The computation of highly oscillatory integrals

M. Van Daele

Department of Applied Mathematics and Computer Science
Ghent University

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Oscillatory integrals

V. LEDOUX AND M. VAN DAELE, *Gauss-type quadrature rules for highly-oscillatory integrals*
submitted for publication in SINUM

$$I[f] = \int_0^h f(x)e^{\mathrm{i}\omega x}dx$$

Gaussian quadrature rules

How to obtain

$$\int_0^h f(x)dx \approx h \sum_{j=1}^{\nu} b_j f(c_j h) \; ?$$

By replacing $f(x)$ by the interpolating polynomial

$$\bar{f}(x) = \sum_{j=1}^{\nu} \ell_j(h) f(c_j h) \, .$$

- nodes $c_1, \ldots, c_\nu \in [0,1] : c_i = 2\hat{c}_i - 1$ with $P_\nu(\hat{c}_i) = 0$
- weights $b_1, \ldots, b_\nu : b_j = \int_0^1 \ell_j(t) dt$
- order $= 2\nu$: the Gauss rule is exact if the integrand $f(x)$ is a polynomial of degree $2\nu - 1$.

Gaussian quadrature rules applied to $I[f]$

$$I[f] = \int_0^h f(x) e^{\mathrm{i}\omega x} dx$$

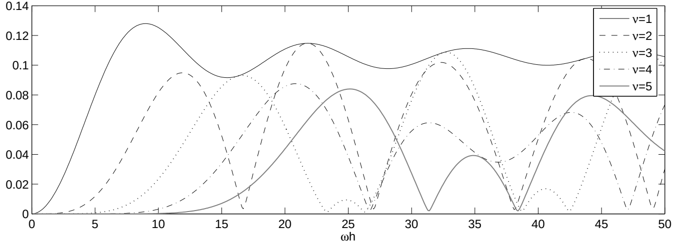
$$I[f] \approx Q_\nu^G[f] = h \sum_{l=1}^{\nu} b_l f(c_l h) e^{\mathrm{i}\omega c_l h}$$

If the integrand oscillates rapidly, and unless we use a huge number of function evaluations, the polynomial interpolation underlying the classical Gauss rule is useless.

Gauss rule applied to oscillatory integrands

Example : $f(x) = exp(x)$ and $h = 1/10$

$$\int_0^h e^x e^{\mathrm{i}\omega x} dx = \frac{-1 + e^{h(1+\mathrm{i}\omega)}}{1 + \mathrm{i}\omega}$$



The absolute error in Gauss-Legendre quadrature for different values of the characteristic frequency $\psi = \omega h$.

Asymptotic expansion

$$
\begin{aligned}
I[f] &= \int_a^b f(x)e^{\mathrm{i}\omega x}dx \\
&= \frac{1}{\mathrm{i}\omega}\left(f(b)\,e^{\mathrm{i}\omega b} - f(a)\,e^{\mathrm{i}\omega a}\right) - \frac{1}{\mathrm{i}\omega}I[f'] \\
&= \frac{1}{\mathrm{i}\omega}\left(f(b)\,e^{\mathrm{i}\omega b} - f(a)\,e^{\mathrm{i}\omega a}\right) \\
&\quad -\frac{1}{(\mathrm{i}\omega)^2}\left(f'(b)\,e^{\mathrm{i}\omega b} - f'(a)\,e^{\mathrm{i}\omega a}\right) + \frac{1}{(\mathrm{i}\omega)^2}I[f'']
\end{aligned}
$$

$$
I[f] = -\sum_{m=0}^{\infty} \frac{1}{(-\mathrm{i}\omega)^{m+1}}\left[e^{\mathrm{i}\omega b}f^{(m)}(b) - e^{\mathrm{i}\omega a}f^{(m)}(a)\right]
$$

Asymptotic rules

$$I[f] = \int_a^b f(x) e^{\mathrm{i}\omega x} dx$$

$$I[f] = -\sum_{m=0}^{\infty} \frac{1}{(-\mathrm{i}\omega)^{m+1}} \left[e^{\mathrm{i}\omega b} f^{(m)}(b) - e^{\mathrm{i}\omega a} f^{(m)}(a) \right]$$

$$Q_s^A[f] = -\sum_{m=0}^{s-1} \frac{1}{(-\mathrm{i}\omega)^{m+1}} \left[e^{\mathrm{i}\omega b} f^{(m)}(b) - e^{\mathrm{i}\omega a} f^{(m)}(a) \right]$$

$$Q_s^A[f] - I[f] \sim O(\omega^{-s-1}) \quad \omega \to +\infty$$

This asymptotic method is of asymptotic order $s+1$.
The asymptotic order gives us the rate at which the error decreases with increasing ω.

Exponential fitting

M. VAN DAELE, G. VANDEN BERGHE AND H. VANDE VYVER, *Exponentially fitted quadrature rules of Gauss type for oscillatory integrands*, Appl. Numer. Math., 53 (2005), pp. 509–526.

