Multiparameter symplectic, symmetric exponentially-fitted modified Runge-Kutta methods of Gauss type

M. Van Daele, D. Hollevoet, G. Vanden Berghe

Department of Applied Mathematics and Computer Science
Ghent University

ICNAAM, Rhodes, September 2010

Outline

Exponential fitting

Aim : build methods which perform very good when the solution has a known exponential of trigonometric behaviour.

Different ways to develop EF methods

- starting from interpolation function

$$p_{n-2}^{(\omega)}(x) = a \cos\omega x + b \sin\omega x + \sum_{i=0}^{n-2} c_i x^i$$

with

$$\lim_{\omega\to 0} p_{n-2}^{(\omega)}(x) = p_n(x) = \text{a polynomial of degree} \le n$$

- starting from linear functional and imposing that for the set of functions $\{\cos\omega x, \sin\omega x, 1, t, t^2, \dots, t^{n-2}\}$ the method produces exact results.

ω which is either real (trigonometric case) or purely imaginary (exponential case), is determined from the expression for the local error.

Different ways to develop EF methods

- starting from interpolation function

$$p_{n-2}^{(\omega)}(x) = a \cos \omega x + b \sin \omega x + \sum_{i=0}^{n-2} c_i x^i$$

with

$$\lim_{\omega \to 0} p_{n-2}^{(\omega)}(x) = p_n(x) = \text{a polynomial of degree} \leq n$$

- starting from linear functional and imposing that for the set of functions $\{\cos \omega x, \sin \omega x, 1, t, t^2, \ldots, t^{n-2}\}$ the method produces exact results.

ω which is either real (trigonometric case) or purely imaginary (exponential case), is determined from the expression for the local error.

Example : Numerov method

$$y'' = f(y) \qquad y(a) = y_a \qquad y(b) = y_b$$

classical Numerov method :

$$y_{n+1} - 2y_n + y_{n-1} = \frac{1}{12} h^2 \left(f(y_{n+1}) + 10 f(y_n) + f(y_{n-1})\right)$$

$$n = 1, 2, \ldots, N \qquad h = \frac{b-a}{N+1}$$

Construction :

impose $\mathcal{L}[z(t); h] = 0$ for $z(t) \in \mathcal{S} = \{1, t, t^2, t^3, t^4\}$ where

$$\begin{aligned}\mathcal{L}[z(t); h] := {} & z(t+h) + a_0 z(t) + a_{-1} z(t-h) \\ & - h^2 \left(b_1 z''(t+h) + b_0 z''(t) + b_{-1} z''(t-h)\right)\end{aligned}$$

$$\mathcal{L}[z(t); h] = -\frac{1}{240} h^6 z^{(6)}(t) + \mathcal{O}(h^8) \qquad \Longrightarrow \text{order } 4$$

Example : Numerov method

$$y'' = f(y) \qquad y(a) = y_a \qquad y(b) = y_b$$

classical Numerov method :

$$y_{n+1} - 2\,y_n + y_{n-1} = \frac{1}{12}\,h^2\,\left(f(y_{n+1}) + 10\,f(y_n) + f(y_{n-1})\right)$$

$$n = 1, 2, \ldots, N \qquad h = \frac{b-a}{N+1}$$

Construction :

impose $\mathcal{L}[z(t);h] = 0$ for $z(t) \in \mathcal{S} = \{1, t, t^2, t^3, t^4\}$ where

$$\begin{aligned}\mathcal{L}[z(t);h] &:= z(t+h) + a_0\,z(t) + a_{-1}\,z(t-h)\\ &\quad -h^2\left(b_1\,z''(t+h) + b_0\,z''(t) + b_{-1}\,z''(t-h)\right)\end{aligned}$$

$$\mathcal{L}[z(t);h] = -\frac{1}{240}h^6\,z^{(6)}(t) + \mathcal{O}(h^8) \qquad \Longrightarrow \text{order } 4$$

