P-stable exponentially fitted Obrechkoff methods for $y'' = f(x, y)$

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Outline

- Introduction on exponentially fitted (EF) methods
- 2-step Obrechkoff methods for $y'' = f(x, y)$
- P-stable Obrechkoff methods for $y'' = f(x, y)$
- P-stable EF Obrechkoff methods for $y'' = f(x, y)$
- Conclusions

Exponentially fitted methods

Since about 1990, the research group of Guido Vanden Berghe at Ghent University did a lot of work on EF methods.

- interpolation : $a\,\cos\omega\,x + b\,\sin\omega\,x + \sum_{i=0}^{n-2} c_i\,x^i$
- quadrature (Newton-Cotes, Gauss)
- differentiation
- integration (linear multistep methods, Runge-Kutta(-Nystrom) methods for IVP and BVP, eigenvalue problems, ...)
- integral equations (Fredholm, Volterra)
- $\dots$

Aim : build methods which perform very good when the solution has a known exponential of trigonometric behaviour.

Exponentially fitted methods

- researchers involved at Ghent University:
  Guido Vanden Berghe, Hans De Meyer, Jan Vanthournout, Marnix Van Daele, Philippe Bocher, Hans Vande Vyver, Veerle Ledoux and Davy Hollevoet
- collaboration with L. Ixaru

Linear multistep methods

A well known method to solve

$$y'' = f(y) \qquad y(a) = y_a \qquad y'(a) = y'_a$$

is the Numerov method (order 4)

$$y_{n+1} - 2\,y_n + y_{n-1} = \frac{h^2}{12}\,\left(f(y_{n-1}) + 10\,f(y_n) + f(y_{n+1})\right)$$

Construction : impose $\mathcal{L}[z(x);h] = 0$ for $z(x) = 1,\ x,\ x^2,\ x^3,\ x^4$

where

$$\begin{aligned}\mathcal{L}[z(x);h] := {} & z(x+h) + \alpha_0\,z(x) + \alpha_{-1}\,z(x-h) \\ & -h^2\left(\beta_1\,z''(x+h) + \beta_0\,z''(x) + \beta_{-1}\,z''(x-h)\right)\end{aligned}$$

Exponential fitting

Consider the initial value problem

$$y'' + \omega^2\, y = g(y) \qquad y(a) = y_a \qquad y(a) = y'_a\,.$$

If $|g(y)| \ll |\omega^2\, y|$ then

$$y(x) \approx \alpha\, \cos(\omega\, x + \phi)$$

To mimic this oscillatory behaviour, one could replace polynomials by trigonometric (in the complex case : exponential) functions.

EF Numerov method

$$\text{Construction : impose } \mathcal{L}[z(x);h] = 0 \text{ for}$$
$$z(x) = 1,\ x,\ x^2,\ \sin(\omega\, x),\ \cos(\omega\, x)$$

$$\begin{aligned}\mathcal{L}[z(x);h] := {} & z(x+h) + \alpha_0\, z(x) + \alpha_{-1}\, z(x-h) \\ & -h^2\left(\beta_1\, z''(x+h) + \beta_0\, z''(x) + \beta_{-1}\, z''(x-h)\right)\end{aligned}$$

$$y_{n+1} - 2\, y_n + y_{n-1} = h^2\ \left(\lambda f(y_{n-1}) + (1 - 2\,\lambda)\, f(y_n) + \lambda\, f(y_{n+1})\right)$$

$$\lambda = \frac{1}{4\,\sin^2\frac{\theta}{2}} - \frac{1}{\theta^2} = \frac{1}{12} + \frac{1}{240}\theta^2 + \frac{1}{6048}\theta^4 + \dots \qquad \theta := \omega\, h$$

EF methods

Generalisation : to determine the coefficients of a method, we impose conditions on a linear functional. These conditions are related to

- polynomials :

$$\{x^q | q = 0, \ \ldots, K\}$$

- exponential or trigonometric functions, multiplied with powers of x :

$$\{x^q \ \exp(\pm \mu x) | q = 0, \ \ldots, P\}$$

or, with $\omega = i\,\mu$,

$$\{x^q \ \cos(\omega x), \ x^q \ \sin(\omega x) | q = 0, \ \ldots, P\}$$

Classical method : $P = -1$

Choice of ω

- local optimization : based on local truncation error (lte)
  e.g. for the EF Numerov method

$$y(x_{n+1}) - y_{n+1} = -\frac{h^6}{240}\left(y^{(6)}(x_n) + \omega^2\, y^{(4)}(x_n)\right) + \ldots$$

$$\Longrightarrow \omega_n^2 = -\frac{y^{(6)}(x_n)}{y^{(4)}(x_n)}$$

  ω is step-dependent

Choice of ω

- global optimization

  Preservation of geometric properties (periodicity, energy, . . .)

  M. VAN DAELE, G. VANDEN BERGHE, GEOMETRIC NUMERICAL INTEGRATION BY MEANS OF EXPONENTIALLY FITTED METHODS, APNUM 57 (2007) 415-435

  Several means to give a value to ω :

  - backward error analysis
  - linearisation : rewrite $y'' = f(y)$ as $y'' + \omega^2 y = g(y)$ with $|g(y)|$ small
  - Hamiltonian : minimize the leading term in $H_{n+1} - H_n$

  ω is constant over the interval of integration

Obrechkoff methods

Nikola Obrechkoff (1896-1963)

Obrechkoff methods (OM) : $^{\circ}1940$ for quadrature

Milne : OM for solving diff. eq. : 1949

Obrechkoff methods for $y'' = f(x,y)$

Two-step methods

$$y_{n+1} - 2y_n + y_{n-1} = \sum_{i=1}^{m} h^{2\,i} \left(\beta_{i0}\, y_{n+1}^{(2i)} + 2\, \beta_{i1}\, y_n^{(2i)} + \beta_{i0}\, y_{n-1}^{(2i)} \right)$$

$$\begin{aligned} \mathcal{L}[z(x);h] &:= z(x+h) - 2\, z(x) + z(x-h) \\ &\quad - \sum_{i=1}^{m} h^{2\,i} \left(\beta_{i0}\, z^{(2i)}(x+h) + 2\, \beta_{i1}\, z^{(2i)}(x) + \beta_{i0}\, z^{(2i)}(x-h) \right) \end{aligned}$$

symmetric method : $\mathcal{L}[z(x);h] \equiv 0$ if $z(x)$ is odd

$$\mathcal{L}[1;h] \equiv 0$$

order $p \Longleftrightarrow \mathcal{L}[x^q;h] = 0,\ q = 0,\ 1,\ \ldots,\ p+1$

lte $= C_{p+2}\, h^{p+2} y^{(p+2)}(x_n) + \mathcal{O}\left(h^{p+3}\right) \qquad C_{p+2} = \dfrac{\mathcal{L}[x^{p+2};h]}{(p+2)!\, h^{p+2}}$

Two-step OM

- $m = 1 : p = 4, C_6 = -\dfrac{1}{240}$ (Numerov method)

$$y_{n+1} - 2y_n + y_{n-1} = \frac{h^2}{12}\left(y^{(2)}_{n+1} + 10\, y^{(2)}_n + y^{(2)}_{n-1}\right)$$

- $m = 2 : p = 8, C_{10} = \dfrac{59}{76204800}$

$$y_{n+1} - 2y_n + y_{n-1} = \frac{h^2}{252}\left(11\, y^{(2)}_{n+1} + 230\, y^{(2)}_n + 11\, y^{(2)}_{n-1}\right) - \frac{h^4}{15120}\left(13\, y^{(4)}_{n+1} - 626\, y^{(4)}_n + 13\, y^{(4)}_{n-1}\right)$$

Two-step OM

- $m = 3 : p = 12,\, C_{14} = -\dfrac{45469}{1697361329664000}$

$$
\begin{aligned}
y_{n+1} - 2y_n + y_{n-1} = {} & \\
& \frac{h^2}{7788}\left(229\, y_{n+1}^{(2)} + 7330\, y_n^{(2)} + 229\, y_{n-1}^{(2)}\right) \\
& - \frac{h^4}{25960}\left(11\, y_{n+1}^{(4)} - 1422\, y_n^{(4)} + 11\, y_{n-1}^{(4)}\right) \\
& + \frac{h^6}{39251520}\left(127\, y_{n+1}^{(6)} + 4846\, y_n^{(6)} + 127\, y_{n-1}^{(6)}\right)
\end{aligned}
$$

- $\ldots$

  method of order $p = 4\, m$

EF 2-step OM : the case $m = 2$

$$y_{n+1} - 2y_n + y_{n-1} = \\ h^2 \left(\beta_{10}\, y^{(2)}_{n+1} + 2\,\beta_{11}\, y^{(2)}_n + \beta_{10}\, y^{(2)}_{n-1}\right) \\ + h^4 \left(\beta_{20}\, y^{(4)}_{n+1} + 2\,\beta_{21}\, y^{(4)}_n + \beta_{20}\, y^{(4)}_{n-1}\right)$$

5 possibilities:

- $P = -1 : \{x^2,\, x^4,\, x^6,\, x^8\}$
- $P = 0 : \{x^2,\, x^4,\, x^6,\, \cos\omega\, x\}$
- $P = 1 : \{x^2,\, x^4,\, \cos\omega\, x, x\, \sin\omega\, x\}$
- $P = 2 : \{x^2,\, \cos\omega\, x,\, x\, \sin\omega\, x,\, x^2\, \cos\omega\, x\}$
- $P = 3 : \{\cos\omega\, x,\, x\, \sin\omega\, x,\, x^2\, \cos\omega\, x,\, x^3\, \sin\omega\, x\}$

EF 2-step OM : the case $m = 2$

$$
\begin{aligned}
y_{n+1} - 2y_n + y_{n-1} = \\
& h^2 \left(\beta_{10}\, y^{(2)}_{n+1} + 2\, \beta_{11}\, y^{(2)}_{n} + \beta_{10}\, y^{(2)}_{n-1} \right) \\
& + h^4 \left(\beta_{20}\, y^{(4)}_{n+1} + 2\, \beta_{21}\, y^{(4)}_{n} + \beta_{20}\, y^{(4)}_{n-1} \right)
\end{aligned}
$$

5 possibilities:

- $P = -1 : \{x^2,\, x^4,\, x^6,\, x^8\}$

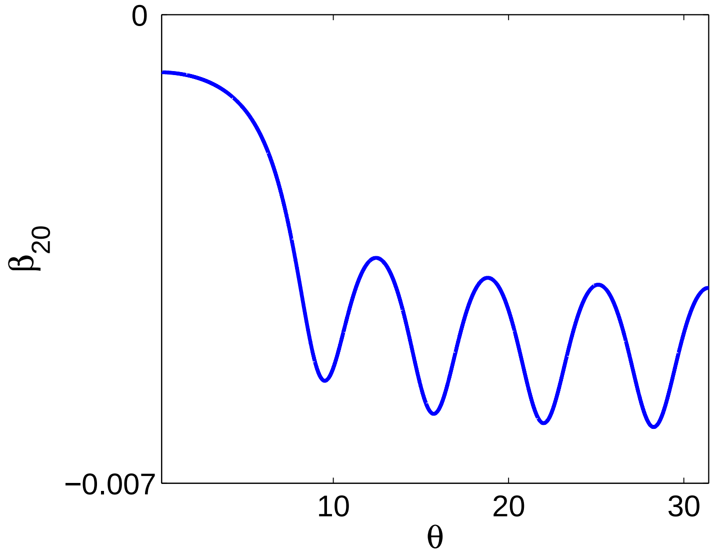
$$
\beta_{21} = \frac{313}{15120} \qquad \beta_{20} = -\frac{13}{15120} \qquad \beta_{11} = \frac{115}{252} \qquad \beta_{10} = \frac{11}{252}
$$