How to compute

$$\int_{-1}^{1} F(t)dt$$

whereby $F(x)$ has an oscillatory behaviour with frequency μ?

$$I[f] = \int_0^h f(x)e^{\mathrm{i}\omega x}dx = \frac{h}{2}e^{\mathrm{i}\mu}\int_{-1}^{1} f(h(t+1)/2)e^{\mathrm{i}\mu t}dt \quad \mu = \frac{\omega h}{2}$$

Exponential fitting

$$\mathcal{L}[F;x;h;\mathbf{a}] = \int_{x-h}^{x+h} F(z)dz - h\sum_{k=1}^{\nu} w_k F(x+\hat{c}_k h), \quad \hat{c}_k \in [-1,1]$$

(put $x = 0$ and $h = 1$ to obtain $\int_{-1}^{1} F(t)dt$)

$\mathcal{L}[F;x;h;\mathbf{a}] = 0$ for a reference set of $K + 2(P+1) + 1 = 2\nu$ functions

$$1, t, t^2, \ldots t^K,$$

$$\exp(\pm i\mu t), t\exp(\pm i\mu t), t^2\exp(\pm i\mu t), \ldots, t^P\exp(\pm i\mu t)$$

In this talk we only consider the case $P = \nu - 1$.

1-node EF rule

$$\int_{-1}^{1} F(x)dx \approx w_1 F(\hat{c}_1)$$

$$\int_{-1}^{1} \exp(\pm \mathrm{i}\mu x)dx - w_1 \exp(\pm \mathrm{i}\,\hat{c}_1\,\mu) = 0$$

$$w_1 = 2\sin(\mu)/\mu \qquad\qquad \hat{c}_1 = 0$$

$$I[f] = \int_0^h F(x)dx = \int_0^h f(x)\exp(\mathrm{i}\omega x)dx$$

$$Q_1^{EF}[F] = \frac{h\sin(\mu)}{\mu}F(h/2) = f(h/2)\frac{e^{\mathrm{i}h\omega}-1}{\mathrm{i}\omega} \qquad \mu = \omega h/2$$

2-node EF rule

$$\int_{-1}^{1} Fx)dx \approx w_1 F(\hat{c}_1) + w_2 F(\hat{c}_2)$$

$$\begin{cases} \int_{-1}^{1} \exp(\pm i\mu x)dx - w_1 \exp(\pm i\ \hat{c}_1\ \mu) - w_2 \exp(\pm i\ \hat{c}_2\ \mu) = 0 \\ \int_{-1}^{1} x\ \exp(\pm i\mu x)dx - w_1\ \hat{c}_1 \exp(\pm i\ \hat{c}_1\ \mu) - w_2 \hat{c}_2 \exp(\pm i\ \hat{c}_2\ \mu) = 0 \end{cases}$$

Assuming $w_1 = w_2$ and $\hat{c}_1 = -\hat{c}_2$:

$$\Longleftrightarrow \begin{cases} w_2 \mu \cos(\mu \hat{c}_2) - \sin(\mu) = 0 \\ w_2 \hat{c}_2 \mu^2 \sin(\mu \hat{c}_2) - \sin(\mu) + \mu \cos(\mu) = 0 \end{cases}$$

$$Q_2^{EF}[f] = \frac{h}{2} w_2 \left[F\left(\frac{h(1+\hat{c}_2)}{2}\right) + F\left(\frac{h(1-\hat{c}_2)}{2}\right)\right] \qquad \mu = \frac{\omega h}{2}$$

2-node EF rule

$$\begin{cases} w_2\mu\cos(\mu\hat{c}_2) - \sin(\mu) = 0 \\ w_2\hat{c}_2\mu^2\sin(\mu\hat{c}_2) - \sin(\mu) + \mu\cos(\mu) = 0 \end{cases}$$

If $\cos(\mu\hat{c}_2) \neq 0$ then $w_2 = \sin\mu/(\mu\cos(\mu\hat{c}_2))$

$$G(\hat{c}_2) := (\sin\mu - \mu\cos\mu)\cos(\mu\hat{c}_2) - \mu\hat{c}_2\sin\mu\sin(\mu\hat{c}_2) = 0$$

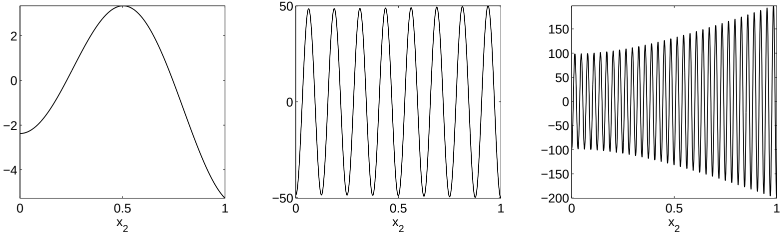


Figure: $G(x_2)$ for $\mu = 5$, $\mu = 50$ and $\mu = 200$.