Example : Numerov method

$$y'' = f(y) \qquad y(a) = y_a \qquad y(b) = y_b$$

classical Numerov method :

$$y_{n+1} - 2\,y_n + y_{n-1} = \frac{1}{12}\,h^2\,\left(f(y_{n+1}) + 10\,f(y_n) + f(y_{n-1})\right)$$

$$n = 1, 2, \ldots, N \qquad h = \frac{b-a}{N+1}$$

Construction :

impose $\mathcal{L}[z(t);h] = 0$ for $z(t) \in \mathcal{S} = \{1, t, t^2, t^3, t^4\}$ where

$$\begin{aligned}\mathcal{L}[z(t);h] &:= z(t+h) + a_0\,z(t) + a_{-1}\,z(t-h)\\ &\quad -h^2\left(b_1\,z''(t+h) + b_0\,z''(t) + b_{-1}\,z''(t-h)\right)\end{aligned}$$

$$\mathcal{L}[z(t);h] = -\frac{1}{240}h^6\,z^{(6)}(t) + \mathcal{O}(h^8) \qquad \Longrightarrow \text{order } 4$$

Example : Numerov method

$$y'' = f(y) \qquad y(a) = y_a \qquad y(b) = y_b$$

classical Numerov method :

$$y_{n+1} - 2\,y_n + y_{n-1} = \frac{1}{12}\,h^2\,\left(f(y_{n+1}) + 10\,f(y_n) + f(y_{n-1})\right)$$

$$n = 1,\,2,\,\ldots,\,N \qquad h = \frac{b-a}{N+1}$$

Construction :

impose $\mathcal{L}[z(t);h] = 0$ for $z(t) \in \mathcal{S} = \{1,\,t,\,t^2,\,t^3,\,t^4\}$ where

$$\begin{aligned}\mathcal{L}[z(t);h] := {} & z(t+h) + a_0\,z(t) + a_{-1}\,z(t-h) \\ & -h^2\left(b_1\,z''(t+h) + b_0\,z''(t) + b_{-1}\,z''(t-h)\right)\end{aligned}$$

$$\mathcal{L}[z(t);h] = -\frac{1}{240}h^6\,z^{(6)}(t) + \mathcal{O}(h^8) \qquad \Longrightarrow \text{order } 4$$

EF Numerov method

Construction : impose $\mathcal{L}[z(t);h] = 0$ for $z(t) \in \mathcal{S}$ with

$$\mathcal{S} = \{1,\, t,\, t^2,\, \sin(\omega\, t),\, \cos(\omega\, t)\}$$

$$\begin{aligned} \mathcal{L}[z(t);h] := {} & z(t+h) + a_0\, z(t) + a_{-1}\, z(t-h) \\ & -h^2 \left(b_1\, z''(t+h) + b_0\, z''(t) + b_{-1}\, z''(t-h) \right) \end{aligned}$$

$$y_{n+1} - 2\, y_n + y_{n-1} = h^2\, (\lambda f(y_{n-1}) + (1 - 2\,\lambda)\, f(y_n) + \lambda\, f(y_{n+1}))$$

$$\lambda = \frac{1}{4\,\sin^2\frac{\theta}{2}} - \frac{1}{\theta^2} = \frac{1}{12} + \frac{1}{240}\theta^2 + \frac{1}{6048}\theta^4 + \ldots \qquad \theta := \omega\, h$$

EF Numerov method

Construction : impose $\mathcal{L}[z(t);h]=0$ for $z(t)\in\mathcal{S}$ with

$$\mathcal{S}=\{1,\, t,\, t^2,\, \sin(\omega t),\, \cos(\omega t)\}$$

$$\begin{aligned}\mathcal{L}[z(t);h] := {} & z(t+h)+a_0\, z(t)+a_{-1}\, z(t-h)\\ & -h^2\left(b_1\, z''(t+h)+b_0\, z''(t)+b_{-1}\, z''(t-h)\right)\end{aligned}$$

$$y_{n+1}-2\,y_n+y_{n-1}=h^2\,\left(\lambda f(y_{n-1})+(1-2\,\lambda)\,f(y_n)+\lambda\, f(y_{n+1})\right)$$

$$\lambda \;=\; \frac{1}{4\sin^2\frac{\theta}{2}}-\frac{1}{\theta^2}=\frac{1}{12}+\frac{1}{240}\theta^2+\frac{1}{6048}\theta^4+\dots \qquad \theta:=\omega h$$