EF 2-step OM : the case $m = 2$

- $P = 0 : \{x^2,\, x^4,\, x^6,\, \cos\omega\, x\}$

$$\beta_{10} = \frac{1}{30} - 12\,\beta_{20} \qquad \beta_{21} = \frac{1}{40} + 5\,\beta_{20} \qquad \beta_{11} = \frac{7}{15} + 12\,\beta_{20}$$

$$\beta_{20} = -\frac{120 - 4\ \cos(\theta)\,\theta^2 - 56\,\theta^2 - 120\ \cos(\theta) + 3\,\theta^4}{120\,\theta^2\,(12\ \cos(\theta) - 12 + \cos(\theta)\,\theta^2 + 5\,\theta^2)}$$



continuous coefficients

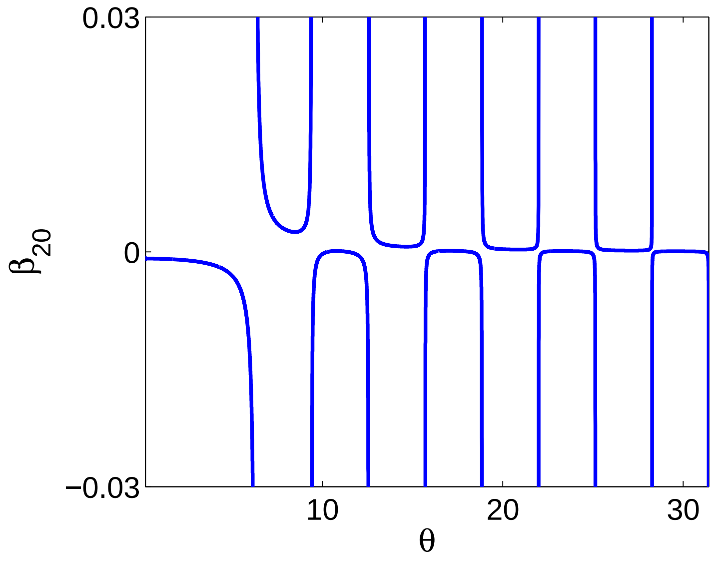
EF 2-step OM : the case $m = 2$

- $P = 1 : \{x^2,\ x^4,\ \cos\omega\, x,\ x\ \sin\omega\, x\}$

$$\beta_{10} = \frac{1}{12} - 2\,\beta_{20} - 2\,\beta_{21} \qquad \beta_{11} = \frac{5}{12} + 2\,\beta_{20} + 2\,\beta_{21}$$

$$\beta_{20} = \frac{\theta^5 \sin\theta + 2\,(\cos\theta + 5)\,\theta^4 + 48\,(\cos\theta - 1)\,\theta^2 + 48\,(\cos\theta - 1)^2}{12\,\theta^4\,(\theta^3\ \sin\theta - 4\,(1 - \cos\theta)^2)}$$

$$\beta_{21} = \frac{5\,\theta^5 \sin\theta - 2\ \cos\theta\,(\cos\theta + 5)\,\theta^4 - 48\ \cos\theta\,(\cos\theta - 1)\,\theta^2 - 48\,(\cos\theta - 1)^2}{12\,\theta^4\,(\theta^3\ \sin\theta - 4\,(1 - \cos\theta)^2)}$$



discontinuous coefficients

EF 2-step OM : the case $m = 2$

- $P = 2 : \{x^2,\, \cos\omega\, x,\, x\, \sin\omega\, x,\, x^2\, \cos\omega\, x\}$
  discontinuous coefficients
- $P = 3 : \{\cos\omega\, x,\, x\, \sin\omega\, x,\, x^2\, \cos\omega\, x,\, x^3\, \sin\omega\, x\}$
  continuous coefficients

Stability of two-step OM

$$y_{n+1} - 2\, y_n + y_{n-1} = \sum_{i=1}^{m} h^{2\,i} \left(\beta_{i0}\, y_{n+1}^{(2i)} + 2\, \beta_{i1}\, y_n^{(2i)} + \beta_{i0}\, y_{n-1}^{(2i)} \right)$$

applied to $y'' = -\lambda^2\, y$ gives

$$y_{n+1} - 2 R_{mm}(\nu^2)\, y_n + y_{n-1} = 0 \qquad \nu := \lambda\, h$$

$$R_{mm}(\nu^2) = \frac{1 + \sum_{i=1}^{m} (-1)^i \beta_{i1}\, \nu^{2i}}{1 + \sum_{i=1}^{m} (-1)^{i+1} \beta_{i0}\, \nu^{2i}}$$

The stabilitity function R_{mm} uniquely determines the method.

A method has the interval of periodicity $(0, \nu_0^2)$ if

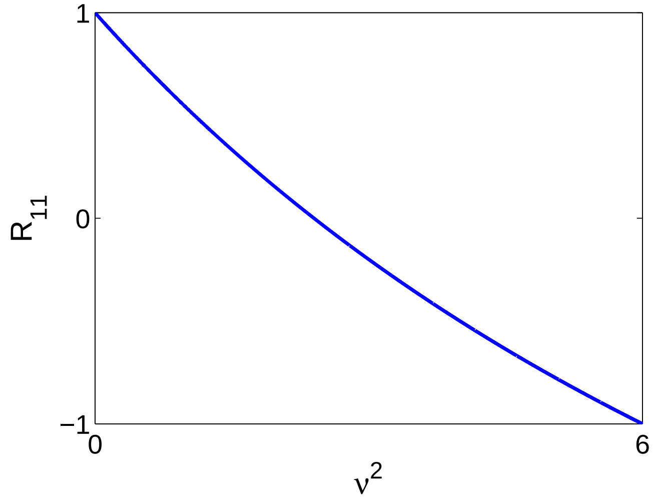
$|R_{mm}(\nu^2)| \leq 1$ for $0 < \nu^2 < \nu_0^2$.

The method is P-stable if $|R_{mm}(\nu^2)| \leq 1$ for all real $\nu \neq 0$.

Stability of two-step OM

- $m = 1 : p = 4, C_6 = -\dfrac{1}{240}$ (Numerov) $\qquad \nu_0^2 = 6$

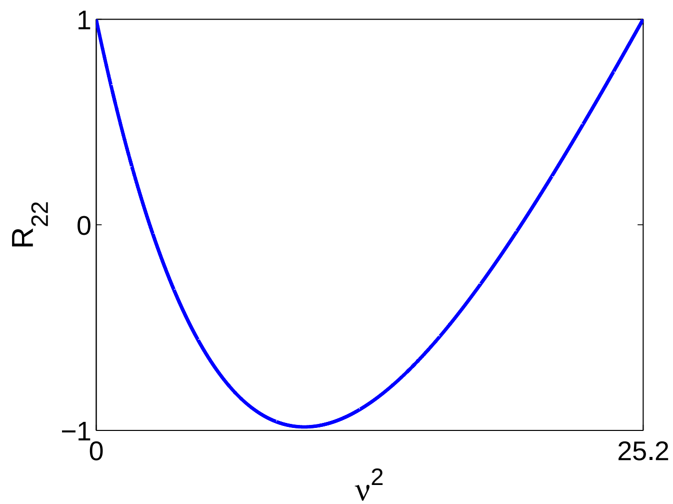
$$y_{n+1} - 2y_n + y_{n-1} = \\ \frac{h^2}{12}\left(y_{n+1}^{(2)} + 10\, y_n^{(2)} + y_{n-1}^{(2)}\right)$$



Stability of two-step OM

- $m = 2 : p = 8, C_{10} = \dfrac{59}{76204800}$ $\qquad\qquad \nu_0^2 = 25.2$

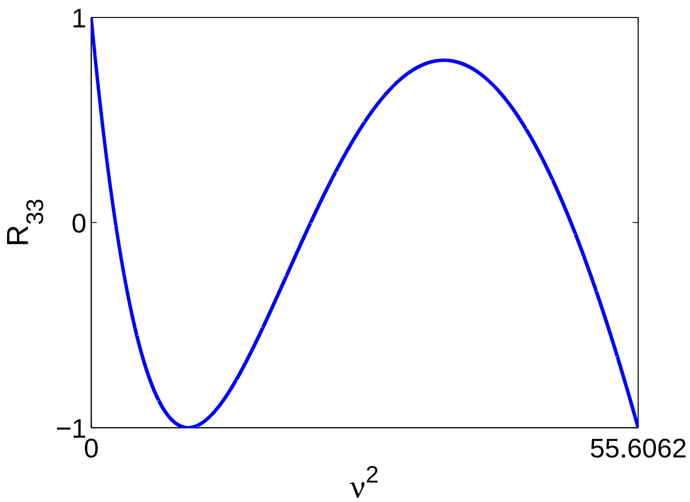
$$y_{n+1} - 2y_n + y_{n-1} =$$
$$\frac{h^2}{252}\left(11\, y_{n+1}^{(2)} + 115\, y_n^{(2)} + 11\, y_{n-1}^{(2)}\right)$$
$$-\frac{h^4}{15120}\left(13\, y_{n+1}^{(4)} - 626\, y_n^{(4)} + 13\, y_{n+1}^{(4)}\right)$$



Stability of two-step OM

- $m = 3 : p = 12,\ C_{14} = -\dfrac{45469}{1697361329664000} \quad \nu_0^2 = 55.60\ldots$

$$
\begin{aligned}
y_{n+1} - 2y_n + y_{n-1} = \\
& \frac{h^2}{7788}\left(229\, y^{(2)}_{n+1} + 7330\, y^{(2)}_{n} + 229\, y^{(2)}_{n-1}\right) \\
& + \frac{h^4}{25960}\left(-11\, y^{(4)}_{n+1} + 1422\, y^{(4)}_{n} - 11\, y^{(4)}_{n-1}\right) \\
& + \frac{h^6}{39251520}\left(127\, y^{(6)}_{n+1} + 4846\, y^{(6)}_{n} + 127\, y^{(6)}_{n-1}\right)
\end{aligned}
$$



P-stable 2-step OM

Ananthakrishnaiah (1987)

idea : use some parameters to obtain P-stability.

$m = 3$ and $m = 4$

e.g. $m = 3$

$$
\begin{aligned}
y_{n+1} - 2y_n + y_{n-1} = \\
& h^2 \left(\beta_{10}\, y_{n+1}^{(2)} + 2\,\beta_{11}\, y_n^{(2)} + \beta_{10}\, y_{n-1}^{(2)} \right) \\
& + h^4 \left(\beta_{20}\, y_{n+1}^{(4)} + 2\,\beta_{21}\, y_n^{(4)} + \beta_{20}\, y_{n-1}^{(4)} \right) \\
& + h^6 \left(\beta_{30}\, y_{n+1}^{(6)} + 2\,\beta_{31}\, y_n^{(6)} + \beta_{30}\, y_{n-1}^{(6)} \right)
\end{aligned}
$$

impose $p = 6 : \{x^2,\, x^4,\, x^6\}$,

then 3 parameters (β_{20}, β_{30} and β_{31}) remain

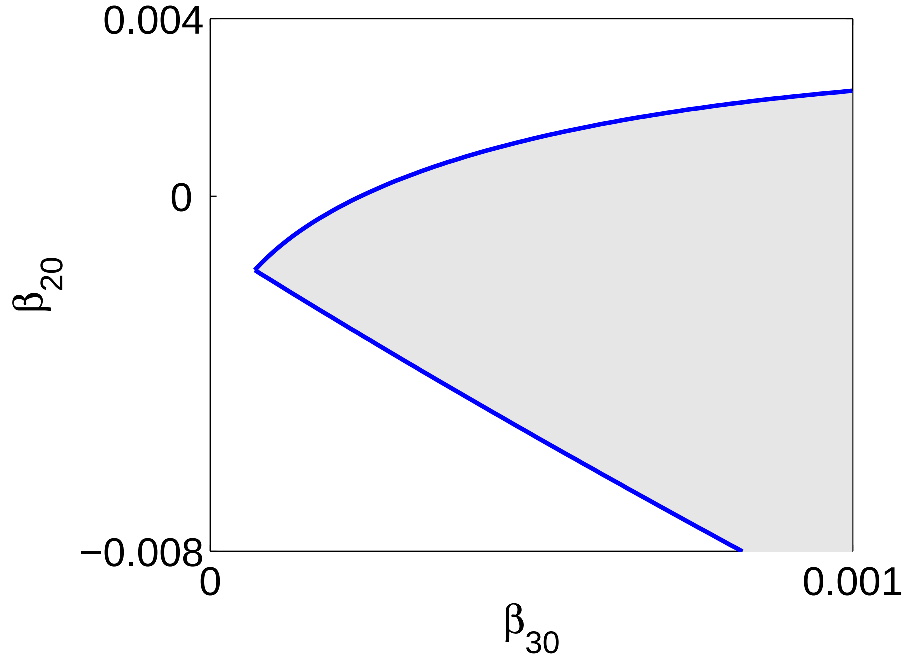
$$R_{33} = \frac{1 - \beta_{11}\,\nu^2 + \beta_{21}\,\nu^4 - \beta_{31}\,\nu^6}{1 + \beta_{10}\,\nu^2 - \beta_{20}\,\nu^4 + \beta_{30}\,\nu^6}$$

let $\beta_{31} = \beta_{30}$

Ananthakrishnaiah's idea

Choose these 2 remaining parameters β_{20}, β_{30} such that the method becomes P-stable

$$|R_{33}| = \left|\frac{N_3}{D_3}\right| < 1 \Longleftrightarrow (D_3 - N_3)\,(D_3 + N_3) > 0$$

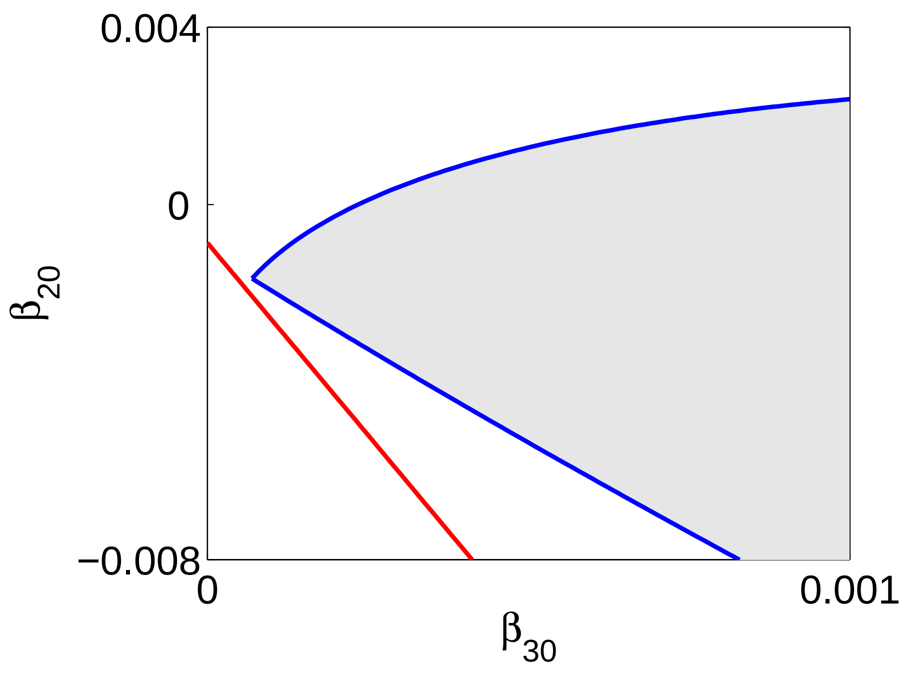


Ananthakrishnaiah's idea

Find the solution for which the phase-lag,

$$\nu - \arccos R_{33} = \left(\frac{13}{604800} + \frac{13}{30}\,\beta_{30} + \frac{1}{40}\,\beta_{20} \right) \nu^7 + \mathcal{O}\left(\nu^9\right)$$

becomes minimal.