2-node EF rule

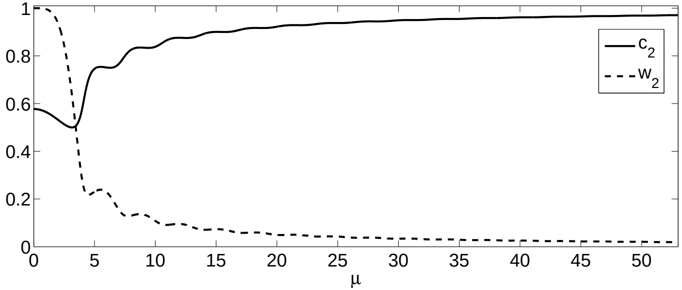


Figure: The $\hat{c}_2(\mu)$ and $w_2(\mu)$ curve for the EF method with $\nu = 2$.

3-node EF rule

$$\hat{c}_1 = 1 - \hat{c}_3 \qquad \hat{c}_2 = 0 \qquad w_1 = w_3$$

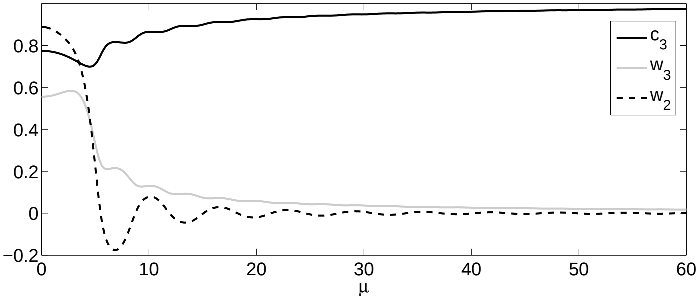


Figure: The $\hat{c}_3(\mu)$, $w_1(\mu) = w_3(\mu)$ and $w_2(\mu)$ curves for the $\nu = 3$ EF rule

4-node EF rule

$$\hat{c}_1 = 1 - \hat{c}_4 \qquad \hat{c}_2 = 1 - \hat{c}_3 \qquad w_1 = w_4 \qquad w_2 = w_3$$

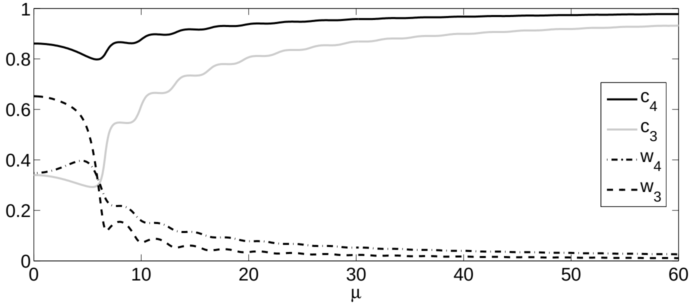


Figure: Nodes and weights of the EF rule with $\nu = 4$ quadrature nodes.

Accuracy of EF rules

All EF rules reduce to the classical ν-point Gauss(-Legendre) method in the limiting case $\mu = 0$.
Thus for small μ : $O(h^{2\nu+1})$
What about the accuracy for larger values of $\mu = \omega h/2$?

J. P. Coleman and L. Gr. Ixaru, *Truncation errors in exponential fitting for oscillatory problems*, SIAM. J. Numer. Anal., 44 (2006), pp. 1441–1465.

for large μ : $O(\mu^{\bar{\nu}-\nu})$ with $\bar{\nu} = \lfloor(\nu - 1)/2\rfloor$

$\nu = 1 : O(\omega^{-1})$ $\nu = 2, 3 : O(\omega^{-2})$ $\nu = 4, 5 : O(\omega^{-3})$

Proof

$$\int_{-1}^{1} F(t)dt \approx \int_{-1}^{1} \bar{F}(t)dt$$

$$\bar{F}(t) \in \mathrm{span}\{\exp(\pm \mathrm{i}\mu t), t\exp(\pm \mathrm{i}\mu t), t^2\exp(\pm \mathrm{i}\mu t), \ldots, t^P\exp(\pm \mathrm{i}\mu t)\}$$

$$I[f] = \int_0^h f(x)e^{\mathrm{i}\omega x}dx = \frac{h}{2}e^{\mathrm{i}\frac{\omega h}{2}}\int_{-1}^{1} f(\frac{h}{2}(t+1))e^{\mathrm{i}\frac{\omega h}{2}t}dt$$

If $\frac{\omega h}{2} = \mu$ then $I[f] \approx I[\bar{f}]$ with $\bar{f}(x) \in \mathrm{span}\{1, x, x^2, \ldots, x^{\nu-1}\}$

$$v(x) := \bar{f}(x) - f(x) = \frac{1}{\nu!}f^{(\nu)}(\xi(x))\prod_{j=1}^{\nu}(x - c_j)$$

