-rT}E^Q[E^Q[(S - K)_+ \mid \Lambda]]
\]

\[
= e^{-rT} \left\{ \int_{-\infty}^{d_\Lambda} E^Q[(S - K)_+ \mid \Lambda = \lambda]dF_\Lambda(\lambda) + \int_{d_\Lambda}^{+\infty} E^Q[S - K \mid \Lambda = \lambda]dF_\Lambda(\lambda) \right\}.
\]  

We shall motivate the choice of \( \Lambda \) and, correspondingly, of \( d_\Lambda \) later.

The second term in (13) can be written out explicitly if for all \( i \), \( (Y_i, \Lambda) \) is bivariate normally distributed. Then, \( Y_i \mid \Lambda = \lambda \) is also normally distributed for all \( i \) with parameters \( \mu(i) = r_i \frac{\sigma_Y}{\sigma_\Lambda} (\lambda - E^Q[\Lambda]) \) and \( \sigma^2(i) = (1 - r_i^2) \sigma_Y^2 \), where \( r_i = r(Y_i, \Lambda) = \frac{\text{cov}(Y_i, \Lambda)}{\sigma_Y \sigma_\Lambda} \) is a correlation between \( Y_i \) and \( \Lambda \), and therefore we easily arrive at

\[
E^Q[S \mid \Lambda] = \sum_{i=1}^{n} a_iE^Q[S_i(T) \mid \Lambda] = \sum_{i=1}^{n} a_iS_i(0)e^{(r - \frac{\sigma_Y^2}{2}r_i^2)T + \sigma_i r_i \sqrt{T} \Phi^{-1}(V)},
\]

where the random variable \( V = \Phi\left( \frac{\Lambda - E^Q[\Lambda]}{\sigma_\Lambda} \right) \) is uniformly distributed on the unit interval and thus, \( \Phi^{-1}(V) = \frac{\Lambda - E^Q[\Lambda]}{\sigma_\Lambda} \) is a standard normal variate. Next we apply the equality

\[
\int_{-\infty}^{d_\Lambda} e^{b\Phi^{-1}(v)}f_\Lambda(\lambda)d\lambda = e^{\frac{b^2}{2}}\Phi(d_\Lambda^* - b), \quad d_\Lambda^* = \frac{d_\Lambda - E^Q[\Lambda]}{\sigma_\Lambda},
\]

where \( f_\Lambda(\cdot) \) is the normal density function for \( \Lambda \), and with \( b = \sigma_i r_i \sqrt{T} \), we can express the second term in (13) in closed-form:

\[
e^{-rT} \int_{d_\Lambda}^{+\infty} E^Q[S - K \mid \Lambda = \lambda]dF_\Lambda(\lambda)
\]

\[
= e^{-rT} \int_{d_\Lambda}^{+\infty} E^Q[S \mid \Lambda = \lambda]f_\Lambda(\lambda)d\lambda - e^{-rT}K(1 - F_\Lambda(d_\Lambda))
\]

\[
= e^{-rT} \sum_{i=1}^{n} a_iS_i(0)e^{(r - \frac{\sigma_Y^2}{2}r_i^2)T} \int_{d_\Lambda}^{+\infty} e^{r_i \sigma_i \sqrt{T} \Phi^{-1}(v)}f_\Lambda(\lambda)d\lambda - e^{-rT}K(1 - \Phi(d_\Lambda^*))
\]

\[
= \sum_{i=1}^{n} a_iS_i(0)\Phi(r_i \sigma_i \sqrt{T} - d_\Lambda^*) - e^{-rT}K\Phi(-d_\Lambda^*). \quad \text{ (16)}
\]

Next we discuss the choice of the conditioning variable \( \Lambda \) which should not only be normally distributed but also be chosen such that \( (Y_i, \Lambda) \) for all \( i \) are bivariate normally distributed.
Hence, we define $\Lambda$ by

$$
\Lambda = \sum_{i=1}^{n} \beta_i \sigma_i W_i(T)
$$

(17)

with $\beta_i$ some real numbers. Since the conditioning variable only enters the exact part (16) via its correlation with the $Y_i$'s, we report these correlations explicitly:

$$
r_i = \frac{\text{cov}^Q(\sigma_i W_i(T), \Lambda)}{\sqrt{T \sigma_i \sigma_\Lambda}} = \frac{\sum_{j=1}^{n} \beta_j \rho_{ij} \sigma_j}{\sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} \beta_i \beta_j \rho_{ij} \sigma_i \sigma_j}}.
$$

(18)

In this paper we consider the following types of conditioning variable $\Lambda$, motivated by the idea that one should put as much available information of $S$ as possible in the conditioning variable. Further on, we shall compare these choices to an optimal choice obtained by an optimization procedure.

- As a first conditioning variable we take a linear transformation of a first order approximation of $S$ (denoted by $FA1$):

$$
FA1 = \sum_{i=1}^{n} e^{(r-\frac{\sigma^2}{2})T} a_i S_i(0) \sigma_i W_i(T),
$$

(19)

and the correlation coefficients then read

$$
r_i = \frac{\sum_{j=1}^{n} a_j S_j(0) e^{(r-\frac{\sigma^2}{2})T} \rho_{ij} \sigma_j}{\sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} a_i S_i(0) e^{(r-\frac{\sigma^2}{2})T} a_j S_j(0) e^{(r-\frac{\sigma^2}{2})T} \rho_{ij} \sigma_i \sigma_j}}.
$$

(20)

- As a second conditioning variable (denoted by $FA2$), we consider

$$
FA2 = \sum_{i=1}^{n} a_i S_i(0) \sigma_i W_i(T).
$$

(21)

In this case, the correlation between $Y_i$ and $\Lambda$ is easily found to be

$$
r_i = \frac{\sum_{j=1}^{n} a_j S_j(0) \rho_{ij} \sigma_j}{\sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} a_i S_i(0) a_j S_j(0) \rho_{ij} \sigma_i \sigma_j}}.
$$

(22)

Note that $FA2$ is also a first order approximation of $S$ and in fact of $FA1$. 