Ananthakrishnaiah's idea

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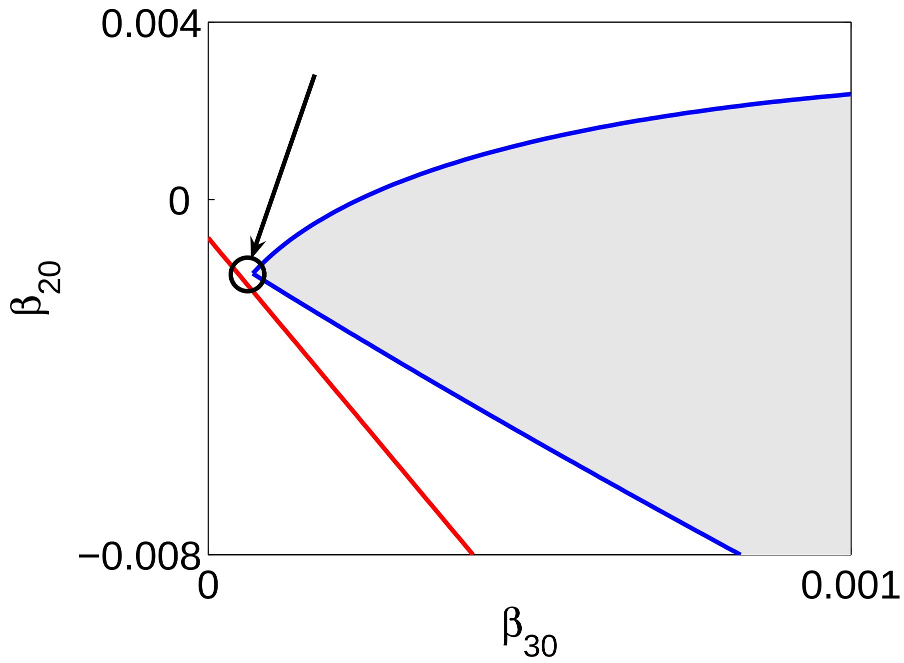


This gives $(\beta_{30}, \beta_{20}) = \left(\frac{1}{14400}, -\frac{1}{600} \right)$.

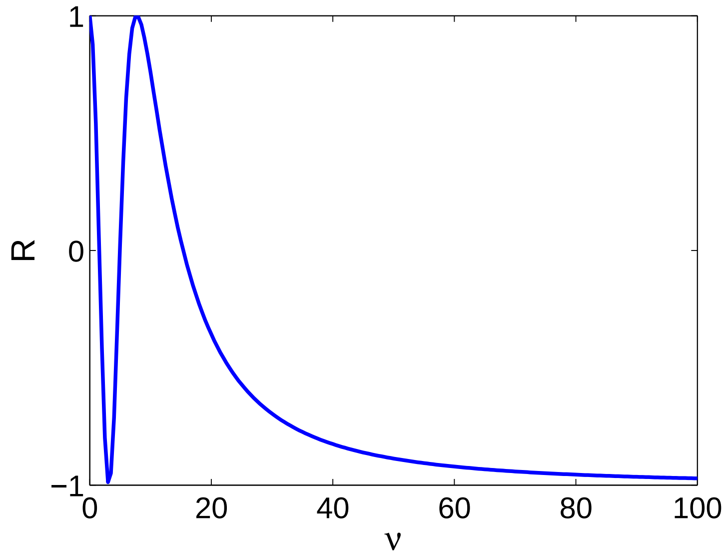
Ananthakrishnaiah's idea

Ananthakrishnaiah $m = 3 : p = 6$, P-stable, $C_8 = -\dfrac{1}{50400}$

$$
\begin{aligned}
& y_{n+1} - 2y_n + y_{n-1} = \\
& \qquad \frac{h^2}{20}\left(y^{(2)}_{n+1} + 18\, y^{(2)}_n + y^{(2)}_{n-1}\right) \\
& \qquad -\frac{h^4}{600}\left(y^{(4)}_{n+1} - 22\, y^{(4)}_n + y^{(4)}_{n-1}\right) \\
& \qquad +\frac{h^6}{14400}\left(y^{(6)}_{n+1} + 2\, y^{(6)}_n + y^{(6)}_{n-1}\right)
\end{aligned}
$$

Ananthakrishnaiah's idea

$$R_{33} = \frac{1 - \frac{9}{20}\,\nu^2 + \frac{11}{600}\,\nu^4 - \frac{1}{14400}\,\nu^6}{1 + \frac{1}{20}\,\nu^2 + \frac{1}{600}\,\nu^4 + \frac{1}{14400}\,\nu^6}$$



$$|R_{33}| = \left|\frac{N_3}{D_3}\right| < 1 \text{ since}$$

$$(D_3 - N_3)\,(D_3 + N_3) = \frac{\nu^2\,(\nu^2 - 10)^2\,(\nu^2 - 60)^2}{360000}$$

P-stable 2-step OM

We were able to generalise Ananthakrishnaiah's idea in

M. VAN DAELE AND G. VANDEN BERGHE,

P-STABLE OBRECHKOFF METHODS OF ARBITRARY ORDER FOR SECOND-ORDER DIFFERENTIAL EQUATIONS, NUMERICAL ALGORITHMS **44**, 2007, 115-131

Algoritm to construct a P-stable OM for a given m :

- impose order $2\,m$
- $D_m - N_m$ and $D_m + N_m$ should both be halves of perfect squares

This leads to a system of non-linear equations.

Theorem : the approximant $R_{mm}(\nu^2)$, obtained by generalising Ananthakrishnaiah's approach, is given by the real part of the (m, m)-Padé approximant of $\exp(i\,\nu)$.

This leads to a system of linear equations.

Example

$$\begin{aligned} R_{33}(\nu^2) &= \frac{1 - \frac{9}{20}\,\nu^2 + \frac{11}{600}\,\nu^4 - \frac{1}{14400}\,\nu^6}{1 + \frac{1}{20}\,\nu^2 + \frac{1}{600}\,\nu^4 + \frac{1}{14400}\,\nu^6} \\ &= \Re\left(\frac{1 + \frac{1}{2}\,i\,\nu - \frac{1}{10}\,\nu^2 - \frac{1}{120}\,i\,\nu^3}{1 - \frac{1}{2}\,i\,\nu - \frac{1}{10}\,\nu^2 + \frac{1}{120}\,i\,\nu^3}\right) \\ &= \frac{\left(1 - \frac{1}{10}\,\nu^2\right)^2 - \left(\frac{1}{2}\,\nu - \frac{1}{120}\,\nu^3\right)^2}{\left(1 - \frac{1}{10}\,\nu^2\right)^2 + \left(\frac{1}{2}\,\nu - \frac{1}{120}\,\nu^3\right)^2} \end{aligned}$$

where $\dfrac{1 + \frac{1}{2}\,\nu + \frac{1}{10}\,\nu^2 + \frac{1}{120}\,\nu^3}{1 - \frac{1}{2}\,\nu + \frac{1}{10}\,\nu^2 - \frac{1}{120}\,\nu^3} = \exp(\nu) + \mathcal{O}\left(\nu^7\right)$

$$|R_{33}(\nu^2)| = \left|\frac{\left(1 - \frac{1}{10}\,\nu^2\right)^2 - \left(\frac{1}{2}\,\nu - \frac{1}{120}\,\nu^3\right)^2}{\left(1 - \frac{1}{10}\,\nu^2\right)^2 + \left(\frac{1}{2}\,\nu - \frac{1}{120}\,\nu^3\right)^2}\right| \leq 1$$

Conclusion

How does the P-stable Obrechkoff method

$$y_{n+1} - 2y_n + y_{n-1} = \sum_{i=1}^{m} h^{2\,i} \left(\beta_{i0}\, y_{n+1}^{(2i)} + 2\,\beta_{i1}\, y_n^{(2i)} + \beta_{i0}\, y_{n-1}^{(2i)} \right)$$

of order $2\,m$ look like ?

$$\left\{ \begin{array}{l} \beta_{i0} = (-1)^{i+1} a_i^2 + 2 \displaystyle\sum_{j=0}^{i-1} (-1)^{j+1} a_j\, a_{2i-j} \\ \beta_{i1} = a_i^2 + 2 \displaystyle\sum_{j=0}^{i-1} a_j\, a_{2i-j} \end{array} \right. \qquad i = 1\ \ldots, m$$

$$\text{where } a_j = \begin{cases} \dfrac{\binom{m}{j}}{\binom{2\,m}{j}} & \text{for } 0 \le j \le m \\ 0 & \text{for } j > m \end{cases}$$

P-stable Expon. fitted OM

How to obtain P-stable exponentially-fitted Obrechkoff methods ?

$$y_{n+1} - 2\,y_n + y_{n-1} = \sum_{i=1}^{m} h^{2\,i} \left(\beta_{i0}\, y_{n+1}^{(2i)} + 2\,\beta_{i1}\, y_n^{(2i)} + \beta_{i0}\, y_{n-1}^{(2i)}\right)$$

applied to $y'' = -\lambda^2\, y$ gives

$$y_{n+1} - 2R_{mm}(\theta, \nu^2)\, y_n + y_{n-1} = 0$$

with $\theta := \omega\, h$ and $\nu := \lambda\, h$

Padé-approximants $\Longrightarrow$ exponentially-fitted Padé approximants

Construction of Polynomially-fitted $(m,\, m)$ Padé approximants :

$$\frac{P_m(x)}{P_m(-x)} = \exp(x) + \mathcal{O}\left(x^{2\,m+1}\right)$$

$$\Longleftrightarrow \quad \exp(x)\, P_m(-x) - P_m(x) = \mathcal{O}\left(x^{2\,m+1}\right)$$

$$\Longleftrightarrow \quad \frac{\mathrm{d}^{2\,q}}{\mathrm{d}x^{2\,q}} \left(\exp(x)\, P_m(-x) - P_m(x)\right)\Big|_{x=0} = 0 \qquad q = 1,\, \ldots,\, m$$

EF Padé approximants

$$\mathcal{F}(x,t) = \exp(t\,x)\,V_m(-t\,x) - V_m(t\,x) \qquad\qquad V_m(x) = 1 + \sum_{j=1}^{m} a_j\,x^j$$

$$\begin{cases} \dfrac{\partial^{2q}}{\partial x^{2q}}\mathcal{F}(x,t)\Big|_{(x,t)=(0,\theta)} = 0 & q = 1,\,\ldots,\,K \\ \Re\left(\dfrac{\partial^{q}}{\partial t^{q}}\mathcal{F}(x,t)\Big|_{(x,t)=(i,\,\theta)}\right) = 0 & q = 0,\,\ldots,\,P \end{cases}$$

where $0 \le K \le m$ and $P + K + 1 = m$.

This leads to a system of m linear equations in the unknowns a_j, $j = 1, \ldots, m$.

The EF (K,P) Padé approximant to $\exp(\nu)$ is then given by ${}^{(K,P)}\hat{P}_m^m(\nu) = V_m(\nu)/V_m(-\nu)$.