EF Numerov method

Construction : impose $\mathcal{L}[z(t);h] = 0$ for $z(t) \in \mathcal{S}$ with

$$\mathcal{S} = \{1,\, t,\, t^2,\, \sin(\omega\, t),\, \cos(\omega\, t)\}$$

$$\begin{aligned}\mathcal{L}[z(t);h] := {} & z(t+h) + a_0\, z(t) + a_{-1}\, z(t-h) \\ & -h^2\left(b_1\, z''(t+h) + b_0\, z''(t) + b_{-1}\, z''(t-h)\right)\end{aligned}$$

$$y_{n+1} - 2\,y_n + y_{n-1} = h^2\,\left(\lambda f(y_{n-1}) + (1-2\,\lambda)\, f(y_n) + \lambda\, f(y_{n+1})\right)$$

$$\lambda = \frac{1}{4\,\sin^2\frac{\theta}{2}} - \frac{1}{\theta^2} = \frac{1}{12} + \frac{1}{240}\theta^2 + \frac{1}{6048}\theta^4 + \dots \qquad \theta := \omega\, h$$

EF Numerov method

Construction : impose $\mathcal{L}[z(t);h] = 0$ for $z(t) \in \mathcal{S}$ with

$$\mathcal{S} = \{1,\, t,\, t^2,\, \sin(\omega\, t),\, \cos(\omega\, t)\}$$

$$\text{or} \;\; \mathcal{S} = \{1,\, t,\, t^2,\, \exp(\mu\, t),\, \exp(-\mu\, t)\} \qquad \mu := \mathrm{i}\,\omega$$

$$\begin{aligned}\mathcal{L}[z(t);h] := & \; z(t+h) + a_0\, z(t) + a_{-1}\, z(t-h) \\ & -h^2 \left(b_1\, z''(t+h) + b_0\, z''(t) + b_{-1}\, z''(t-h)\right)\end{aligned}$$

$$y_{n+1} - 2\,y_n + y_{n-1} = h^2\, \left(\lambda f(y_{n-1}) + (1 - 2\,\lambda)\, f(y_n) + \lambda\, f(y_{n+1})\right)$$

$$\lambda \;\; = \;\; \frac{1}{4\,\sin^2\frac{\theta}{2}} - \frac{1}{\theta^2} = \frac{1}{12} + \frac{1}{240}\theta^2 + \frac{1}{6048}\theta^4 + \ldots \qquad\qquad \theta := \omega\, h$$

EF Numerov method

Construction : impose $\mathcal{L}[z(t);h]=0$ for $z(t)\in\mathcal{S}$ with

$$\mathcal{S}=\{1,\, t,\, t^2,\, \sin(\omega\, t),\, \cos(\omega\, t)\}$$

$$\text{or}\;\; \mathcal{S}=\{1,\, t,\, t^2,\, \exp(\mu\, t),\, \exp(-\mu\, t)\} \qquad \mu := \mathrm{i}\,\omega$$

$$\begin{aligned}\mathcal{L}[z(t);h] &:= z(t+h)+a_0\, z(t)+a_{-1}\, z(t-h)\\ &\quad -h^2\left(b_1\, z''(t+h)+b_0\, z''(t)+b_{-1}\, z''(t-h)\right)\end{aligned}$$

$$y_{n+1}-2\,y_n+y_{n-1}=h^2\,\left(\lambda f(y_{n-1})+(1-2\,\lambda)\,f(y_n)+\lambda\, f(y_{n+1})\right)$$

$$\begin{aligned}\lambda &= \frac{1}{4\,\sin^2\frac{\theta}{2}}-\frac{1}{\theta^2}=\frac{1}{12}+\frac{1}{240}\theta^2+\frac{1}{6048}\theta^4+\ldots && \theta:=\omega\, h\\ &= -\frac{1}{4\,\sinh^2\frac{\nu}{2}}+\frac{1}{\nu^2}=\frac{1}{12}-\frac{1}{240}\nu^2+\frac{1}{6048}\nu^4+\ldots && \nu:=\mu\, h\end{aligned}$$