As a third conditioning variable (denoted by $GA$), we look at the standardized logarithm of the geometric average $G$ which is defined by

$$G = \prod_{i=1}^{n} S_i(T)^{a_i} = \prod_{i=1}^{n} \left( S_i(0)e^{(r-\frac{\sigma_i^2}{2})T+\sigma_i W_i(T)} \right)^{a_i}. \quad (23)$$

Indeed, we can consider

$$GA = \frac{\ln G - E^Q[\ln G]}{\sqrt{\text{var}^Q[\ln G]}} = \frac{\sum_{i=1}^{n} a_i \sigma_i W_i(T)}{\sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \sigma_i \sigma_j \rho_{ij} T}}, \quad (24)$$

since

$$E^Q[\ln G] = \sum_{i=1}^{n} a_i \left( \ln(S_i(0)) + (r - \frac{\sigma_i^2}{2})T \right) \quad (25)$$

$$\text{var}^Q[\ln G] = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \sigma_i \sigma_j \rho_{ij} T. \quad (26)$$

The correlation coefficients in this case are given by

$$r_i = \frac{\sum_{j=1}^{n} a_j \sigma_j \rho_{ij}}{\sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \rho_{ij} \sigma_i \sigma_j}}. \quad (27)$$

We can determine the integration bound $d_\Lambda$ for each of the three different $\Lambda$’s (19), (21) and (23), such that $\Lambda \geq d_\Lambda$ implies that $S \geq K$. Bounding the exponential function $e^x$ below by its first order approximation $1 + x$ with $x = \sigma_i W_i(T)$, respectively $x = (r - \frac{\sigma_i^2}{2})T + \sigma_i W_i(T)$, the integration bound corresponding to $\Lambda = FA_1$ given by (19), respectively to $\Lambda = FA_2$ given by (21), is found to be

$$d_{FA_1} = K - \sum_{i=1}^{n} a_i S_i(0)e^{(r-\frac{\sigma_i^2}{2})T}, \quad (28)$$

$$d_{FA_2} = K - \sum_{i=1}^{n} a_i S_i(0)(1 + (r - \frac{\sigma_i^2}{2})T). \quad (29)$$

When $\Lambda$ is the standardized logarithm of the geometric average (GA), see (24), we use the relationship $S \geq G \geq K$ and (25)–(26) in order to arrive at

$$d_{GA} = \frac{\ln(K) - \sum_{i=1}^{n} a_i \left( \ln(S_i(0)) + (r - \frac{\sigma_i^2}{2})T \right)}{\sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \sigma_i \sigma_j \rho_{ij} T}}. \quad (30)$$
4. Bounds

In Section 3 we decomposed the basket option price into the exact part (16) and the remaining part. Deriving bounds for the latter according to Section 2, and adding up to the exact part we obtain the bounds for the basket option price (1).

4.1. Lower bound.

By means of Jensen’s inequality, the first term in (13) can be bounded below as follows:

\[
\int_{-\infty}^{d_\Lambda} E^Q[(S - K)_+ | \Lambda = \lambda] dF_\Lambda(\lambda) \geq \int_{-\infty}^{d_\Lambda} (E^Q[S | \Lambda = \lambda] - K)_+ dF_\Lambda(\lambda).
\]

By adding the exact part (16) and introducing notation (5), we end up with the inequality

\[
E^Q[(S - K)_+] \geq E^Q[(S^\ell - K)_+],
\]

where we recall that $S^\ell$ is in convex order a lower bound of $S$. When $S^\ell$ is a sum of $n$ comonotonic risks we can apply the results of Section 2.2. Hereto the correlations $r_i$ (22) should all have the same sign. For the time being we assume that the correlations are positive for all $i$, we shall come back to this issue later on. Invoking (7) and (10)-(11), and substituting $\alpha_i$ and the standard deviation of $Y_i$, we find the following lower bound for the price of the basket call option:

\[
BC(n, K, T) \geq \sum_{i=1}^{n} a_i S_i(0) \Phi \left[ \sigma_i \sqrt{T} r_i - \Phi^{-1}(F_{S^\ell}(K)) \right] - e^{-rT} K \left( 1 - F_{S^\ell}(K) \right)
\]

which holds for any $K > 0$ and where $F_{S^\ell}(K)$, according to (6), solves

\[
\sum_{i=1}^{n} a_i S_i(0) e^{\left( r - \frac{1}{2} \sigma_i^2 \right) T + r \sigma_i \sqrt{T} \Phi^{-1}(F_{S^\ell}(K))} = K.
\]

The lower bound (32)-(33) can be formulated as an average of Black & Scholes formulae with new underlying assets and new exercise prices. The new assets $\tilde{S}_i$ are with $\tilde{S}_i(0) = S_i(0)$ and with new volatilities $\tilde{\sigma}_i = \sigma_i r_i$ for $i = 1, \ldots, n$. The new exercise prices $\tilde{K}_i$, $i = 1, \ldots, n$,
are given by
\[ \tilde{K}_i = \tilde{S}_i(0)e^{(r-\frac{\tilde{\sigma}^2}{2})T+\tilde{\sigma}i\sqrt{T}\Phi^{-1}(F_{x_i}(K))}. \] (34)

Indeed,
\[ BC(n,K,T) \geq \sum_{i=1}^{n} a_i \left( \tilde{S}_i(0) \Phi (d_{1i}) - e^{-rT} \tilde{K}_i \Phi (d_{2i}) \right) \] (35)

with
\[ d_{1i} = \frac{\ln \left( \frac{\tilde{S}_i(0)}{\tilde{K}_i} \right) + (r+\tilde{\sigma}^2)T}{\tilde{\sigma}\sqrt{T}} \quad \text{and} \quad d_{2i} = d_{1i} - \tilde{\sigma}\sqrt{T}, \quad \text{for } i = 1, \ldots, n. \]

We stress that (7) is an equality and thus the decomposition in a sum of \( n \) terms is an optimal one. Hence also in this sense, given a conditioning variable \( \Lambda \), (32) or (35) is the best lower bound that can be obtained if one wants the lower bound in the form of a linear combination of Vanilla European call options. However, the conditioning variable is still a free parameter and its choice can be optimized. Hereby note that the integration bound \( d_{\Lambda} \) is just a technical tool for making the link between the exact part and the lower bound, but does not appear in the final expression (32) or (35) of this lower bound because this is based on the inequality (31).

Further remark that the lower bound only depends on the conditioning variable \( \Lambda \) through the correlations \( r_i \). In case \( r_i \) equals one we obtain the exact price, but unfortunately, it will not always be possible to find a conditioning variable \( \Lambda \) such that \( r_i = 1 \) for all \( i \). But we do have that for the conditioning variables (19), (21) and (24) the lower bound is quite good. Beißer (2001), who has obtained the same lower bound by using other arguments, chooses along intuitive arguments the numerator of the standardized logarithm of the geometric average (24). This is indeed a good choice since the geometric average and arithmetic average are based on the same information. In this case, the correlation coefficients in the formulae for the lower bound are given by (27). Note however that these correlation coefficients are independent of the initial value of the assets in the basket which can lead to a lower quality of the lower bound when the assets in the basket have different initial values. Beißer (2001) therefore considers also the conditioning variable \( \Lambda \) given by (21) with correlations \( r_i \) (22), which depend on the
weights, the initial values and the volatilities of the assets in the basket. It is easily seen that the lower bound will coincide for the three different choices of $\Lambda$ when the initial values as well as the volatilities are equal for the different assets. When only the volatilities $\sigma_i$ are equal for all $i$ then the correlation coefficients (20) and (22) coincide and hence also the corresponding lower bounds. Similarly, when only the initial values are equal the correlation coefficients (22) and (27) lead to the same lower bound.

Next we go deeper into the assumption of positiveness for the correlation coefficients $r_i$ (18). This condition is needed for $S^\ell$ (14) to be a comonotonic sum and thus for (7) to hold and to be applied.