EF Padé approximants : $m = 1$

$$\mathcal{F}(x,t) = \exp(t\,x)\,V_1(-t\,x) - V_1(t\,x) \qquad\qquad V_m(x) = 1 + a_1\,x$$

- $(K = 1, P = -1)$

$$\frac{\partial^2}{\partial x^2}\mathcal{F}(x,t)\Big|_{(x,t)=(0,\theta)} = 0 \iff \theta^2\,(1 - 2\,a_1) = 0 \iff a_1 = \frac{1}{2}$$

$${}^{(1,-1)}\hat{P}_1^1(\nu) = \frac{1 + \frac{1}{2}\,x}{1 - \frac{1}{2}\,x}$$

- $(K = 0, P = 0)$

$$\Re\left(\mathcal{F}(i,\,\theta)\right) = 0 \iff \Re\left(e^{i\,\theta}\,(1 - i\,a_1\,\theta) - (1 + i\,a_1\theta)\right) = 0$$

$$\iff a_1 = \frac{\sin\theta}{\theta\,(\cos\theta + 1)} = \frac{1}{2}\,\frac{\tan(\theta/2)}{\theta/2}$$

$${}^{(0,0)}\hat{P}_1^1(\nu) = \frac{1 + \frac{1}{2}\,\frac{\tan(\theta/2)}{\theta/2}\,x}{1 - \frac{1}{2}\,\frac{\tan(\theta/2)}{\theta/2}\,x}$$

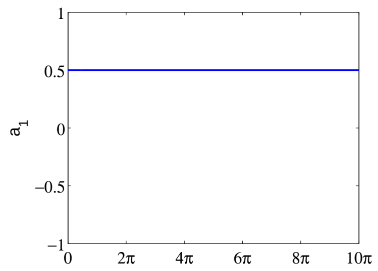
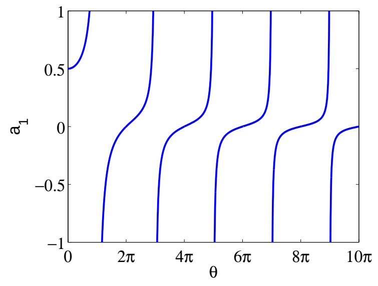
EF Padé approximants : $m = 1$

$m = 1$	
(K, P)	a_1
$(1, -1)$	$\frac{1}{2}$
$(0, 0)$	$\frac{\tan\frac{\theta}{2}}{\theta}$

EF Padé approximants : $m = 1$

$m = 1$	
(K, P)	a_1
$(1, -1)$	$\frac{1}{2}$
$(0, 0)$	$\frac{1}{2} + \frac{\theta^2}{24} + \mathcal{O}(\theta^4)$

EF Padé approximants : $m = 1$

$m = 1$	
(K, P)	a_1
$(1, -1)$	
(0,0)	

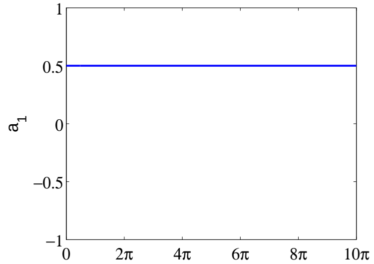
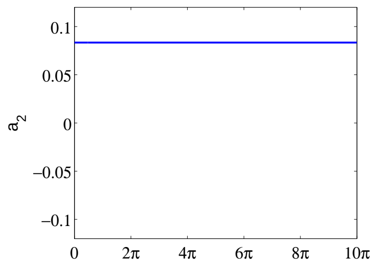
EF Padé approximants : $m = 2$

$m = 2$		
(K, P)	a_1	a_2
$(2, -1)$	$\dfrac{1}{2}$	$\dfrac{1}{12}$
$(1, 0)$	$\dfrac{1}{2}$	$\dfrac{2\tan\frac{\theta}{2} - \theta}{2\,\theta^2 \tan\frac{\theta}{2}}$
$(0, 1)$	$\dfrac{2\,(1 - \cos\theta)}{(\theta + \sin\theta)\,\theta}$	$\dfrac{\theta - \sin\theta}{(\theta + \sin\theta)\,\theta^2}$

EF Padé approximants : $m = 2$

$m = 2$		
(K,P)	a_1	a_2
$(2,-1)$	$\frac{1}{2}$	$\frac{1}{12}$
$(1,0)$	$\frac{1}{2}$	$\frac{1}{12} + \frac{\theta^2}{720} + \mathcal{O}\left(\theta^4\right)$