Exponential Fitting

L. Ixaru and G. Vanden Berghe
Exponential fitting
Kluwer Academic Publishers, Dordrecht, 2004

$$\xi(Z) = \begin{cases} \cos(|Z|^{1/2}) & \text{if } Z < 0 \\ \cosh(Z^{1/2}) & \text{if } Z \geq 0 \end{cases}$$

$$\eta(Z) = \begin{cases} \sin(|Z|^{1/2})/|Z|^{1/2} & \text{if } Z < 0 \\ 1 & \text{if } Z = 0 \\ \sinh(Z^{1/2})/Z^{1/2} & \text{if } Z > 0 \end{cases}$$

$$Z := (\mu h)^2 = -(\omega h)^2$$

Exponential Fitting

L. Ixaru and G. Vanden Berghe
Exponential fitting
Kluwer Academic Publishers, Dordrecht, 2004

$$\xi(Z) = \begin{cases} \cos(|Z|^{1/2}) & \text{if } Z < 0 \\ \cosh(Z^{1/2}) & \text{if } Z \geq 0 \end{cases}$$

$$\eta(Z) = \begin{cases} \sin(|Z|^{1/2})/|Z|^{1/2} & \text{if } Z < 0 \\ 1 & \text{if } Z = 0 \\ \sinh(Z^{1/2})/Z^{1/2} & \text{if } Z > 0 \end{cases}$$

$$Z := (\mu h)^2 = -(\omega h)^2$$

EF Numerov method

Construction : impose $\mathcal{L}[z(t);h]=0$ for $z(t)\in\mathcal{S}$ with

$$\mathcal{S}=\{1,\, t,\, t^2,\, \sin(\omega t),\, \cos(\omega t)\}$$

$$\text{or}\;\; \mathcal{S}=\{1,\, t,\, t^2,\, \exp(\mu t),\, \exp(-\mu t)\} \qquad \mu := \mathrm{i}\,\omega$$

$$\begin{aligned}\mathcal{L}[z(t);h] &:= z(t+h)+a_0\,z(t)+a_{-1}\,z(t-h)\\ &\quad -h^2\left(b_1\,z''(t+h)+b_0\,z''(t)+b_{-1}\,z''(t-h)\right)\end{aligned}$$

$$y_{n+1}-2\,y_n+y_{n-1}=h^2\,\left(\lambda f(y_{n-1})+(1-2\,\lambda)\,f(y_n)+\lambda\,f(y_{n+1})\right)$$

$$\begin{aligned}\lambda &= \frac{1}{4\,\sin^2\frac{\theta}{2}}-\frac{1}{\theta^2}=\frac{1}{12}+\frac{1}{240}\theta^2+\frac{1}{6048}\theta^4+\dots && \theta := \omega\,h\\ &= -\frac{1}{4\,\sinh^2\frac{\nu}{2}}+\frac{1}{\nu^2}=\frac{1}{12}-\frac{1}{240}\nu^2+\frac{1}{6048}\nu^4+\dots && \nu := \mu\,h\\ &= \frac{1}{Z}\left(1-\frac{1}{\eta^2(\frac{Z}{4})}\right)=\frac{1}{12}-\frac{1}{240}Z+\frac{1}{6048}Z^2+\dots && Z := \nu^2=-\theta^2\end{aligned}$$

EF Numerov method

$$\mathcal{S} = \{1, t, t^2, \sin(\omega t), \cos(\omega t)\}$$

$$y_{n+1} - 2y_n + y_{n-1} = h^2 \left(\lambda f(y_{n-1}) + (1 - 2\lambda) f(y_n) + \lambda f(y_{n+1})\right)$$

How to choose ω ?

$$\mathcal{L}[z(t); h] = -\frac{1}{240} h^6 \left(z^{(6)}(t) + \omega^2 z^{(4)}\right) + \mathcal{O}(h^8) \qquad \Longrightarrow \text{order } 4$$