When the correlations $\rho_{ij}$ (8) are positive for all $i$ and $j$ then it suffices to take all coefficients $\beta_i$ also with a positive sign in order to satisfy the assumption. However when a $\rho_{ij}$ is negative a general discussion is much more involved. Therefore, we first look at the special case when $n = 2$ and $\rho_{12} = \rho_{21} = \rho \leq 0$. The conditions $r_1, r_2 \geq 0$ are equivalent to

\[
\begin{cases}
\beta_1 \sigma_1 - \beta_2 \sigma_2 |\rho| \geq 0 \\
\beta_2 \sigma_2 - \beta_1 \sigma_1 |\rho| \geq 0
\end{cases}
\Leftrightarrow \beta_2 \sigma_2 |\rho| \leq \beta_1 \sigma_1 \leq |\rho|, \quad (36)
\]

and imply that $\beta_1$ and $\beta_2$ should have the same sign and differ from zero. For simplicity assume that $\beta_1$ and $\beta_2$ are both strictly positive, then the condition (36) can be rewritten as

\[
|\rho| \leq \frac{\beta_1 \sigma_1}{\beta_2 \sigma_2} \leq \frac{1}{|\rho|}. \quad (37)
\]

Note that since $|\rho| \leq 1$, the second inequality is trivially fulfilled when $\beta_1 \sigma_1 \leq \beta_2 \sigma_2$ while in the case $\beta_1 \sigma_1 \geq \beta_2 \sigma_2$ the first inequality is trivial. Hence only one of these inequalities has to be checked. Beißer (2001) made a similar reasoning but only for the particular correlation coefficients (22).

When $\rho$ is negative, it can happen that for none of the three choices for $\Lambda$, namely (19), (21) and (24), relation (37) is satisfied. However, since we derived a lower bound for any $\Lambda$ given by (17) we are not restricted to the three choices. Indeed, it is always possible to find a $\beta_1$ and
\[ \beta_2 \text{ since the interval } [|\rho|\sigma_2, \frac{1}{|\rho|}\sigma_1] \text{ is non-empty.} \]

In fact, one might search for the \( \beta_1 \) and \( \beta_2 \) which leads to an optimal lower bound. When we write \( r_i, i = 1, 2 \) in function of \( x = \frac{\beta_2\sigma_2}{\beta_1\sigma_1} \), we find that \( r_1 = \frac{1+\rho x}{\sqrt{1+x^2+2x\rho}} \) and \( r_2 = \frac{\rho+x}{\sqrt{1+x^2+2x\rho}} \).

If one assumes that \( r_2 \neq 1 \) and if we rewrite the equation defining \( r_2 \), we find the relation \( r_1 - r_2 \rho = \sqrt{(1-\rho^2)(1-r_2^2)} \). As a consequence, the optimal lower bound becomes the solution to the optimization program

\[
\max_{r_1, r_2} \text{LB}(r_1, r_2) = \sum_{i=1}^{2} a_i[\tilde{S}_i(0)\Phi(d_{1i}) - e^{-r^T\tilde{K}_i}\Phi(d_{1i})]
\]

such that \( 0 \leq r_1 \leq 1, 0 \leq r_2 < 1 \)

\[ \sum_{i=1}^{2} a_i \tilde{K}_i = K \]

\[ r_1 - r_2 \rho = \sqrt{1-\rho^2}\sqrt{1-r_2^2} \]

where we used the notation introduced in (34)-(35). Solving this problem by Lagrange optimization leads to a conclusion of three cases:

1. If \( r_2 = 0 \) and \( r_1 = \sqrt{1-\rho^2} \), which is only possible if \( \rho < 0 \), the lower bound is maximized under the above conditions if

\[ a_2\tilde{K}_2\sigma_2 + \rho\sigma_1(K - a_2\tilde{K}_2) \leq 0 \text{ with } \tilde{K}_2 = S_2(0)e^{rT}. \]

2. If \( r_1 \) and \( r_2 \) are strictly between 0 and 1, \( r_1 \) and \( r_2 \) are solutions to the equations:

\[
\begin{align*}
& e^{-r^T\varphi(\Phi^{-1}(F_{S\ell}(K)))}a_1\tilde{K}_1\sigma_1\sqrt{T} - \lambda_2 = 0 \\
& e^{-r^T\varphi(\Phi^{-1}(F_{S\ell}(K)))}a_2\tilde{K}_2\sigma_2\sqrt{T} - \lambda_2 \left(-\rho + r_2\sqrt{1-\rho^2}\right) = 0 \\
& a_1\tilde{K}_1 + a_2\tilde{K}_2 = K \\
& r_1 - r_2 \rho = \sqrt{1-\rho^2}\sqrt{1-r_2^2},
\end{align*}
\]

which are four non-linear equations in four unknowns \( r_1, r_2, \lambda_2, \Phi^{-1}(F_{S\ell}(K)) \) and where \( \varphi \) is the density function of the \( N(0,1) \) distribution.
3. If \( r_1 = 0 \) and \( r_2 = \sqrt{1 - \rho^2} \), which is only possible if \( \rho < 0 \), the lower bound is maximized under the above conditions if

\[
a_1 \tilde{K}_1 \sigma_1 + \rho \sigma_2 (K - a_1 \tilde{K}_1) \leq 0 \quad \text{with} \quad \tilde{K}_1 = S_1(0)e^{rT}.
\]

We now turn to the general case for \( n \geq 3 \) where at least one correlation \( \rho_{ij} \), defined in (8), is supposed to be strictly negative. As a conclusion of the following statement we see that a lower bound can be computed also in a general case. However, the optimization program will be much more involved when there are more than two assets in the basket.

**Theorem 2.** There always exist coefficients \( \beta_i \in \mathbb{R} \), \( i = 1, \ldots, n \), in (17) such that all correlations \( r_i \), \( i = 1, \ldots, n \), (18) are positive.

**Proof.** Denoting \( A \) for the correlation matrix \((\rho_{ij})_{1 \leq i,j \leq n}\) and putting \( \tilde{\beta}^T = (\beta_1 \sigma_1, \ldots, \beta_n \sigma_n) \), the conditions \( r_i \geq 0 \), \( i = 1, \ldots, n \) are equivalent to \( A \tilde{\beta} \geq 0 \).

As all variance-covariance matrices, this matrix \( A \) is symmetric and positive semi-definite. Moreover it is non-singular and positive definite since we assume that the market is complete. By a reasoning ex absurdo we show that at least one of the coefficients \( \beta_i \) is strictly positive: Assume that all \( \beta_i \) are negative then from \( A \tilde{\beta} \geq 0 \) it follows that \( \tilde{\beta}^T A \tilde{\beta} \leq 0 \) which is a contradiction to the positive definiteness of \( A \).

Finally, using the link between the primal and the dual of a linear programming problem, and again by the positiveness of matrix \( A \) the assertion can then be proved. \( \square \)

For a more detailed discussion we refer to Section 7.