$(0,1)$	$\frac{1}{2} + \mathcal{O}(\theta^4)$	$\frac{1}{12} + \frac{\theta^2}{360} + \mathcal{O}(\theta^4)$

EF Padé approximants : $m = 2$

<table>
<tr><th colspan="3">$m = 2$</th></tr>
<tr><th>(K, P)</th><th>a_1</th><th>a_2</th></tr>
<tr><td>$(2, -1)$</td><td>
</td><td>
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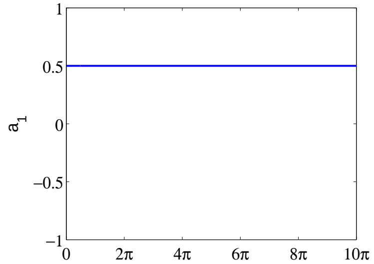
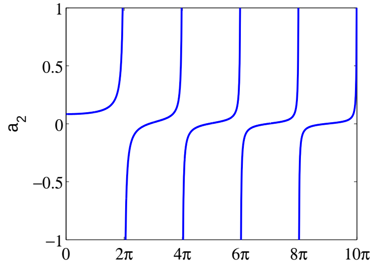
<tr><td>$(1, 0)$</td><td>
</td><td>
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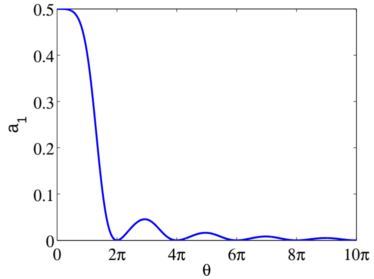
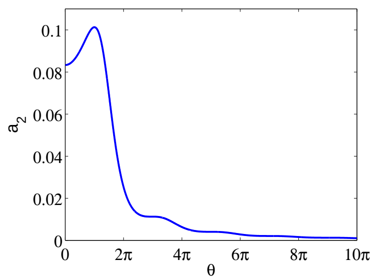
<tr><td>$(0, 1)$</td><td>
</td><td>
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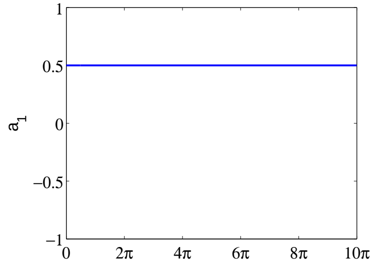
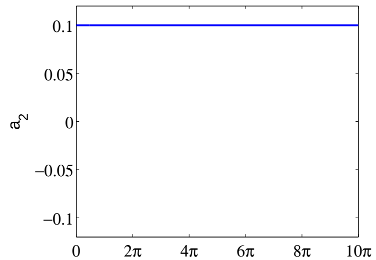
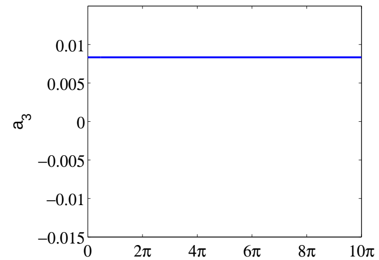
<table>
<tr><th colspan="4">$m = 3$</th></tr>
<tr><th>(K, P)</th><th>a_1</th><th>a_2</th><th>a_3</th></tr>
<tr><td>$(3, -1)$</td><td>
</td><td>
</td><td>
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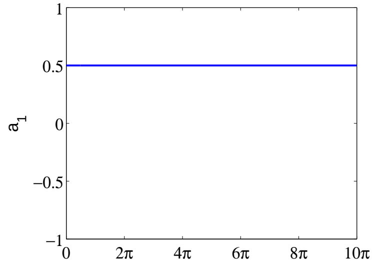
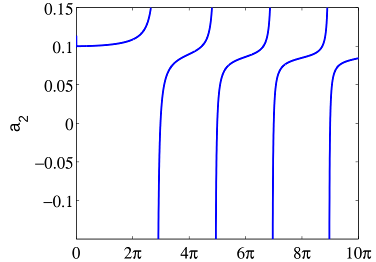
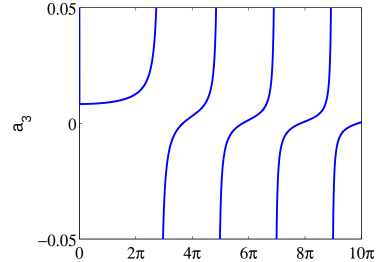
<tr><td>$(2, 0)$</td><td>
</td><td>
</td><td>
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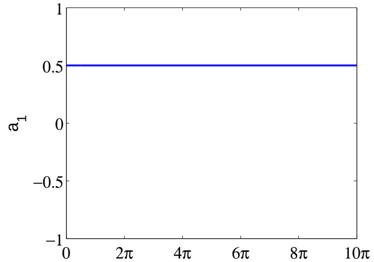
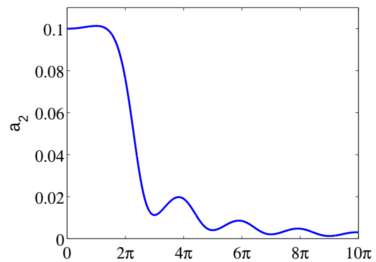
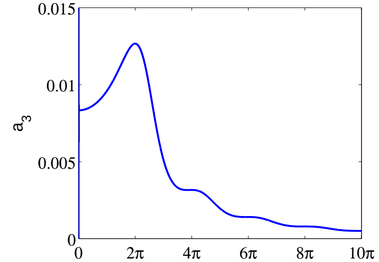
<tr><td>$(1, 1)$</td><td>
</td><td>
</td><td>
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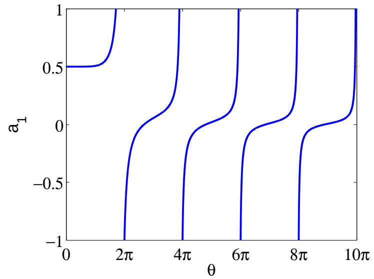
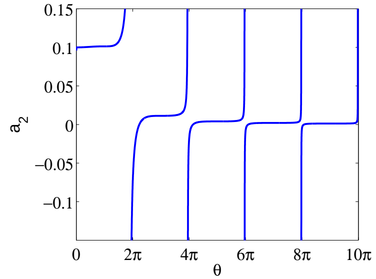
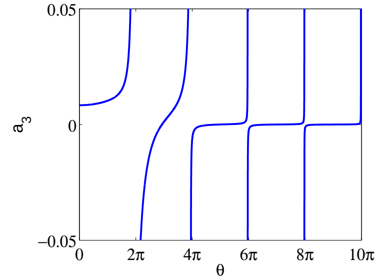
<tr><td>$(0, 2)$</td><td>
</td><td>
</td><td>
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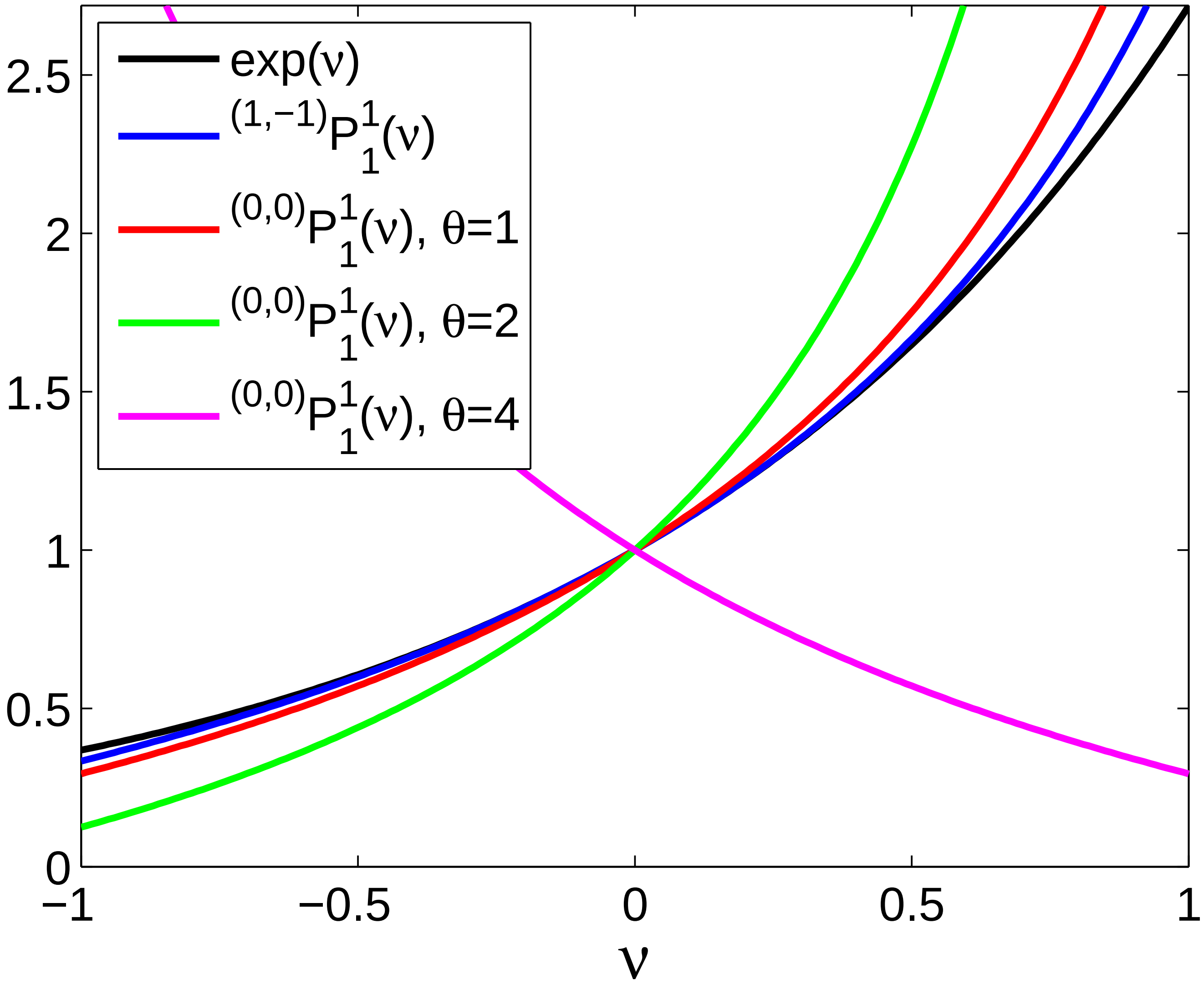
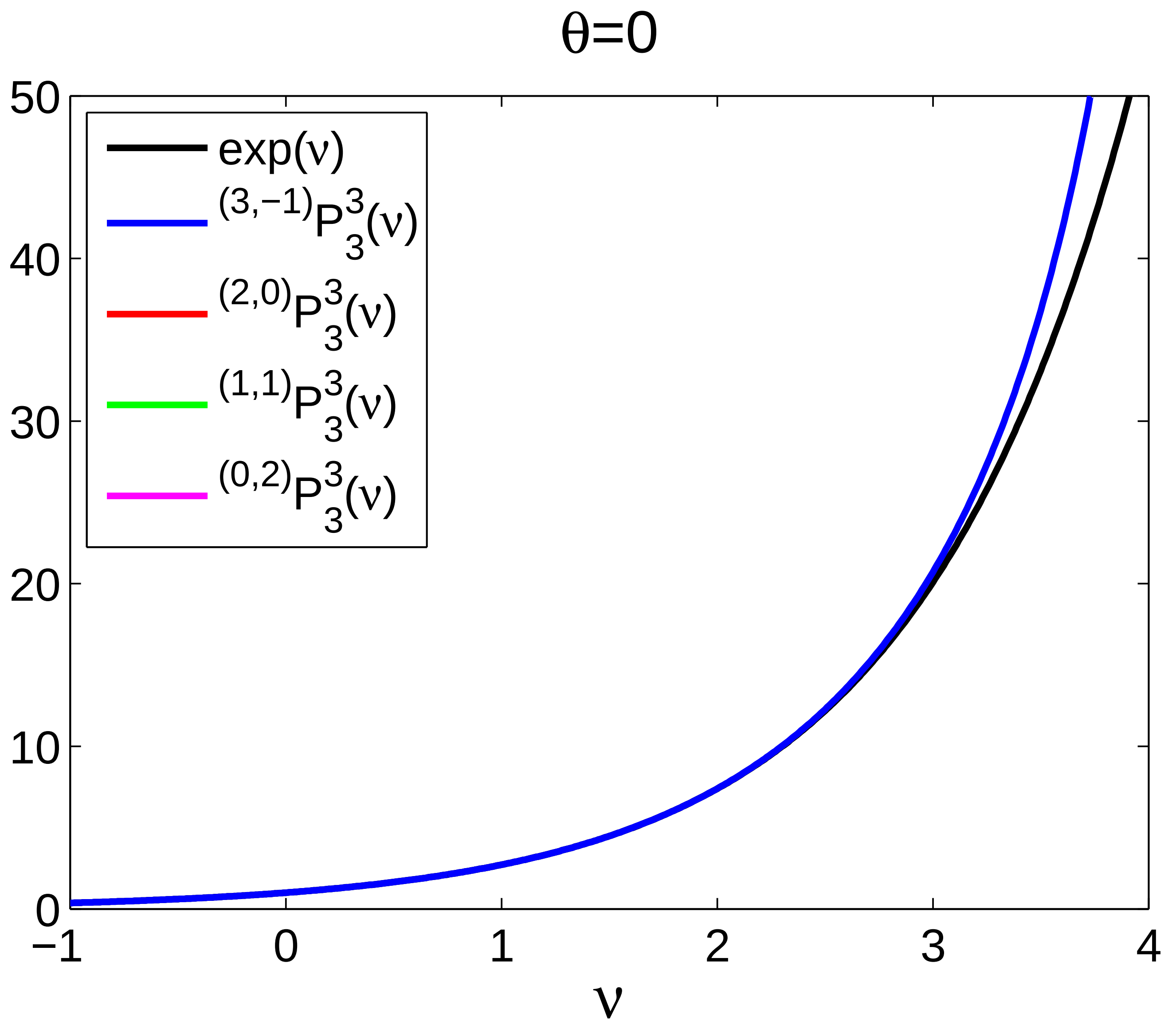
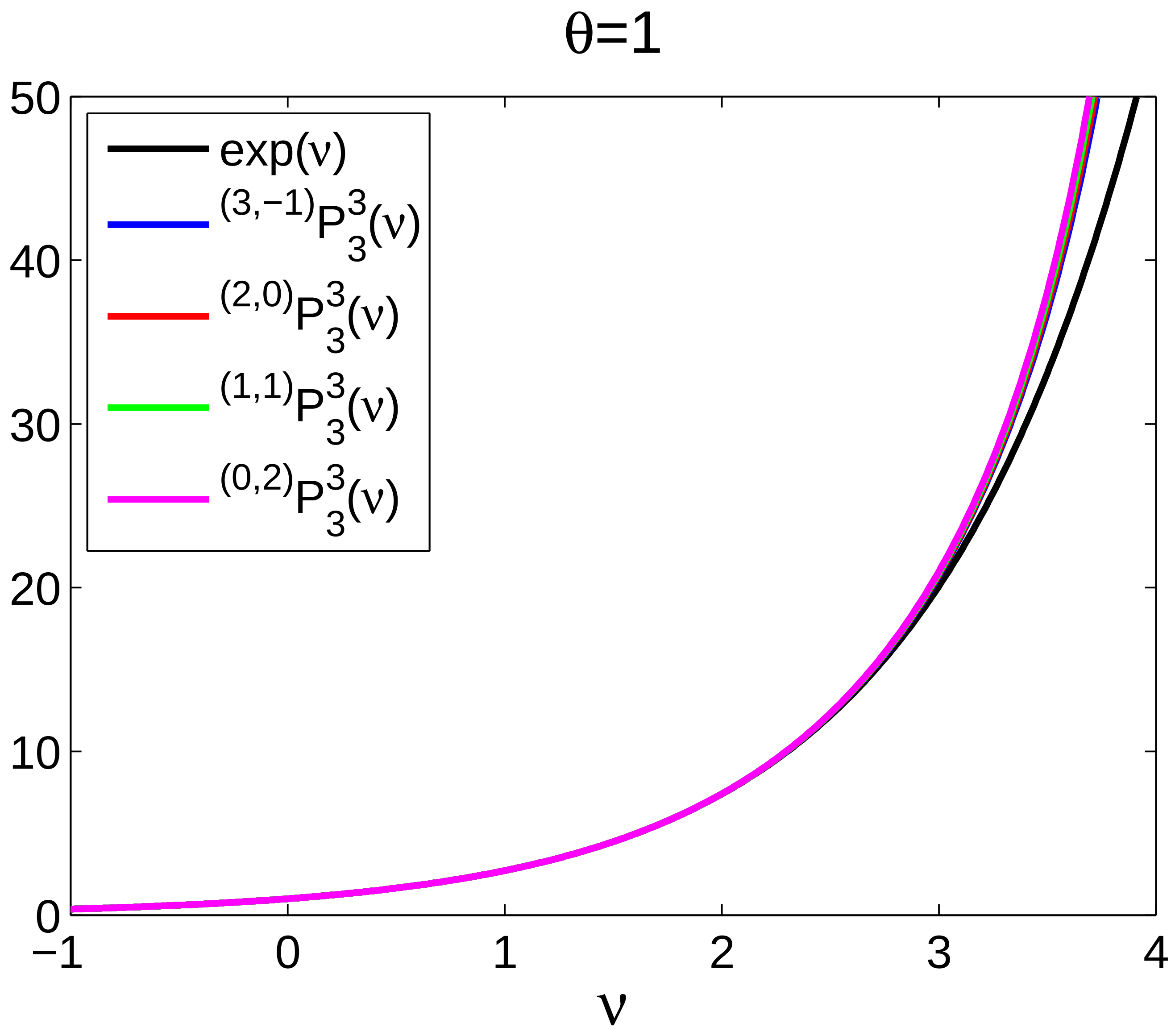
EF Padé approximants : $m = 1$