Proof

$$v(x) := \bar{f}(x) - f(x) = \frac{1}{\nu!} f^{(\nu)}(\xi(x)) \prod_{j=1}^{\nu} (x - c_j)$$

$$I[f] = \int_a^b f(x) e^{\mathrm{i}\omega x} dx = -\sum_{m=0}^{\infty} \frac{1}{(-\mathrm{i}\omega)^{m+1}} \left[e^{\mathrm{i}\omega b} f^{(m)}(b) - e^{\mathrm{i}\omega a} f^{(m)}(a) \right]$$

$$\begin{aligned} Q_\nu^{EF}[f] - I[f] &= I[\bar{f}] - I[f] = I[v] \\ &= -\sum_{m=0}^{\infty} \frac{1}{(-\mathrm{i}\omega)^{m+1}} \left[e^{\mathrm{i}\omega b} v^{(m)}(b) - e^{\mathrm{i}\omega a} v^{(m)}(a) \right] \end{aligned}$$

Proof

Suppose ν is even

$$a < c_1 < c_2 < \ldots < c_\nu < b$$

$$c_j = a + \lambda_j/\omega \qquad c_{\nu-j+1} = b - \lambda_j/\omega \quad j = 1, \ldots, \nu/2$$

$$v(x) = s(x) \prod_{i=1}^{\nu/2} (x - b + \lambda_i/\omega) \qquad s(x) = \frac{f^\nu(\xi(x))}{\nu!} \prod_{j=1}^{\nu/2} (x - a - \lambda_j/\omega)$$

$$v(b) = s(b) \prod_{i=1}^{\nu/2} (\lambda_i/\omega) = O(\omega^{-\nu/2})$$

Proof

$$c_j = a + \lambda_j/\omega \qquad c_{\nu-j+1} = b - \lambda_j/\omega \quad j = 1, \ldots, \nu/2$$

$$v(x) = s(x)\prod_{i=1}^{\nu/2}(x - b + \lambda_i/\omega) \qquad s(x) = \frac{f^{\nu}(\xi(x))}{\nu!}\prod_{j=1}^{\nu/2}(x - a - \lambda_j/\omega)$$

$$v'(x) = s(x)\sum_{k=1}^{\nu/2}\prod_{i\neq k}(x - b + \lambda_i/\omega) + s'(x)\prod_{i=1}^{\nu/2}(x - b + \lambda_i/\omega)$$

$$v'(b) = s(b)\omega^{-\nu/2+1}\sum_{k=1}^{\nu/2}\prod_{i\neq k}\lambda_i + O(\omega^{-\nu/2}) = O(\omega^{-\nu/2+1})$$

Exponential fitting

$$v(b) = O(\omega^{-\nu/2}) \qquad v'(b) = O(\omega^{-\nu/2+1})$$

$$v^{(n)}(b) = O(\omega^{-\nu/2+n}), \; n = 0, 1, \ldots, \nu/2 - 1$$

$$v^{(n)}(a) = O(\omega^{-\nu/2+n}), \; n = 0, 1, \ldots, \nu/2 - 1$$

$$\begin{aligned} Q_\nu^{EF}[f] - I[f] &= I[\bar{f}] - I[f] = I[v] \\ &= -\sum_{m=0}^{\infty} \frac{1}{(-\mathrm{i}\omega)^{m+1}} \left[e^{\mathrm{i}\omega b} v^{(m)}(b) - e^{\mathrm{i}\omega a} v^{(m)}(a) \right] \\ &= -\sum_{m=0}^{\nu/2-1} \frac{1}{(-\mathrm{i}\omega)^{m+1}} O(\omega^{-\nu/2+m}) + O(\omega^{-\nu/2-1}) \\ &= O(\omega^{-\nu/2-1}) = O(\omega^{\lfloor(\nu-1)/2\rfloor-\nu}) \end{aligned}$$

Filon-type

L. N. G FILON, *On a quadrature formula for trigonometric integrals*, Proc. Royal Soc. Edinburgh, 49 (1928), pp. 38–47.

Interpolate only the function $f(x)$ at $c_1h, \ldots, c_\nu h$ by a polynomial $\bar{f}(x)$

$$I[f] \approx Q_\nu^F[f] = \int_0^h \bar{f}(x)e^{\mathrm{i}\omega x}dx = h\sum_{l=1}^{\nu} b_l(\mathrm{i}h\omega)f(c_lh)$$

$$b_l(\mathrm{i}h\omega) = \int_0^1 \ell_l(x)e^{\mathrm{i}h\omega x}dx$$

ℓ_l is the lth cardinal polynomial of Lagrangian interpolation.

1-node Filon-type rule

$$I[f] = \int_0^h F(x)dx = \int_0^h f(x)\exp(\mathrm{i}\omega x)dx$$

$$Q_1^F[f] = \frac{\exp(\mathrm{i}h\,\omega) - 1}{\mathrm{i}\omega} f(c_1\,h)$$

$$Q_1^{EF}[F] = f(h/2)\frac{e^{\mathrm{i}h\omega} - 1}{\mathrm{i}\omega}$$

$$Q_1^F[f] = Q_1^{EF}[F] \text{ iff } c_1 = \frac{1}{2}$$

2-node Filon-type rule

$$I[f] = \int_0^h F(x)dx = \int_0^h f(x)\exp(\mathrm{i}\omega x)dx$$

If f is interpolated at $c_1 h$ and $c_2 h$, then

$$\begin{aligned} Q_2^F[f] &= h\Bigg[\left(\frac{i\left((e^{i\psi}-1)c_2 - e^{i\psi}\right)}{(c_1-c_2)\psi} + \frac{e^{i\psi}-1}{(c_1-c_2)\psi^2}\right) f(c_1 h) \\ &\quad + \left(\frac{i\left((e^{i\psi}-1)c_1 - e^{i\psi}\right)}{(c_2-c_1)\psi} + \frac{e^{i\psi}-1}{(c_2-c_1)\psi^2}\right) f(c_2 h)\Bigg] \end{aligned}$$