**4.2. Upper bounds.**

4.2.1. Partially exact/comonotonic upper bound. Recalling that \( Y_i \mid \Lambda = \lambda \) is normally distributed for all \( i \) with parameters \( \mu(i) \) and \( \sigma^2(i) \), see Section 3, we bound the first term of (13) above by replacing \( S \mid \Lambda = \lambda \) by its comonotonic upper bound \( S^c_{\Lambda=\lambda} \) (in convex order sense),
given by
\[ S_{\Lambda=\lambda}^c = (S \mid \Lambda = \lambda)^c = \frac{d}{\sum a_i e^{r_i \sigma_i \Phi^{-1}(v) + \sqrt{1-r_i^2 \sigma_i^2 \Phi^{-1}(U)}} \]

where \( U \) and \( V = \Phi\left(\frac{\Lambda - E^Q[\Lambda]}{\sigma_\Lambda}\right) \) are mutually independent uniform\((0,1)\) random variables.

Combining (3)-(4) with (10)-(11) and substituting \( \alpha_i \) and the standard deviation of \( Y_i \), the comonotonic upper bound for the first term of (13) reads:
\[ e^{-rT} \int_{-\infty}^{d_\Lambda} E^Q[(S - K) \mid \Lambda = \lambda] f_\Lambda(\lambda) d\lambda \]
\[ \leq e^{-rT} \int_{-\infty}^{d_\Lambda} E^Q[(S_{\Lambda=\lambda}^c - K) \mid \Lambda = \lambda] f_\Lambda(\lambda) d\lambda \]
\[ = e^{-rT} \int_0^{\Phi(d_\Lambda^*)} E^Q[(S_{\Lambda=\lambda}^c - K) \mid \Lambda = \lambda] dv \]
\[ = \sum_{i=1}^{n} a_i S_i(0) e^{-\frac{1}{2} \sigma^2_i r^2 T} \Phi(d_\Lambda^*) e^{r_i \sigma_i \sqrt{T} \Phi^{-1}(v)} \Phi\left(\sqrt{1-r_i^2 \sigma_i^2 \sqrt{T} - \Phi^{-1}\left(F_{S_{\Lambda=\lambda}^c}(K)\right)}\right) dv \]
\[ - e^{-rT} K \left( \Phi(d_\Lambda^*) - \int_0^{\Phi(d_\Lambda^*)} F_{S_{\Lambda=\lambda}^c}(K) dv \right), \] (39)

where we recall that \( d_\Lambda^* \) is defined as in (15), and the cumulative distribution \( F_{S_{\Lambda=\lambda}^c}(K) \) is, according to (3), determined by
\[ \sum_{i=1}^{n} a_i S_i(0) \exp \left[ \left( r - \frac{\sigma^2_i}{2} T + r_i \sigma_i \sqrt{T} \Phi^{-1}(v) + \sqrt{1-r_i^2 \sigma_i^2 \sqrt{T} \Phi^{-1}\left(F_{S_{\Lambda=\lambda}^c}(K)\right)} \right) \right] = K. \]

Adding (39) to the exact part (16) of the decomposition (13) results in the so-called partially exact/comonotonic upper bound for \( BC(n, K, T) \).

For the random variables \( \Lambda \) given by (19), (21) and (24) we derived a \( d_\Lambda \), see (28), (29) and (30), and thus we can compute this upper bound.

Note that we can rewrite the comonotonic upper bound, applied to the first term in (13), as a linear combination of Black & Scholes prices for European call options. For this purpose, given \( \Lambda = \lambda \) or equivalently given \( V = v \), we introduce some artificial underlying assets \( \tilde{S}_{i,v} \) having volatilities \( \tilde{\sigma}_{i,v} = \sigma_i \sqrt{1-r_i^2} \) and with initial value
\[ \tilde{S}_{i,v}(0) = S_i(0) e^{-\frac{1}{2} \sigma^2_i r^2 T + r_i \sigma_i \sqrt{T} \Phi^{-1}(v)}. \]
We also consider new exercise prices:

\[
\widetilde{K}_{i,v} = S_i(0)e^{(r - \frac{\sigma_i^2}{2})T + r_i \sigma_i \sqrt{T} \Phi^{-1}(v) + \sqrt{1 - r_i^2} \sigma_i \sqrt{T} \Phi^{-1}(\bar{F}_{S_i,\lambda}(K))}.
\]

We also stress here that in view of the equality in (4) and given \( \Lambda = \lambda \) or equivalently, given \( V = v \), this linear combination of Black & Scholes prices is the best, i.e. the smallest, upper bound constructed in this way. As for the lower bound, the choice of the conditioning variable is an additional parameter to further optimize this upper bound but is limited by the knowledge of an integration bound \( d_\Lambda \).

4.2.2. Upper bound based on lower bound. In this section we follow the ideas of Rogers and Shi (1995) in order to derive an upper bound based on the lower bound. By Jensen’s inequality and according to (12) we can find an error bound to the first term in (13):

\[
0 \leq E^Q \left[ E^Q \left[ \left( S - K \right)_+ \mid \Lambda \right] - \left( S' - K \right)_+ \right]
\]

\[
= \int_{-\infty}^{d_\Lambda} \left( E^Q \left[ \left( S - K \right)_+ \mid \Lambda = \lambda \right] - \left( E^Q \left[ S \mid \Lambda = \lambda \right] - K \right)_+ \right) f_\Lambda(\lambda) d\lambda
\]

\[
\leq \frac{1}{2} \int_{-\infty}^{d_\Lambda} \left( \text{var}^Q \left( S \mid \Lambda = \lambda \right) \right)^{\frac{1}{2}} f_\Lambda(\lambda) d\lambda
\]

\[
\leq \frac{1}{2} \left( E^Q \left[ \text{var}^Q \left( S \mid \Lambda \right) 1_{\{\Lambda < d_\Lambda\}} \right] \right)^{\frac{1}{2}} \left( E^Q \left[ 1_{\{\Lambda < d_\Lambda\}} \right] \right)^{\frac{1}{2}}, \tag{40}
\]

where Hölder’s inequality has been applied in the last inequality and where \( 1_{\{\Lambda < d_\Lambda\}} \) is the indicator function.

Now we shall derive an easily computable expression for (40).

The second expectation factor in the product (40) equals \( \Phi(d_\Lambda^*) \), where \( d_\Lambda^* \) is defined in (15).