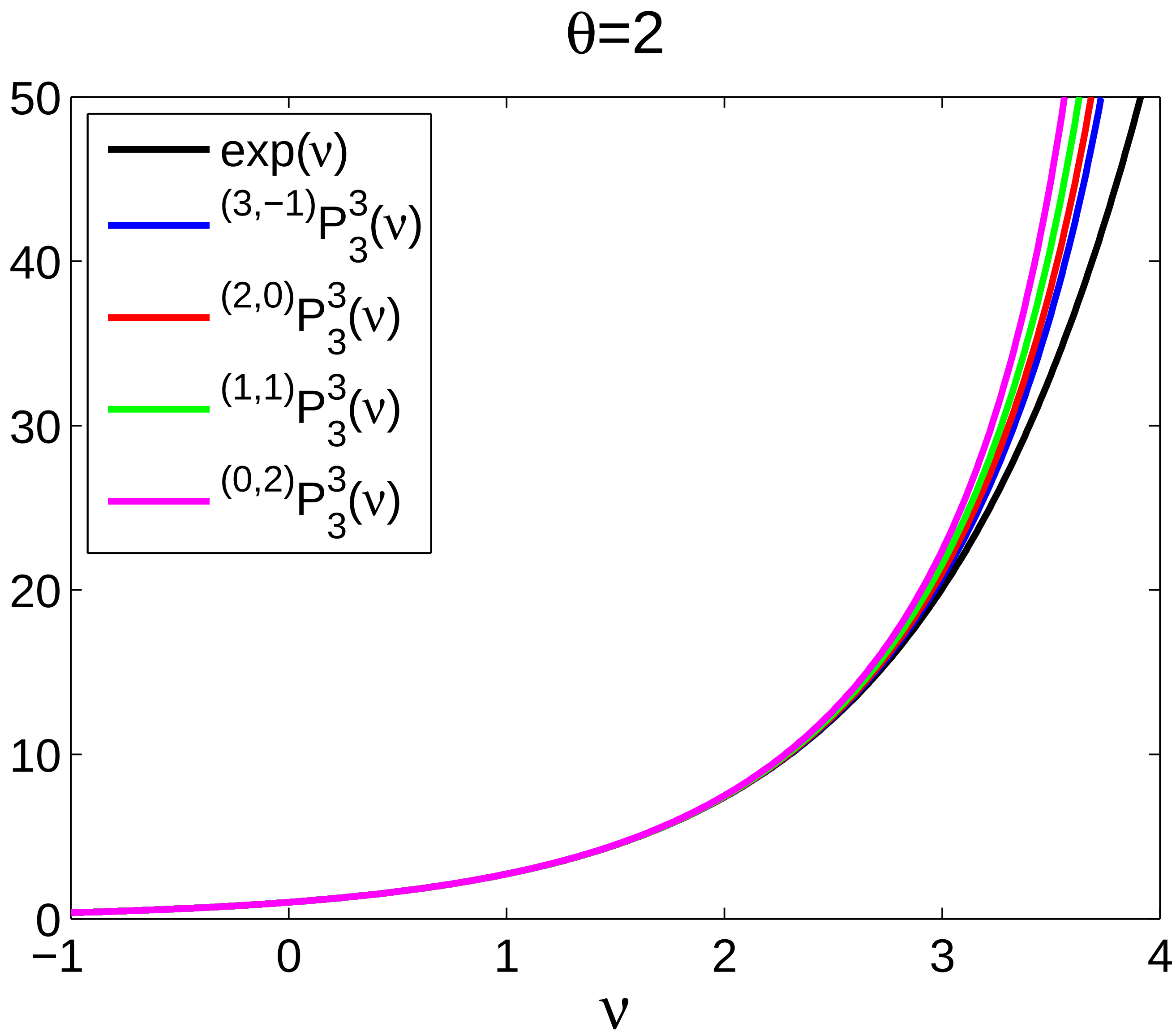
EF Padé approximants : $m = 3$



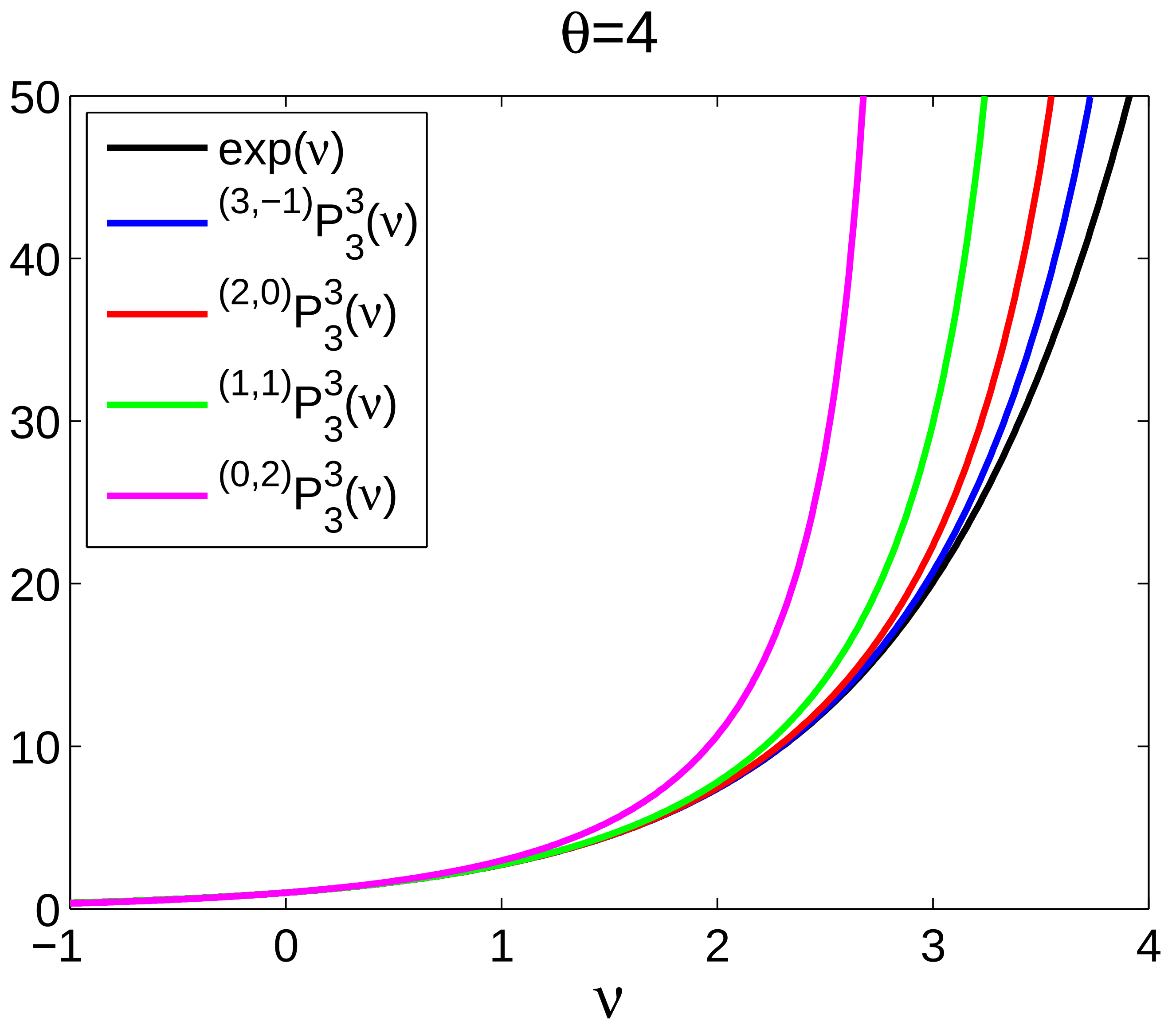
EF Padé approximants : $m = 3$



EF Padé approximants : $m = 3$



EF Padé approximants : $m = 3$



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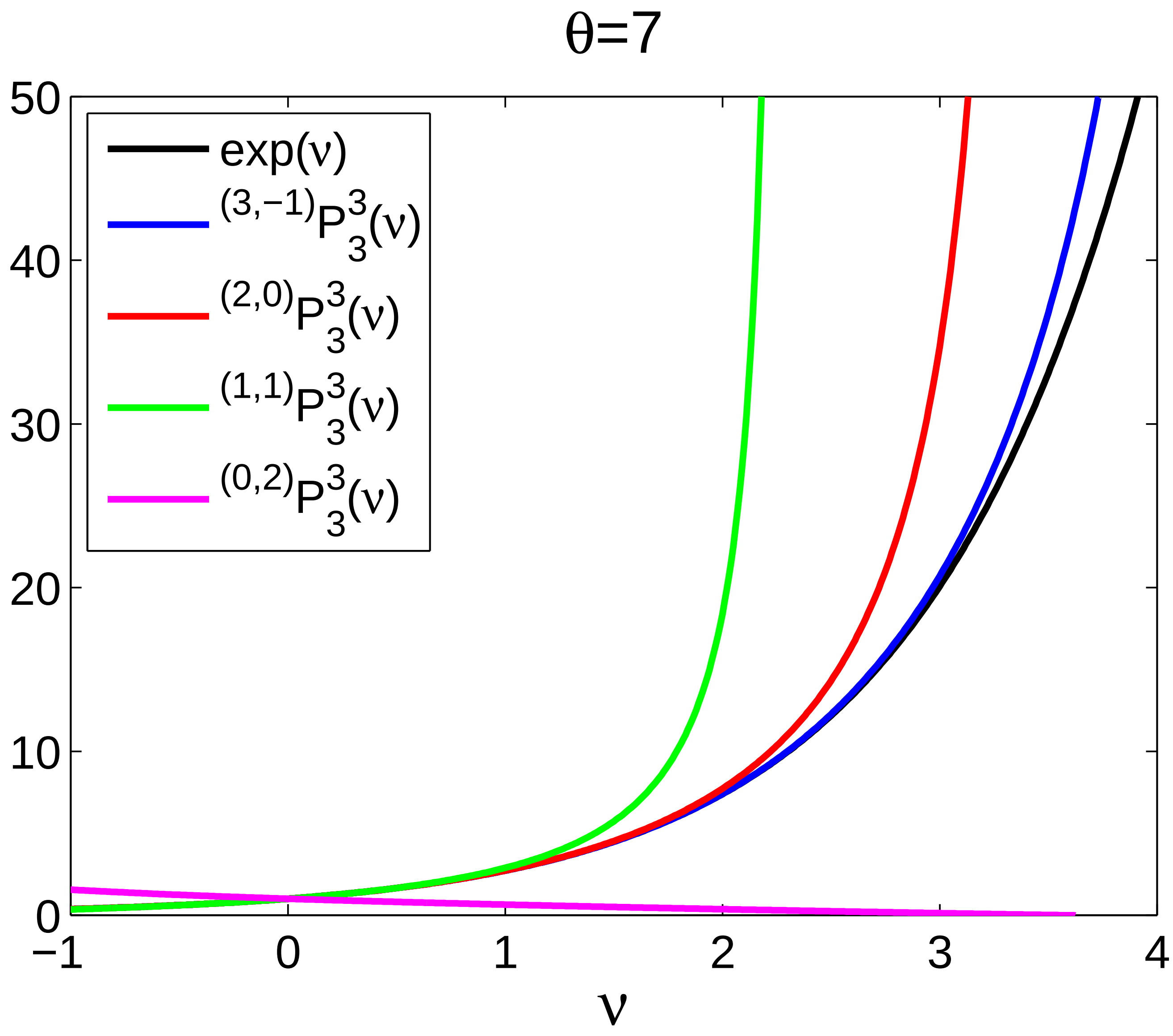
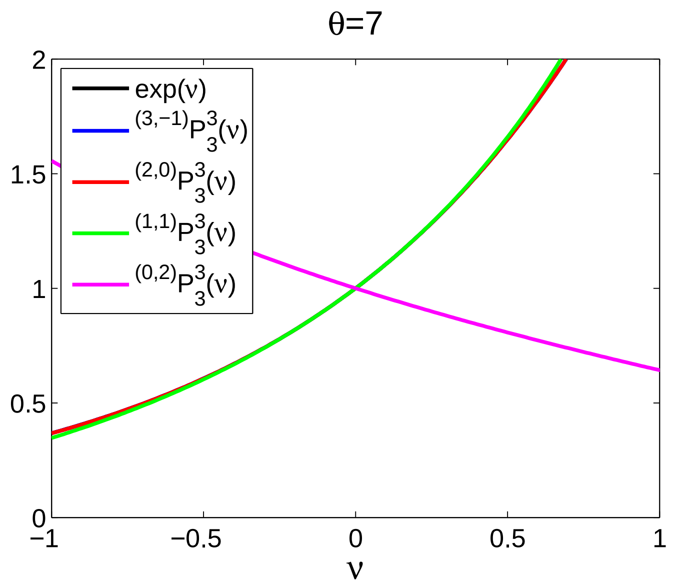
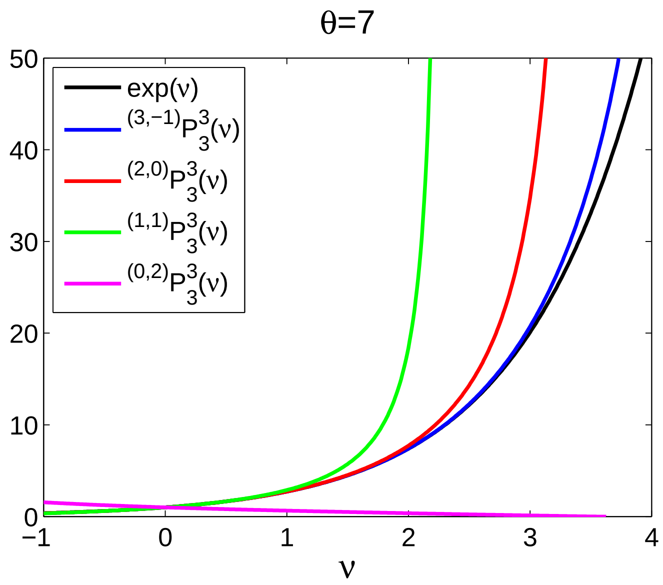


EF Padé approximants : $m = 3$





The Stiefel-Bettis problem

$$z'' + z = 0.001\exp(ix) \quad z(0) = 1 \quad z'(0) = 0.995i \qquad 0 \le x \le 40\pi$$

equivalent real form

$$\begin{cases} u'' + u = 0.001\cos(x), \quad u(0) = 1, \quad u'(0) = 0, \\ v'' + v = 0.001\sin(x), \quad v(0) = 0, \quad v'(0) = 0.9995\,. \end{cases}$$

$$z(x) = u(x) + i\,v(x)$$

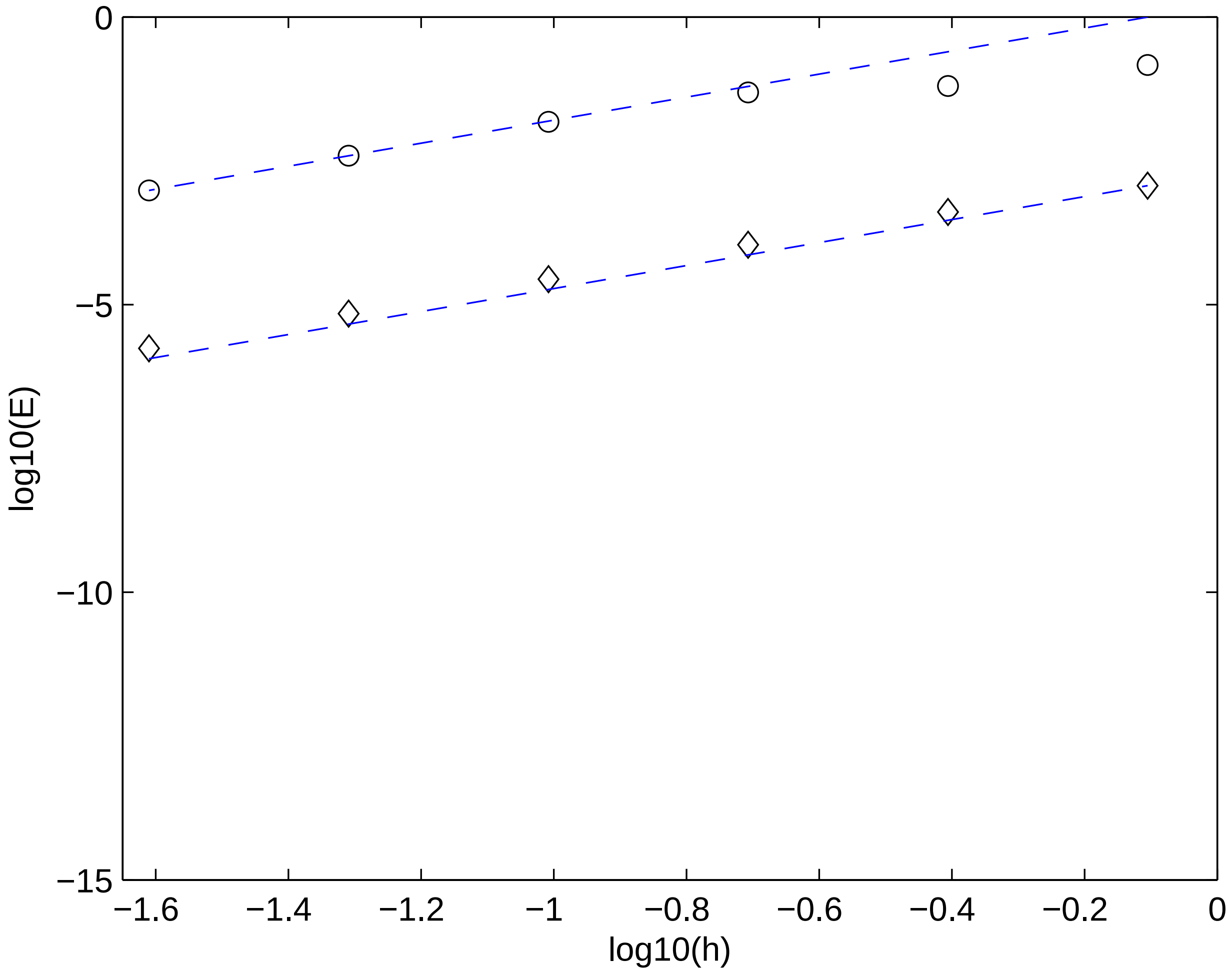
$$\begin{aligned} u(x) &= \cos(x) + 0.0005x\sin(x) \\ v(x) &= \sin(x) - 0.0005x\cos(x) \end{aligned}$$

$$\gamma(x) = \sqrt{u^2(x) + v^2(x)} = \sqrt{1 + (0.0005x)^2}$$

$$E := \max_{x_j \in [0, 40\,\pi]} \|y(x_j) - y_j\| \qquad h = 2^{-i}\,\pi \qquad \omega = 1$$