$Q_2^F[f] = Q_2^{EF}[F]$ iff the same nodes are used

Accuracy of Filon-type rules

A. Iserles, *On the numerical quadrature of highly-oscillating integrals. I. Fourier transforms*, IMA J. Numer. Anal., 24 (2004), pp. 365–391.

For small ω, a Filon-type quadrature method has an order as if $\omega = 0$.

Legendre nodes : order 2ν Lobatto nodes : order $2\nu - 2$

For large ω :

$$Q_\nu^F[f] - I[f] \sim \begin{cases} O(\omega^{-1}) & c_1 > 0 \text{ or } c_\nu < 1 \\ O(\omega^{-2}) & c_1 = 0, c_\nu = 1 \end{cases}$$

Accuracy of Filon-type rules

$$Q_\nu^F[f] - I[f] \sim \begin{cases} O(\omega^{-1}) & c_1 > 0 \text{ or } c_\nu < 1 \\ O(\omega^{-2}) & c_1 = 0, c_\nu = 1 \end{cases}$$

$$\begin{aligned} Q_\nu^F[f] - I[f] &= I[\bar{f}] - I[f] = I[v] \\ &= -\sum_{m=0}^{\infty} \frac{1}{(-\mathrm{i}\omega)^{m+1}} \left[e^{\mathrm{i}\omega h} v^{(m)}(h) - v^{(m)}(0) \right] \end{aligned}$$

If $(c_1, c_\nu) = (0, 1)$ then $v(h) = v(0) = 0$

$\Longrightarrow Q_\nu^F[f] - I[f] = O(\omega^{-2})$.

$Q_1^F[f] = Q_1^{EF}[F]$ illustrated

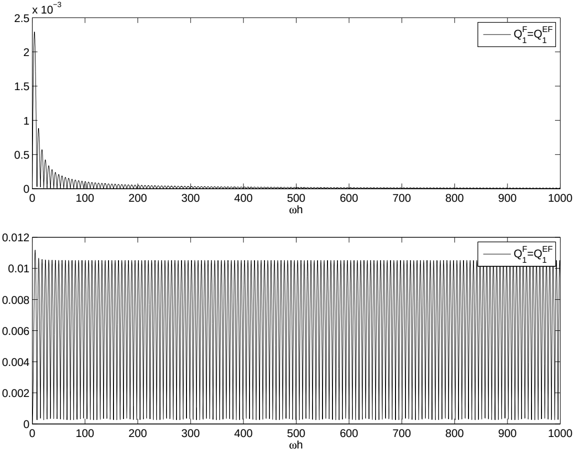


Figure: The errors in $Q_1^F[f] = Q_1^{EF}[F]$ for $f(x) = e^x$, $h = 1/10$ as a function of $\psi = \omega h$. The top graph shows the absolute error E, the bottom graph shows the normalised error $(\omega h)E$.

$Q_2^F[f]$ and $Q_2^{EF}[F]$ illustrated

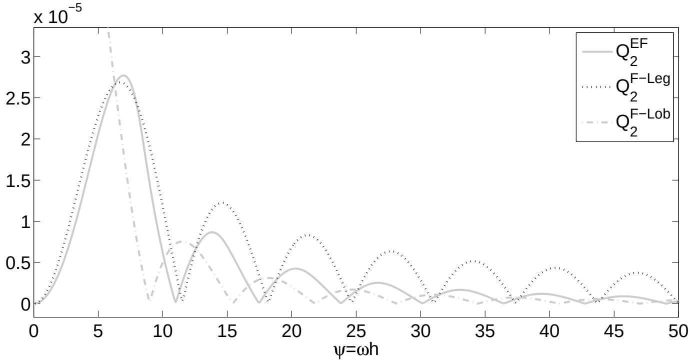


Figure: The absolute error in some $\nu = 2$ Filon-type schemes for $f(x) = e^x, h = 1/10$ and different values of ωh.

How to improve the accuracy of Filon-rules ?