The first expectation factor in the product (40) can be expressed as

\[
E^Q \left[ \text{var}^Q \left( S \mid \Lambda \right) 1_{\{\Lambda < d_\Lambda\}} \right] = E^Q \left[ E^Q[S^2 \mid \Lambda] 1_{\{\Lambda < d_\Lambda\}} \right] - E^Q \left[ (E^Q[S \mid \Lambda])^2 1_{\{\Lambda < d_\Lambda\}} \right]. \tag{41}
\]
The second term of the right-hand side of (41) can according to (14) be rewritten as

\[
E^Q \left[ (E^Q [S|\Lambda])^2 1_{\{\Lambda < d_A\}} \right] = \int_{-\infty}^{d_A} (E^Q [S|\Lambda] = \lambda)^2 f_\Lambda(\lambda) d\lambda
\]

\[
= \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j S_i(0)S_j(0) e^{(2r - \frac{\sigma_i^2 + \sigma_j^2}{2}) T + \frac{\sigma_i^2 + \sigma_j^2}{2} T} \int_{-\infty}^{d_A} e^{\sigma_i r_i + \sigma_j r_j} \sqrt{T} \Phi^{-1}(v) f_\Lambda(\lambda) d\lambda,
\]

where we recall that \( \Phi^{-1}(v) = \frac{\lambda - E^Q[\Lambda]}{\sigma_\Lambda} \) and \( \Phi(\cdot) \) is the cumulative distribution function of a standard normal variable. Applying the equality (15) with \( b = (\sigma_i r_i + \sigma_j r_j) \sqrt{T} \) we can express \( E^Q \left[ (E^Q [S|\Lambda])^2 1_{\{\Lambda < d_A\}} \right] \) as

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j S_i(0)S_j(0) e^{(2r + \sigma_i r_i + \sigma_j r_j) T} \Phi \left( d_\Lambda^* - (r_i \sigma_i + r_j \sigma_j) \sqrt{T} \right).
\]

To transform the first term of the right-hand side of (41) we can rely on the fact that the product of two lognormal random variables is again lognormal:

\[
E^Q [S^2 | \Lambda] = \sum_{i=1}^{n} \sum_{j=1}^{n} E^Q [a_i a_j S_i(T)S_j(T) | \Lambda]
\]

\[
= \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j S_i(0)S_j(0) e^{(2r - \frac{\sigma_i^2 + \sigma_j^2}{2}) T + r_i \sigma_i + r_j \sigma_j \sqrt{T} \Phi^{-1}(V) + \frac{1}{2} (1 - r_i^2) T \sigma_i^2 + \frac{1}{2} (1 - r_j^2) T \sigma_j^2},
\]

with \( \sigma_{ij}^2 = \sigma_i^2 + \sigma_j^2 + 2 \sigma_i \sigma_j \rho_{ij} \) and \( r_{ij} = \frac{\sigma_i}{\sigma_{ij}} r_i + \frac{\sigma_j}{\sigma_{ij}} r_j \), and where \( V = \Phi(\frac{\Lambda - E^Q[\Lambda]}{\sigma_\Lambda}) \) is uniformly distributed on the interval \((0, 1)\). Next we apply (15) again with \( b = r_{ij} \sigma_{ij} \sqrt{T} = (\sigma_i r_i + \sigma_j r_j) \sqrt{T} \):

\[
E^Q \left[ E^Q [S^2 | \Lambda] 1_{\{\Lambda < d_A\}} \right]
\]

\[
= \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j S_i(0)S_j(0) e^{(2r - \frac{\sigma_i^2 + \sigma_j^2}{2}) T + \frac{1}{2} (1 - r_i^2) T \sigma_i^2 T} \int_{-\infty}^{d_A} e^{r_{ij} \sigma_{ij} \sqrt{T} \Phi^{-1}(v)} f_\Lambda(\lambda) d\lambda
\]

\[
= \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j S_i(0)S_j(0) e^{(2r + \sigma_i r_i + \sigma_j r_j) T} \Phi \left( d_\Lambda^* - (r_i \sigma_i + r_j \sigma_j) \sqrt{T} \right).
\]
Combining (43) and (45) into (41), and then substituting $\Phi(d^*_\Lambda)$ and (41) into (40) we get the following expression for the error bound, shortly denoted by $\varepsilon(d \Lambda)$

$$
\varepsilon(d \Lambda) = \frac{1}{2} \{\Phi(d^*_\Lambda)\}^{1/2} \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j S_i(0) S_j(0) e^{2r_i \sigma_i \sigma_j r_j} T \Phi\left( d^*_\Lambda - (r_i \sigma_i + \sigma_j r_j) \sqrt{T} \right) \times \\
\times \left( e^{\sigma_i \sigma_j (r_i \sigma_j - r_i r_j)} T - 1 \right) \right\}^{1/2}.
$$

The upper bound for the basket option price $BC(n, K, T)$ is then given by adding the lower bound (32) (which contains the exact part (16)) and the error term (46) multiplied with the discount factor $e^{-rT}$.

5. Moment based approximations

In this section we investigate approximations to the option price by using a moment matching method.

From Section 3 we know that $S | \Lambda = \lambda$ is a sum of $n$ lognormal random variables with parameters $\mu(i) + \ln(\alpha_i)$ and $\sigma^2(i)$, $i = 1, \ldots, n$. We approximate this sum by assuming that it is also lognormally distributed with parameters $\mu$ and $\sigma$ (depending on $\Lambda = \lambda$ or equivalently on $V = v = \Phi\left( \frac{\lambda - E^Q[\Lambda]}{\sigma_\Lambda} \right)$) and having the same first two moments as the sum itself:

$$
e^{\mu + \frac{1}{2} \sigma^2} = E^Q[S | \Lambda = \lambda] \quad (47)
$$

$$
e^{2(\mu + \sigma^2)} = E^Q[S^2 | \Lambda = \lambda]. \quad (48)
$$

Explicit expressions for the right-hand sides are given by (14) and (44) for $V = v$.

Taking the logarithm of both sides of (47)-(48) and solving for $\sigma^2$ yields

$$
\sigma^2 = 2(\mu + \sigma^2) - 2(\mu + \frac{1}{2} \sigma^2) = \ln E^Q[S^2 | \Lambda = \lambda] - 2 \ln E^Q[S | \Lambda = \lambda].
$$

Then, the integrand of the first integral in (13) can be approximated by a European call option price for which the standard Black & Scholes formula (10)-(11) applies, leading to:

$$
\int_{-\infty}^{d^*_\Lambda} E^Q[(S - K)_+ | \Lambda = \lambda] dF_{\Lambda}(\lambda) \approx \int_{-\infty}^{d^*_\Lambda} [E^Q[S | \Lambda = \lambda] \Phi(d_1(\lambda)) - K \Phi(d_2(\lambda))] dF_{\Lambda}(\lambda),
$$
with
\[ d_1(\lambda) = \frac{\frac{1}{2} \ln E^Q[S^2 | \Lambda = \lambda] - \ln(K)}{\sigma}, \quad d_2(\lambda) = d_1(\lambda) - \sigma. \]

Instead of approximating \( S | \Lambda = \lambda \) when \( \Lambda \) is given by (19) or (21), we consider the difference \( H \) between the sum \( S \) and its approximation \( \Lambda + K - d_\Lambda \) (as in the derivation of \( d_\Lambda \)):
\[ S - K = (S - (\Lambda + K - d_\Lambda)) - (K - (\Lambda + K - d_\Lambda)) \not= H - \tilde{K}, \]
and approximate \( H | \Lambda = \lambda \) by a lognormal with parameters \( \mu_H \) and \( \sigma^2_H \) (depending on \( \Lambda = \lambda \), or equivalently on \( V = v \)) using a moment matching method:
\[ e^{\mu_H + \frac{1}{2} \sigma^2_H} = E^Q[H | \Lambda = \lambda] = E^Q[S | \Lambda = \lambda] - (\lambda + K - d_\Lambda), \]
\[ e^{2(\mu_H + \sigma^2_H)} = E^Q[H^2 | \Lambda = \lambda] \]
\[ = E^Q[S^2 | \Lambda = \lambda] - 2(\lambda + K - d_\Lambda)E^Q[S | \Lambda = \lambda] + (\lambda + K - d_\Lambda)^2, \]
where the right-hand sides are expressed in terms of the expectations (14) and (44) for \( V = v \).