The Stiefel-Bettis problem

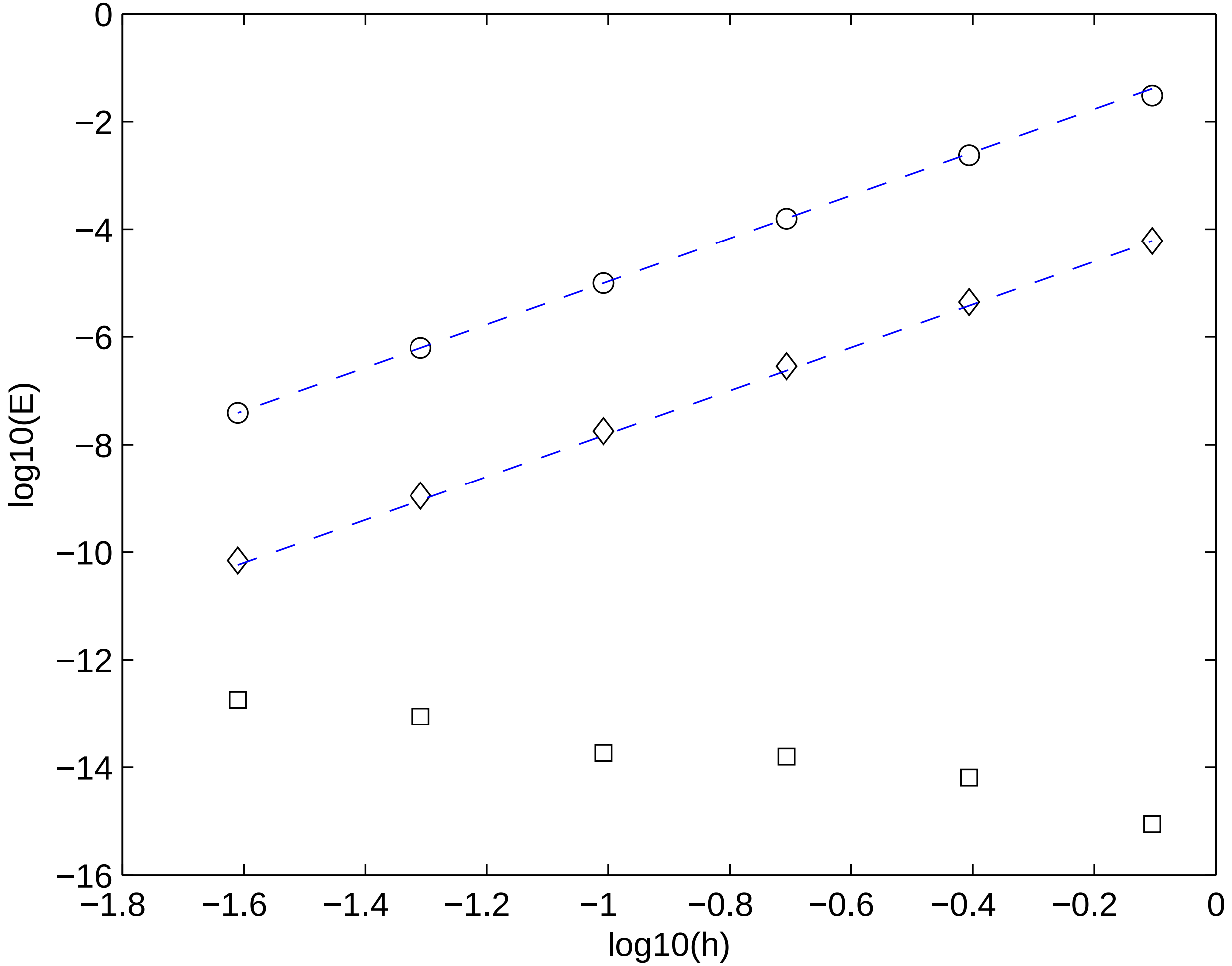
$$m = 1$$



$$\circ : P = -1 \qquad \diamond : P = 0$$

The Stiefel-Bettis problem

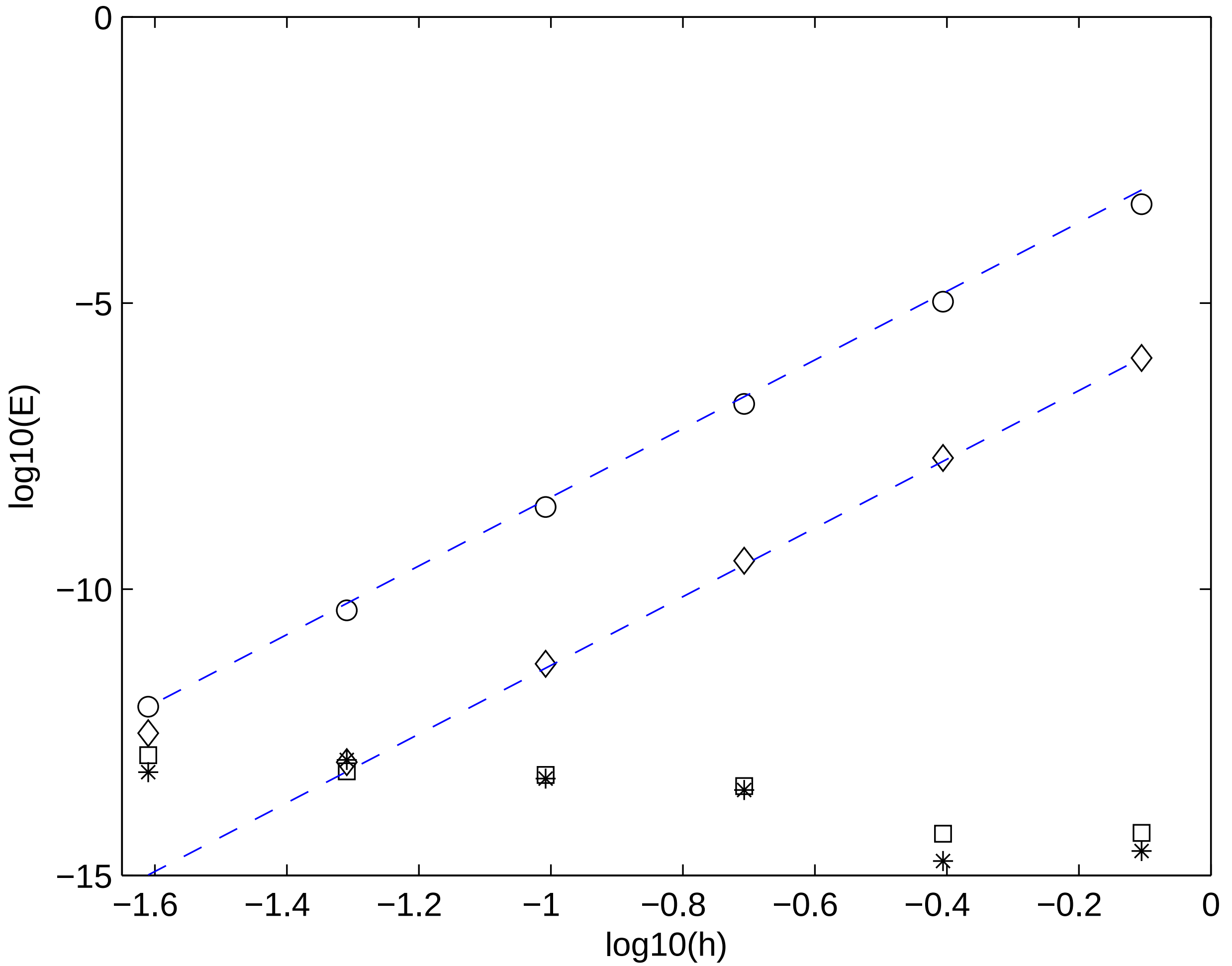
$$m = 2$$



$\circ : P = -1 \qquad \diamond : P = 0 \qquad \square : P = 1$

The Stiefel-Bettis problem

$$m = 3$$



$\circ : P = -1 \qquad \diamond : P = 0 \qquad \square : P = 1 \qquad \star : P = 2$

The Duffing problem

$$y''(x) = -y(x) - y(x)^3 + \cos(\Omega x) \qquad\qquad 0 \le x \le 50\pi$$

$$B = 0.002 \qquad\qquad \Omega = 1.01$$

$$y(x) = \sum_{i=0}^{4} K_{2i+1} \cos[(2i+1)\Omega x] \ ,$$

$$(K_1, K_3, K_5, K_7, K_9) =$$

$$\big(0.200179477536661502, 2.46946143255559\,10^{-4},$$
$$3.04014985196924\,37\,10^{-7}, 3.74349070160924\,7\,10^{-10},$$
$$4.60968294962269\,7\,10^{-13}\big)$$

The Duffing problem

$$
\begin{aligned}
y^{(3)}(x) &= -(1+3\,y^2(x))\,y'(x) - B\,\Omega\sin(\Omega x) \\
y^{(4)}(x) &= -(1+3\,y^2(x))\,y''(x) - 6y(x)y'(x)^2 - B\,\Omega^2\cos(\Omega x) \\
y^{(5)}(x) &= -(1+3\,y^2(x))\,y'''(x) - 6y'(x)^3 - 18y(x)y'(x)y''(x) \\
&\quad +B\,\Omega^3\sin(\Omega x) \\
y^{(6)}(x) &= -(1+3\,y^2(x))\,y^{(4)}(x) - 24y(x)y'(x)y^{(3)}(x) - 18y(x)y''(x)^2 \\
&\quad -36y'(x)^2y''(x) + B\,\cos(\Omega\,x)\,\Omega^4 \\
y^{(7)}(x) &= -(1+3\,y^2(x))\,y^{(5)}(x) - 30\,y(x)\,y^{(4)}(x) \\
&\quad -60\,(y'(x)+y(x)\,y''(x))\,y'''(x) - B\,\sin(\Omega\,x)\,\Omega^5 - 90\,y'(x)\,y''(x)^2
\end{aligned}
$$

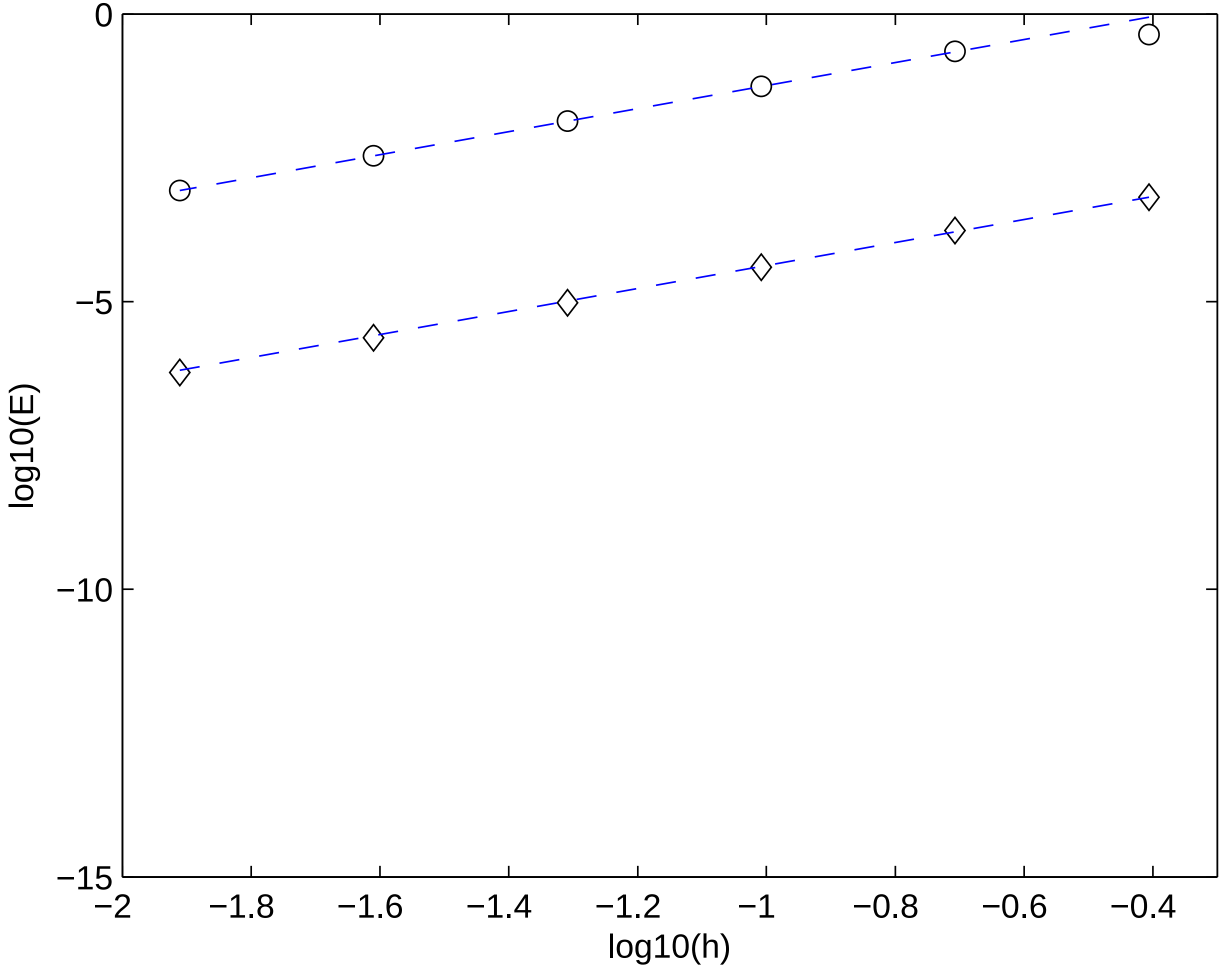
$$
z'' = \begin{pmatrix} y \\ y' \end{pmatrix}'' = \begin{pmatrix} f(x,y) \\ g(x,y,y') \end{pmatrix} \qquad g(x,y,y') = \frac{\mathrm{d}}{\mathrm{d}x} f(x,y(x))
$$