- by using Hermite interpolation : asymptotic order $p+1$ can be reached where p is the number of derivatives at the endpoints:

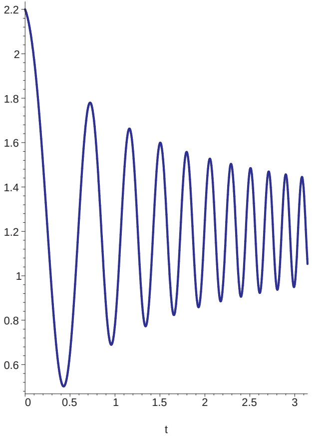
$$\bar{f}^{(l)}(h) = f^{(l)}(h), \bar{f}^{(l)}(0) = f^{(l)}(0), l = 0, \ldots, p-1$$

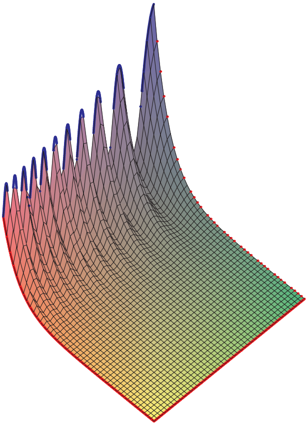
$$Q_{\nu}^{F}[f] - I[f] = O(\omega^{-p-1})$$

- by using adaptive Filon-type methods : allowing the interpolation point to depend on ω (is discussed later)
- by using nodes in the complex plane (=method of steepest descent)

Method of steepest descent

D. Huybrechs and S. Vandewalle, *On the evaluation of highly oscillatory integrals by analytic continuation*, SIAM J. Numer. Anal., 44 (2002) pp 1026–1048.





Method of steepest descent

$$\begin{aligned}
&\int_a^b f(x)e^{\mathrm{i}\omega x}dx \\
&= e^{\mathrm{i}\omega a}\int_0^{\infty} f(a+\mathrm{i}p)e^{-\omega p}dp - e^{\mathrm{i}\omega b}\int_0^{\infty} f(b+\mathrm{i}p)e^{-\omega p}dp \\
&= \frac{e^{\mathrm{i}\omega a}}{\omega}\int_0^{\infty} f(a+\mathrm{i}\frac{q}{\omega})e^{-q}dq - \frac{e^{\mathrm{i}\omega b}}{\omega}\int_0^{\infty} f(b+\mathrm{i}\frac{q}{\omega})e^{-q}dq
\end{aligned}$$

This leads to the numerical evaluation of the two resulting integrals with classical Gauss-Laguerre quadrature.

High asymptotic order is obtained : using ν points for each integral, the error behaves as $O(\omega^{-2\nu-1})$.

Method of steepest descent

$$\int_a^b f(x)e^{\mathrm{i}\omega x}dx$$
$$= \frac{e^{\mathrm{i}\omega a}}{\omega}\int_0^\infty f(a+\mathrm{i}\frac{q}{\omega})e^{-q}dq - \frac{e^{\mathrm{i}\omega b}}{\omega}\int_0^\infty f(b+\mathrm{i}\frac{q}{\omega})e^{-q}dq$$

One ends up evaluating f at the points

$$a+\mathrm{i}\frac{x_{nj}}{\omega}, \ \text{and} \ b+\mathrm{i}\frac{x_{nj}}{\omega}, \ j=1,\ldots,n,$$

where x_{nj} are the n roots of the Laguerre polynomial of degree n.

This approach is equivalent to using a Filon rule with the same interpolation points.

Adaptive Filon-type rules

Idea : combine best properties of EF and Filon quadrature

- EF
  - \+ accurate for small ωh since the method reduces to Gauss-Legendre quadrature
  - \+ good results for large ωh since the nodes tend to the endpoints (at a rate proportional to ω^{-1})
  - \- but : difficult to determine the nodes and weights for a given ωh (iteration needed and ill-conditioned)
- Filon
  - \+ any set of nodes can be used
  - \- there is no optimal set of nodes for all ωh
    - most accurate for small ωh if the method is built on Legendre nodes
    - most accurate for large ωh if the endpoints are included in the set of nodes

Adaptive Filon-type methods

$$S(\psi; r; n) = \frac{1 - \frac{\psi^n - r^n}{1 + |\psi^n - r^n|}}{1 + \frac{r^n}{1 + r^n}}$$

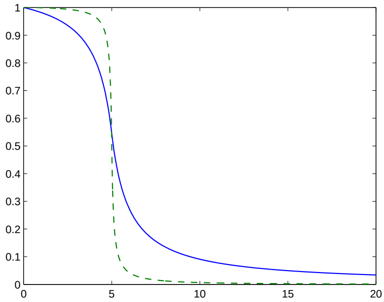


Figure: $S(x, r, 1)$ and $S(x, r, 2)$ (dashed) for $r = 5$ in $[0, 20]$

Adaptive Filon-type methods

- $\nu = 2 : c_1(\psi) = \dfrac{3-\sqrt{3}}{6} S(\psi; 2\pi; 1); c_2(\psi) = 1 - c_1(\psi)$
- $\nu = 3 : c_1(\psi) = \dfrac{10-\sqrt{15}}{5} S(\psi; 3\pi; 1); c_3(\psi) = 1 - c_1(\psi)$

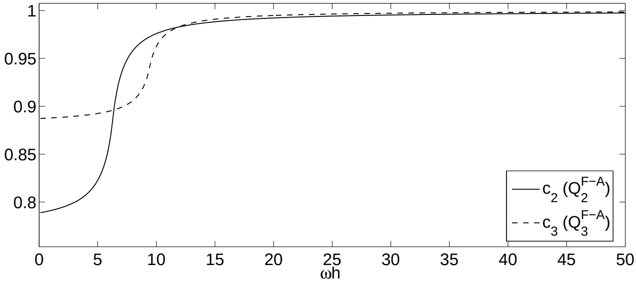


Figure: $c_2(\psi)$ of the adaptive Filon method $Q_2^{F\text{-}A}$ and $c_3(\psi)$ of the adaptive Filon method $Q_3^{F\text{-}A}$.