Note that \( \tilde{K} = d_\Lambda - \Lambda \) is known and positive for a given \( \Lambda = \lambda \) with \( \lambda \leq d_\Lambda \).

An approximation to the first integral in (13) then reads
\[ \int_{-\infty}^{d_\Lambda} E^Q[(H - \tilde{K})_+ | \Lambda = \lambda]dF_\Lambda(\lambda) \approx \int_{-\infty}^{d_\Lambda} \left[ (E^Q[S | \Lambda = \lambda] - (\lambda + K - d))\Phi(d_1(\lambda)) - (d_\Lambda - \lambda)\Phi(d_2(\lambda)) \right] dF_\Lambda(\lambda) \]
\[ = \int_0^{\Phi(d_\Lambda^* \sigma_H)} \left[ (E^Q[S | V = v] - (\sigma_\Lambda \Phi^{-1}(v) + E^Q[\Lambda] + K - d_\Lambda))\Phi(d_1(v)) \]
\[ - (d_\Lambda - \sigma_\Lambda \Phi^{-1}(v) - E^Q[\Lambda])\Phi(d_2(v)) \right] dv \]
with \( d_\Lambda^* \) defined in (15) and with, recalling that \( \lambda = \sigma_\Lambda \Phi^{-1}(v) + E^Q[\Lambda] \),
\[ d_1(\lambda) = d_1(v) = \frac{\mu_H + \sigma^2_H - \ln(d_\Lambda - \sigma_\Lambda \Phi^{-1}(v) - E^Q[\Lambda])}{\sigma_H}, \quad d_2(v) = d_1(v) - \sigma_H. \]

We can further approximate this result by assuming that the parameters \( \mu_H \) and \( \sigma^2_H \) are independent of \( \lambda \) and thus are constant. Since the main contribution to this integral comes
from values \( \lambda \) close to \( d_A \) (for \( \lambda \ll d_A \), \( (S - K)_+ \mid \Lambda = \lambda \) equals zero by definition of \( d_A \)), we now put

\[
\mu_H + \frac{1}{2} \sigma^2_H = \frac{1}{2} \ln E^Q [H^2 \mid \Lambda = d_A]
\]

\[
\sigma^2_H = 2(\mu_H + \frac{1}{2} \sigma^2_H) - 2 \ln E^Q [H \mid \Lambda = d_A].
\]

Note that \( d_1 \) and \( d_2 \) however remain dependent on \( \lambda \) through the term \( -\ln(\lambda + K - d_A) \) in de numerator.

In case \( \Lambda = GA = \ln G - \mathbb{E}[\ln G] \sqrt{\text{var}[\ln G]} \) (24), we follow the approach of Curran (1994) and consider the difference between the sum \( S \) and the geometric average \( G \):

\[
S - K = (S - G) - (K - G) \not= H - \tilde{K},
\]

where \( H = S - G \) and \( \tilde{K} = K - G \), and approximate \( H \mid G \) by a lognormal variable using a moment matching method. Note that we have a new strike price \( \tilde{K} = (K - G) \) which is known once \( G \) (or equivalently \( \Lambda \)) is known, and which is positive for \( G = g \) and \( g \leq K \). The latter is also equivalent with \( \Lambda = \lambda \) and \( \lambda \leq d_{GA} \). Hence the first integral in (13) can be transformed to:

\[
\int_{-\infty}^{d_{GA}} E^Q [(S - K)_+ \mid \Lambda = \lambda] \, dF_A(\lambda) = \int_0^K E^Q [(S - K)_+ \mid G = g] \, dF_G(g).
\]

Thus we assume that \( H \mid G = g \) is lognormal with parameters \( \mu_H \) and \( \sigma^2_H \) (depending on \( G = g \)) which are such that

\[
e^{\mu_H + \frac{1}{2} \sigma^2_H} = E^Q [H \mid G = g] = E^Q [S \mid G = g] - g,
\]

\[
e^{2(\mu_H + \sigma^2_H)} = E^Q [H^2 \mid G = g] = E^Q [S^2 \mid G = g] - 2gE^Q [S \mid G = g] - g^2,
\]

where the first two moments of \( S \mid G = g \) are given by (14) and (44) with \( \Phi^{-1}(v) = \frac{\ln g - E^Q [\ln G]}{\text{var}[\ln G]} \) ((25)-(26)). Taking the logarithm of both sides of (49)–(50) and solving for \( \sigma^2_H \), we find

\[
\sigma^2_H = 2(\mu_H + \sigma^2_H) - 2(\mu_H + \frac{1}{2} \sigma^2_H)
\]

\[
= \ln E^Q [H^2 \mid G = g] - 2 \ln E^Q [H \mid G = g].
\]
By assuming that the distribution of $H \mid G = g$ is lognormal, the integrand of the first integral in (13) can be approximated by a European call option price for which the standard Black & Scholes formula (10)-(11) applies, leading to:

$$\int_0^K \mathbb{E}_Q[(S - K)_+ \mid G = g] dF_G(g) \approx \int_0^K [(\mathbb{E}_Q[S \mid G = g] - g)\Phi(d_1(g)) - (K - g)\Phi(d_2(g))] dF_G(g)$$

with

$$d_1(g) = \frac{\mu_H + \sigma_H^2 - \ln \tilde{K}}{\sigma_H} = \frac{1}{2} \ln \mathbb{E}_Q[H^2 \mid G = g] - \ln(K - g)$$

and

$$d_2(g) = d_1(g) - \sigma_H.$$

Note that the moments depend in fact on $g$ and have to be recomputed for each value that $g$ takes. One could avoid this by assuming that the moments are constant and equal to the moments for $G = K$. Thus the first integral in (13) is in that case approximated by

$$\int_0^K [(\mathbb{E}_Q[S \mid G = K] - g)\Phi(d_1(g)) - (K - g)\Phi(d_2(g))] dF_G(g)$$

with

$$d_1(g) = \frac{1}{2} \ln \mathbb{E}_Q[H^2 \mid G = K] - \ln(K - g)$$

and

$$d_2(g) = d_1(g) - \sigma_H,$$

where $\sigma_H^2 = \ln \mathbb{E}_Q[H^2 \mid G = K] - 2 \ln \mathbb{E}_Q[H \mid G = K]$.