$$
E := \max_{x_j\in[0,50\,\pi]} \|z(x_j)-z_j\|
$$

$$
h = \pi/2^i,\ i = 2,\ 3,\ \ldots,\ 7 \qquad \omega = \Omega
$$

The Duffing problem

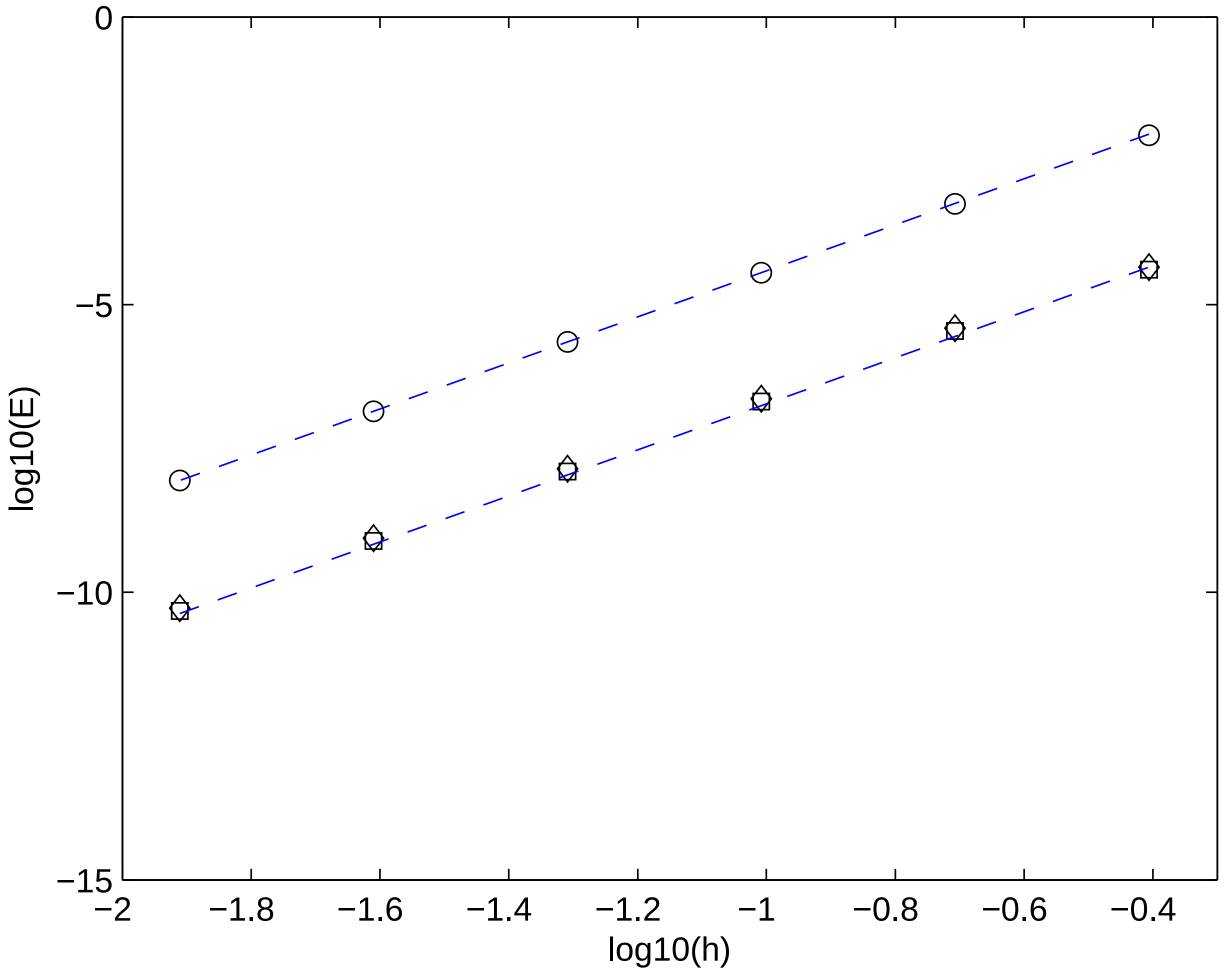
$$m = 1$$



$$\circ : P = -1 \qquad \diamond : P = 0$$

The Duffing problem

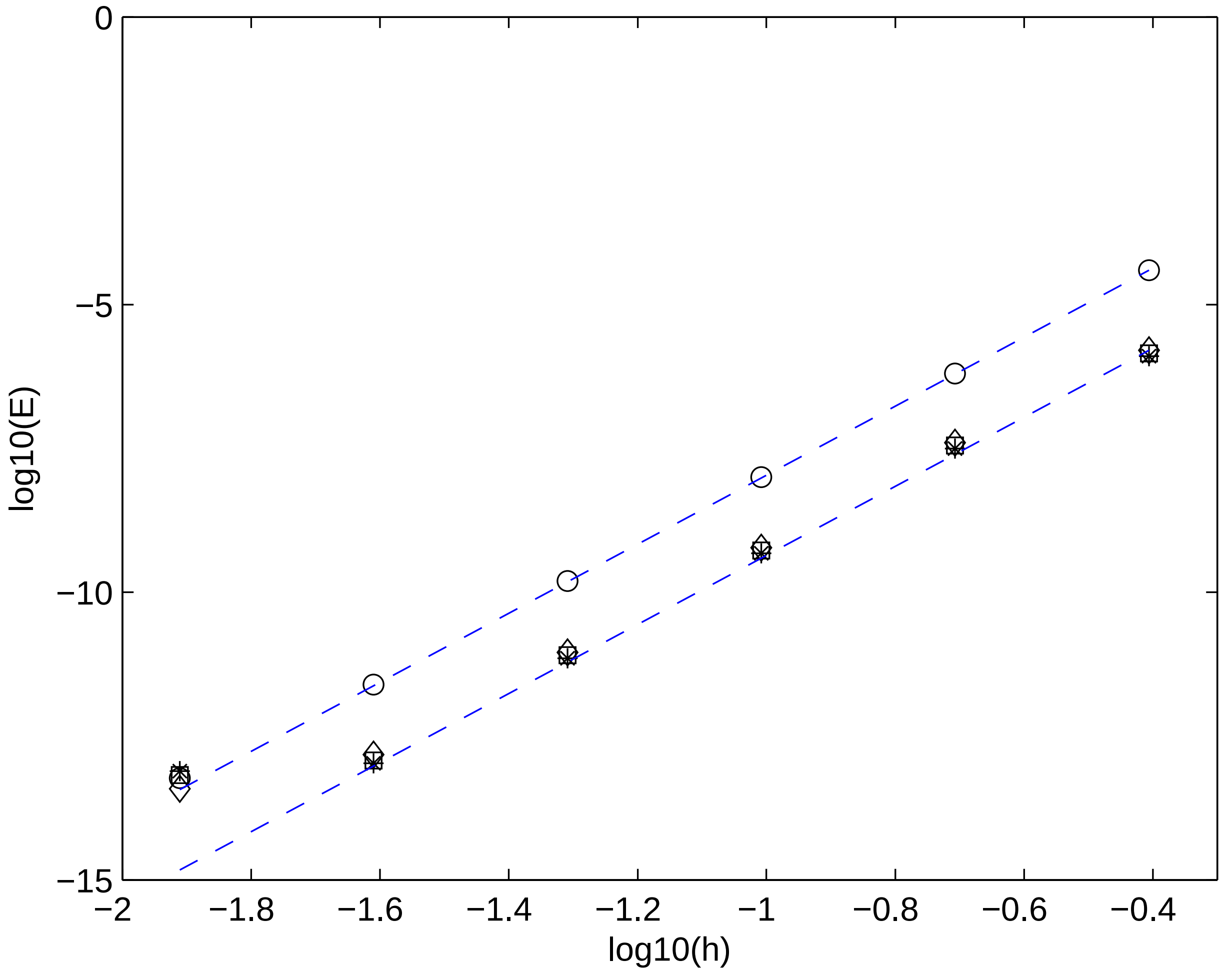
$$m = 2$$



$\circ : P = -1 \qquad \diamond : P = 0 \qquad \Box : P = 1$

The Duffing problem

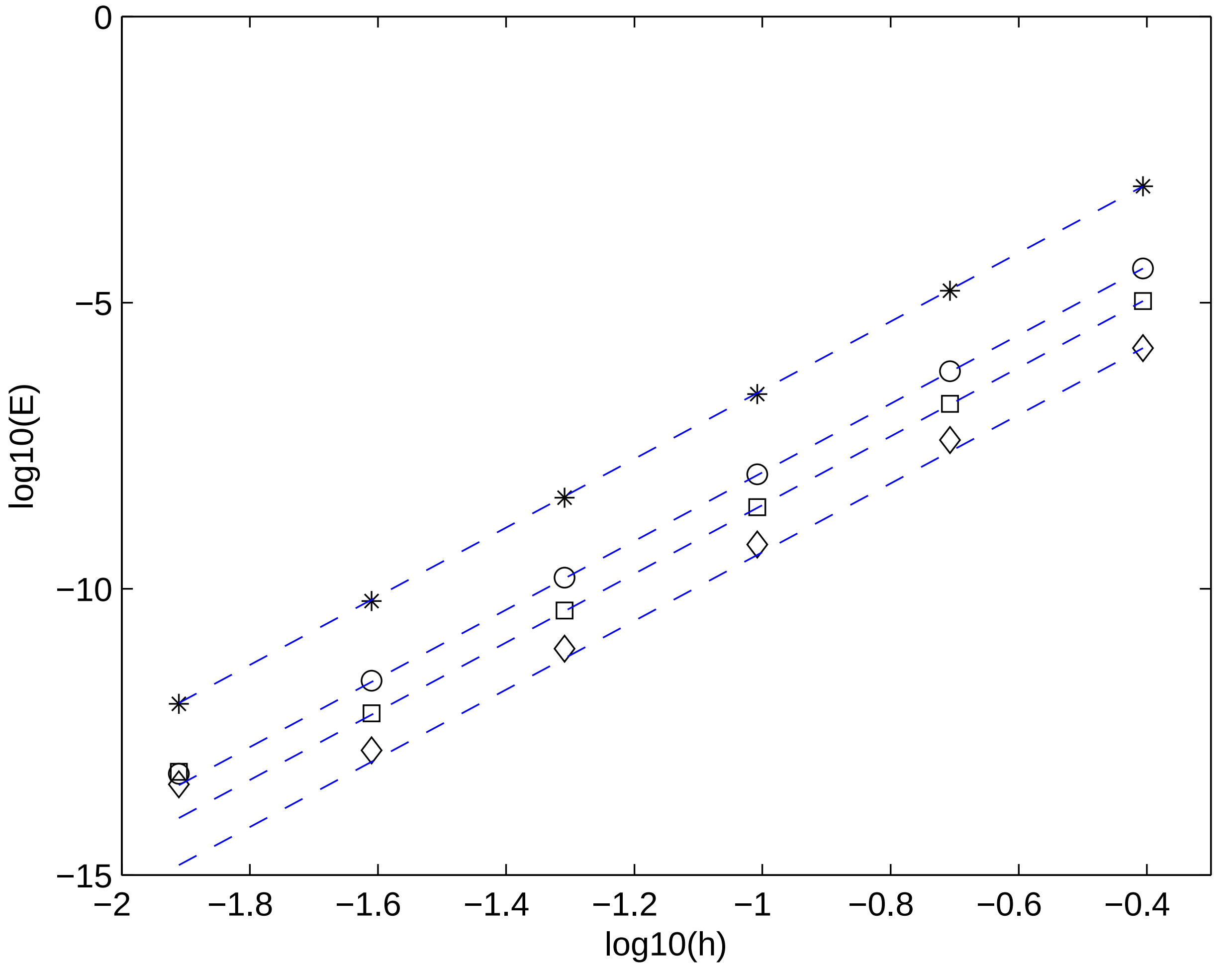
$m = 3$



$\circ : P = -1 \qquad \diamond : P = 0 \qquad \square : P = 1 \qquad \star : P = 2$

The Duffing problem

$$m = 3,\ (K, P) = (2, 0)$$



$\omega = 0$ (circles) $\omega = 1.01$ (diamonds) $\omega = 0.7$ (squares)

$\omega = 2$ (asterisks)

Conclusion

Two-step P-stable exponentially fitted Obrechkoff methods :

- for any given m, P-stable EF (K, P)-methods of order $2\,m$ exist
- the construction is based on EF Padé approximants to the exponential function
- the coefficients depend on a parameter θ
- the coefficients are continous functions of θ iff P is odd
- numerical examples are given as an illustration

References :

M. Van Daele and G. Vanden Berghe, P-stable Obrechkoff methods of arbitrary order for second-order differential equations, Numerical Algorithms **44**, 2007, 115-131

M. Van Daele and G. Vanden Berghe, P-stable exponentially fitted Obrechkoff methods of arbitrary order for second-order differential equations, accepted for publication in Numerical Algorithms