Asymptotic analysis for $Q_2^{F\text{-}A}$

$\tilde{c}_1 = c_1 h = \sigma_1(\omega)$ and $\tilde{c}_2 = c_2 h = h + \sigma_2(\omega)$ with $\sigma_{1,2}(\omega) \sim \omega^{-1}$

$$v(x) = s_h(x)(x - h - \sigma_2) \qquad s_h(x) = \frac{f''(\xi_h(x))}{2}(x - \sigma_1)$$

$$\begin{aligned} v'(x) &= s_h(x) + s_h'(x)(x - h - \sigma_2) \\ v''(x) &= 2s_h'(x) + s_h''(x)(x - h - \sigma_2) \\ &\vdots \end{aligned}$$

$$\begin{aligned} v(h) &= -s_h(h)\sigma_2 \\ v'(h) &= s_h(h) - s_h'(h)\sigma_2 \\ v''(h) &= 2s_h'(h) - s_h''(h)\sigma_2 \\ &\vdots \end{aligned}$$

Similar results for the other endpoint.

Asymptotic analysis for $Q_2^{F\text{-}A}$

$$Q_2^{F-A}[f] - I[f] = I[v] \sim \sum_{m=0}^{\infty} \frac{1}{(-\mathrm{i}\omega)^{m+1}} \left[e^{\mathrm{i}\omega h} v^{(m)}(h) - v^{(m)}(0) \right]$$

Reordering for $s_h(h) = f''(\xi_h(x))$, $s_h'(h) = f^{(iii)}(\xi_h(x))$, ...

$$\begin{aligned} I[v] &\sim s_h(h) e^{\mathrm{i}\psi} \left[\frac{\sigma_2}{\mathrm{i}\omega} - \frac{1}{\omega^2} \right] + s_h'(h) e^{\mathrm{i}\psi} \left[\frac{\sigma_2}{\omega^2} + \frac{2}{\mathrm{i}\omega^3} \right] + \dots \\ &+ s_0(0) \left[\frac{\sigma_1}{\mathrm{i}\omega} - \frac{1}{\omega^2} \right] + s_0'(0) \left[\frac{\sigma_1}{\omega^2} + \frac{2}{\mathrm{i}\omega^3} \right] + \dots \end{aligned}$$

$\sigma_2 = -\sigma_1$ with $\sigma_{1,2}(\omega) \sim \psi^{-1} \iff Q_2^{F-A}[f] - I[f] \sim O(\psi^{-2})$

A complex adaptive Filon-rule : $Q_2^{F\text{-}C}$

Are there better options than choosing $\sigma_2 = -\sigma_1$?

$$
\begin{aligned}
I[v] \quad &\sim \quad s_h(h)e^{\mathrm{i}\psi}\left[\frac{\sigma_2}{\mathrm{i}\omega} - \frac{1}{\omega^2}\right] + s_h'(h)e^{\mathrm{i}\psi}\left[\frac{\sigma_2}{\omega^2} + \frac{2}{\mathrm{i}\omega^3}\right] + \dots \\
&+ \quad s_0(0)\left[\frac{\sigma_1}{\mathrm{i}\omega} - \frac{1}{\omega^2}\right] + s_0'(0)\left[\frac{\sigma_1}{\omega^2} + \frac{2}{\mathrm{i}\omega^3}\right] + \dots
\end{aligned}
$$

Yes : Suppose $\sigma_1 = \sigma_2 = i/\omega \Longrightarrow Q_2^{F-C}[f] - I[f] \sim O(\psi^{-3})$.

$$
Q_2^{F\text{-}C} = \frac{\mathrm{i}h\left[f(\mathrm{i}h/\psi) - e^{\mathrm{i}\psi} f\left((i+\psi)h/\psi\right)\right]}{\psi}, \quad \psi = \omega h
$$

Illustration

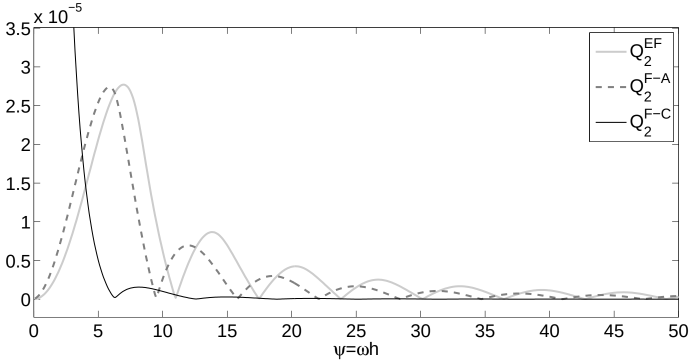


Figure: The errors in some $\nu = 2$ Filon-type schemes for $f(x) = e^x, h = 1/10$ and different values of ω.