In the formulas (14) and (44) we then have that $\Phi^{-1}(v) = \frac{\ln K - \mathbb{E}_Q[\ln G]}{\text{var}^{1/2}[\ln G]} = d_{G_A}$. Fixing the moments on $G = K$ is motivated by the fact that the main contribution to the integral comes from values of $g$ close to $K$ since for $g \ll K$ it holds that $(S - K)_+ \mid G = g$ equals zero.

The appearing integrals have to be computed by means of numerical quadrature. Numerical experiments are carried out for the different conditioning variables and we consider both cases of variable moments and of constant moments.
6. General remarks

In this section we summarize some general remarks:

- The price of the basket put option with exercise date \( T \), \( n \) underlying assets and fixed exercise price \( K \), given by \( BP(n, K, T) = e^{-rT} E^Q [(K - S(T))]_+ \) satisfies the put-call parity at the present: \( BC(n, K, T) - BP(n, K, T) = S(0) - e^{-rT}K \). Hence, we can derive bounds for the basket put option from the bounds for the call. These bounds for the put option coincide with the bounds that are obtained by applying the theory of comonotonic bounds or the Rogers and Shi approach directly to basket put options. This stems from the fact that the put-call parity also holds for these bounds.

- The case of a continuous dividend yield \( q_i \) can easily be dealt with by replacing the interest rate \( r \) by \( r - q_i \).

- For \( n = 1 \) there is only one asset in the basket and hence the comonotonic sums \( S_{\Lambda=\lambda}^c \) and \( S_c^e \) coincide with the sum \( S \) which consists of only one term: this asset. In this case, the comonotonic upper and lower bound reduce to the well-known Black & Scholes price for an option on a single asset. This is also true for the bound based on the Rogers & Shi approach since the error bound is zero.

- As for the Asian options (see Vanmaele et al. (2002)), we can easily derive the hedging Greeks for the upper and lower bounds as well as for the approximations of a basket option since we found analytical expressions for them. Moreover the expressions are in terms of Black & Scholes prices.

7. Numerical illustration

In this section we give a number of numerical examples on basket options in the Black & Scholes setting. We first concentrate on the bounds proposed in Section 4 and afterwards on
the approximations in Section 5. Therefore, we first discuss Tables 1, 3 and 5 and secondly, Tables 2, 4 and 6.

We introduce the following notations where \( \Lambda \) can be \( FA_1, FA_2, GA \): \( LB\Lambda \) for lower bound, \( PECUB\Lambda \) for partially exact/comonotonic upper bound, and \( UB\Lambda \) for upper bound based on lower bound.

The first set of input data was taken from Arts (1999). Note that we consider here the forward-moneyness, which is defined as the ratio of the forward price of the basket and the exercise price \( K \). The input parameters correspond to a two-dimensional basket. We first consider equal weights and afterwards, unequal weights. The spot prices are first assumed to be equal to 100 units, and then allowed to vary. The risk-free interest rate is fixed at 5% and we assume no dividends. Moneyness ranges from 10% in-the-money to 10% out-of-the-money. For the time to maturity \( T \) two cases are considered (\( T = 1, 3 \) years). For the correlation (8), two values are considered, representing low and high correlation respectively. We consider equal volatilities (high and low) for both individual assets in the basket.

Concerning the upper bounds, we present only the results that lead to the best upper bound together with the corresponding type of the bound. That is, the upper bound given in the Tables 1, 3 and 5 is the bound which satisfies \( \min(UB\Lambda, PECUB\Lambda) \), where the bounds were computed for all three choices \( FA_1, FA_2 \) and \( GA \) of the conditioning variable \( \Lambda \). The detailed numerical results for all bounds are available upon request. Notice that, in general, the Monte Carlo (MC) price is closer to the best lower bound than to the best upper bound. One can also note that the relative difference between the best lower and upper bound is smaller for higher correlation.

We start by discussing Table 1 which corresponds to the case of equal weights, spot prices and volatilities for both assets. In this case the lower bound (32)-(33) applied with \( \Lambda \) given by (19), (21) and (24), which are denoted by \( LBFA_1, LBFA_2 \) and \( LBG_4 \), are equal. The
optimized lower bound \( \text{LB}_{\text{opt}} \), which is obtained by solving the optimization program (38), gave practically the same values, therefore it is not reported in the table. From all the upper bounds considered, \( \text{UB}_{\text{GA}} \) uniformly performs the best.

Table 3 refers to the case of unequal weights and spot prices with equal volatilities. From Table 3, we notice that \( \text{LB}_{\text{FA}1} = \text{LB}_{\text{FA}2} \) gives sharper results than \( \text{LB}_{\text{GA}} \). The lower bound \( \text{LB}_{\text{opt}} \) only slightly improves the lower bound \( \text{LB}_{\text{FA}1} \). However, for high volatilities, small correlation \( \rho \) and long maturity, the improvement is significant. As for the upper bounds, we could observe some pattern, namely for out- and at-the-money options, and in-the-money options with the maturity of three years, the upper bound \( \text{UB}_{\text{FA}1} \) performs the best for smaller volatility (0.2), whereas \( \text{UB}_{\text{FA}2} \) is the best for larger volatility (0.4) with the exception of three years to maturity out-of-the-money option. In the latter case the partially exact/comonotonic upper bound \( \text{PECUB}_{\text{FA}2} \) based on the standardized logarithm of the geometric average outperforms the other bounds for larger volatility. For in-the-money options with the maturity of one year, the pattern is reversed compared to that of in-the-money options with three years to maturity.

The second set of input data was taken from Brigo et al. (2002). Here we consider two assets with weights 0.5956 and 0.4044, and spot prices of 26.3 and 42.03, respectively. Maturity is equal to 5 years. The discount factor at payoff is 0.783895779. This example refers to a realistic basket, for which we allow the volatilities and correlations of individual assets to vary in order to facilitate the comparative price analysis. From Table 5 we see that the optimized lower bound gives the best value, and that the lower bound \( \text{LB}_{\text{GA}} \) leads to the closest values to \( \text{LB}_{\text{opt}} \). The lower bound \( \text{LB}_{\text{FA}2} \) led to the worst results and is therefore not reported. For this example the partially exact/comonotonic upper bound \( \text{PECUB}_{\text{FA}2} \), i.e. with \( \Lambda \) given by \( \text{FA}2 \) (21), turns out to be the sharpest upper bound, except for very high correlation when \( \text{PECUB}_{\text{GA}} \) is to be preferred, and for \( \sigma_1 = 0.1, \sigma_2 = 0.3 \) (for both \( \rho = 0.2 \) and \( \rho = 0.6 \)) when \( \text{UB}_{\text{GA}} \) is the best.
As mentioned before for a negative correlation between the assets in the basket the lower bound (32) is not applicable if any of the correlations $r_1$ and $r_2$ is negative. If this happens, one should turn to the optimization procedure which enables to choose the coefficients $\beta_1$ and $\beta_2$ such that $r_1$ and $r_2$ would be positive. Consider a case where $\sigma_1 = 0.3$, $\sigma_2 = 0.6$, and $\rho = -0.6$. In this instance we have that the correlations $r_1$ and $r_2$ are positive for the conditioning variables $FA_1$ and $GA$ and therefore we can find the lower bounds based on those variables: $LBFA_1 = 29.39746493$, $LBGA = 29.77084284$. The optimization procedure (38) gives $LB_{opt} = 29.773172314$, which shows again that the geometric average is a fairly good choice for a conditioning variable when $a_1 = a_2$, and $S_1(0) = S_2(0)$.

We now turn to the discussion of the approximations. In Tables 2 and 4, we used as in Tables 1 and 3 the dataset of Arts respectively with equal weights and spot prices and with different weights and spot prices, but in both cases with equal volatility. The numerical results clearly show that the approximations improve for shorter maturities and this for in-the-money, at-the-money as well as for out-of-the-money, and always lie between the corresponding bounds. For long maturities (3 year) and high volatility (0.4), the approximation by using $\mathbb{H}$ and constant moments turns out to be disappointing. However, the other approximation methods seem to deliver approximating prices with an equivalent level of high quality.

In Table 6, we discuss the approximations for the dataset of Brigo et al. (2002) used in Table 5 with different weights, different spot prices and different correlations. The numerical results for a very high correlation (0.99) are excellent since in that case, the proportion of the exact part in the price is more than 99%. We have attempted to prove this fact analytically by using the derivative of the exact part with respect to the correlation, but the expression of the derivative is too long and too complicated to obtain straightforward general conclusions. For this dataset, the moment matching method with the first order approximation $FA_1$ and with fixed moments turns out to be somewhat poor, especially for high volatility and low correlation. For the first order approximation $FA_2$ and with fixed moments, nearly all approximation values
are close to the MC value but are smaller than the lower bound $LB_{opt}$. In general, all moment matching methods with variable moments superperform the methods with fixed moments and deliver approximating prices which are close to the Monte Carlo estimates. Since nowadays even quite simple PC’s can treat numerical quadratures with high speed, we therefore recommend to use moment matching approximations with variable moments.

8. Asian basket options

An Asian basket option is an option whose payoff depends on an average of values at different dates of a portfolio (or basket) of assets, or which is equivalent on the portfolio value of an average of asset prices taken at different dates. The price of a discrete arithmetic Asian basket call option at current time $t = 0$ is given by

$$ABC(n, K, T) = e^{-rT}E^{Q} \left[ \left( \sum_{t=1}^{n} \sum_{j=0}^{m-1} a_{t} b_{j} S_{t}(T - j) - K \right)_{+} \right]$$

with $a_{t}$ and $b_{j}$ positive coefficients. For $T \leq m - 1$ we call this Asian basket call option in progress and for $T > m - 1$, we call it forward starting. Remark that the double sum $S = \sum_{t=1}^{n} a_{t} \sum_{j=0}^{m-1} b_{j} S_{t}(T - j)$ is a sum of lognormal distributed variables:

$$S = \sum_{i=1}^{mn} X_{i} = \sum_{i=1}^{mn} \alpha_{i} e^{Y_{i}}$$

with

$$\alpha_{i} = a_{\lceil \frac{i}{m} \rceil} b_{(i-1) \mod m} S_{\lceil \frac{i}{m} \rceil}(0) e^{(r - \frac{1}{2} \sigma^{2})_{\lceil \frac{i}{m} \rceil}(T - (i-1) \mod m)}$$

and

$$Y_{i} = \sigma_{\lceil \frac{i}{m} \rceil} W_{\lceil \frac{i}{m} \rceil}(T - (i-1) \mod m) \sim N \left( 0, \sigma_{\lceil \frac{i}{m} \rceil}^{2}(T - (i-1) \mod m) \right)$$

for all $i = 1, \ldots, mn$, where $\lceil x \rceil = \inf \{ k \in \mathbb{Z} \mid x \leq k \}$.

Hence, we can apply the general formulae for lognormals from Section 3 (see also Vanmaele et
9. Conclusion

We derived lower and upper bounds and approximations for the price of the arithmetic basket call options by decomposing the option price into an exact and an approximating part, and applying different techniques to the latter, such as firstly results based on comonotonicity and bounds for stop-loss premiums of sums of dependent random variables as in Kaas, Dhaene and Goovaerts (2000), and secondly, conditioning on some random variable as in Rogers and Shi (1995), and finally using a moment matching method. Notice that all bounds and approximations have analytical and easily computable expressions. For the numerical illustration it was important to find and motivate a good choice of the conditioning variables appearing in the formulae. We also managed to find the best lower bound through an optimization procedure.

Acknowledgements

G. Deelstra, J. Liinev and M. Vanmaele would like to acknowledge the financial support by the BOF-project 001104599 of the Ghent University. G. Deelstra also thanks Fortis for financial support.

References


Pricing of arithmetic basket options by conditioning


Table 1: Comparing bounds, equal weights and spot prices.

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Table 2: Comparing approximations, equal weights and spot prices.

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Table 3: Comparing bounds, different weights and spot prices.

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<th>vol</th>
<th>MC</th>
<th>LBGA</th>
<th>$\text{LBFAI} = \text{LBFAI}_2$</th>
<th>$\text{LBopt}$</th>
<th>UB</th>
<th>Type</th>
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<td>6.4823</td>
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<td>0.2</td>
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<td>11.0985</td>
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<td>11.1778</td>
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<tr>
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Table 4: Comparing approximations, different weights and spot prices.

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<th>$K$</th>
<th>$T$</th>
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<th>vol</th>
<th>MC</th>
<th>$\mathbb{S}^I_{FAI} = \lambda$</th>
<th>$\mathbb{H}^I_{FAI} = \lambda$</th>
<th>$\mathbb{S}^I_{FAI} = \lambda$</th>
<th>$\mathbb{H}^I_{FAI} = \lambda$</th>
<th>$\mathbb{S}^I_{FAI} = \lambda$</th>
<th>$\mathbb{H}^I_{FAI} = \lambda$</th>
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<td>0.2</td>
<td>5.69</td>
<td>5.6838</td>
<td>5.6842</td>
<td>5.6842</td>
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<td>0.2</td>
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Table 5: Comparing bounds, different weights and spot prices, different correlations.

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<tr>
<th>Data</th>
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<th>$\sigma_2$</th>
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<th>LBFA1</th>
<th>LBGA</th>
<th>LBopt</th>
<th>UB</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.2</td>
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<td>26.2206</td>
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<td>27.4400</td>
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Table 6: Comparing approximations, different weights and spot prices, different correlations.

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<th>MC</th>
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<th>$H_{FA1-\lambda\lambda}$</th>
<th>$S_{FA2-\lambda\lambda}$</th>
<th>$H_{FA2-\lambda\lambda}$</th>
<th>$S_{G-g\lambda\lambda}$</th>
<th>$H_{G-g\lambda\lambda}$</th>
<th>$g$</th>
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<td>27.4427</td>
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