Illustration

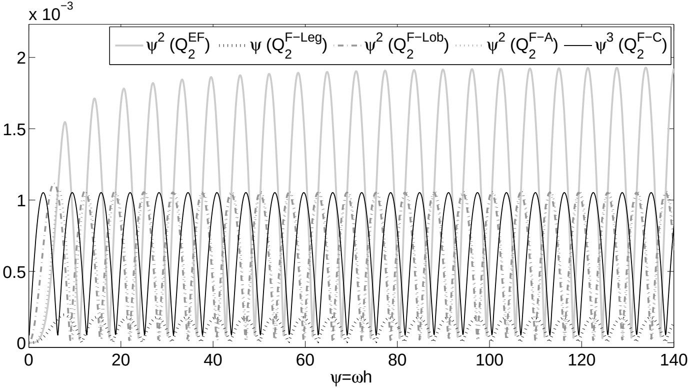


Figure: The normalised errors in some $\nu = 2$ Filon-type schemes for $f(x) = e^x, h = 1/10$ and different values of ω.

Error control for $Q_2^{F\text{-}C}$

$$Q_2^{F\text{-}C} = \frac{\mathrm{i}h\left[f(\mathrm{i}h/\psi) - e^{\mathrm{i}\psi} f\left((i+\psi)h/\psi\right)\right]}{\psi}, \quad \psi = \omega h.$$

Obtained by replacing f by interpolating polynomial $\bar{f}$ in nodes $\mathrm{i}\,h/\omega$ and $h + \mathrm{i}\,h/\omega$ (for large $\psi : \sim \psi^{-3}$)
Similarly : $Q_3^{F\text{-}C}$ by replacing f by interpolating polynomial $\tilde{f}$ in nodes $\mathrm{i}\,h/\omega$, $h/2$ and $h + \mathrm{i}\,h/\omega$ (for large ψ : also $\sim \psi^{-3}$ but about 100 times more accurate)

$$I[\tilde{f}] - I[\bar{f}] = \frac{(1-e^{\mathrm{i}\psi})2h}{\psi^2(4+\psi^2)} \times \left((2-\mathrm{i}\psi)\,f(\frac{\mathrm{i}}{\omega}) - (2+\mathrm{i}\psi)\,f(h+\frac{\mathrm{i}}{\omega}) + (2\mathrm{i}\psi)f(\frac{h}{2})\right)$$

Illustration

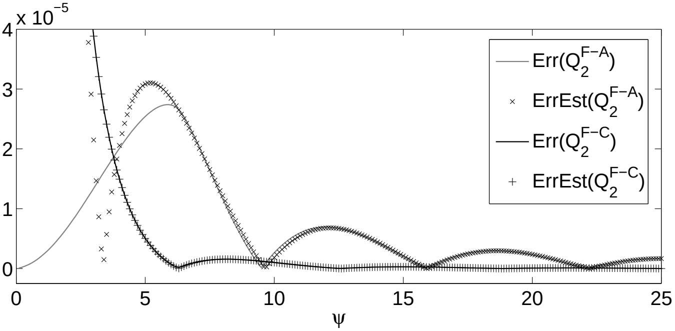


Figure: Error estimations for the $Q_2^{F\text{-}A}$ and $Q_2^{F\text{-}C}$ method applied on the problem with $f(x) = e^x, h = 2$.

Illustration

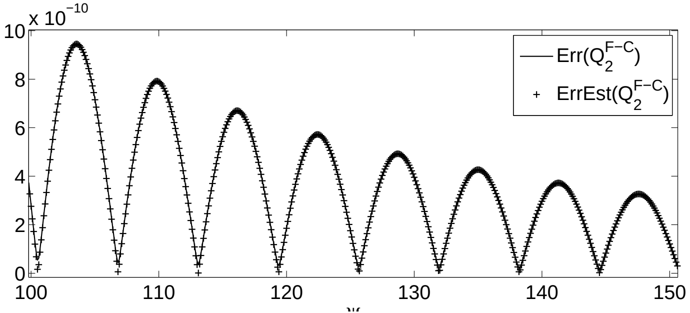


Figure: Error estimations for the $Q_2^{F\text{-}A}$ and $Q_2^{F\text{-}C}$ method applied on the problem with $f(x) = e^x, h = 2$.

Conclusions

- Filon rules, EF rules, and steepest descent rules are built up starting from different points op view, the basic underlying idea is the same : $f(x)$ is interpolated by a polynomial.
- Different choices can be made for the interpolation nodes.
- A choice of the (complex) interpolation nodes can improve the asymptotic behaviour of the quadrature rule.
- Even better asymptotic behaviour is obtained if the nodes are frequency dependent.
- Cheap error estimation is possible.