A value for the parameter ω can be obtained from the expression for the lte :

$$y_n^{(6)} + \omega_n^2 y_n^{(4)} = 0 \, .$$

EF Numerov method

$$\mathcal{S} = \{1, t, t^2, \sin(\omega t), \cos(\omega t)\}$$

$$y_{n+1} - 2y_n + y_{n-1} = h^2\,(\lambda f(y_{n-1}) + (1-2\lambda) f(y_n) + \lambda f(y_{n+1}))$$

How to choose ω ?

$$\mathcal{L}[z(t); h] = -\frac{1}{240} h^6 \left(z^{(6)}(t) + \omega^2 z^{(4)}\right) + \mathcal{O}(h^8) \qquad \Longrightarrow \text{order } 4$$

A value for the parameter ω can be obtained from the expression for the lte :

$$y_n^{(6)} + \omega_n^2 y_n^{(4)} = 0\,.$$

Generalisations

To determine the coefficients of a method, we impose conditions on a linear functional. These conditions are related to the fitting space $\mathcal{S}$ which contains $\{1, t, t^2, \ldots, t^K\}$ and

- possibility 1 (Calvo et al.) : trigonometric polynomials $\{\exp(\pm\mu t), \exp(\pm 2\mu t), \ldots, \exp(\pm(P+1)\mu t)\}$
- possibility 2 (Ixaru, Vanden Berghe, V.D., ...) : exponential-fitting $\{\exp(\pm\mu t), t\ \exp(\pm\mu t), \ldots, t^P \exp(\pm\mu t)\}$

A method can be characterized by the couple (K, P)

Here, we consider a generalisation of both classes :

- possibility 3 : $\{\exp(\pm\mu_0 t), \exp(\pm\mu_1 t), \ldots, \exp(\pm\mu_P t)\}$

Generalisations

To determine the coefficients of a method, we impose conditions on a linear functional. These conditions are related to the fitting space $\mathcal{S}$ which contains $\{1, t, t^2, \ldots, t^K\}$ and

- possibility 1 (Calvo et al.) : trigonometric polynomials $\{\exp(\pm\mu t), \exp(\pm 2\mu t), \ldots, \exp(\pm(P+1)\mu t)\}$
- possibility 2 (Ixaru, Vanden Berghe, V.D., ...) : exponential-fitting $\{\exp(\pm\mu t), t\ \exp(\pm\mu t), \ldots, t^P \exp(\pm\mu t)\}$

A method can be characterized by the couple (K, P)

Here, we consider a generalisation of both classes :

- possibility 3 : $\{\exp(\pm\mu_0 t), \exp(\pm\mu_1 t), \ldots, \exp(\pm\mu_P t)\}$

Generalisations

To determine the coefficients of a method, we impose conditions on a linear functional. These conditions are related to the fitting space $\mathcal{S}$ which contains $\{1, t, t^2, \dots, t^K\}$ and

- possibility 1 (Calvo et al.) : trigonometric polynomials $\{\exp(\pm\mu t), \exp(\pm 2\mu t), \dots, \exp(\pm(P+1)\mu t)\}$
- possibility 2 (Ixaru, Vanden Berghe, V.D., ...) : exponential-fitting $\{\exp(\pm\mu t), t\ \exp(\pm\mu t), \dots, t^P \exp(\pm\mu t)\}$

A method can be characterized by the couple (K, P)

Here, we consider a generalisation of both classes :

- possibility 3 : $\{\exp(\pm\mu_0 t), \exp(\pm\mu_1 t), \dots, \exp(\pm\mu_P t)\}$

Generalisations

To determine the coefficients of a method, we impose conditions on a linear functional. These conditions are related to the fitting space $\mathcal{S}$ which contains $\{1, t, t^2, \ldots, t^K\}$ and

- possibility 1 (Calvo et al.) : trigonometric polynomials $\{\exp(\pm\mu t), \exp(\pm 2\mu t), \ldots, \exp(\pm(P+1)\mu t)\}$
- possibility 2 (Ixaru, Vanden Berghe, V.D., . . .) : exponential-fitting $\{\exp(\pm\mu t), t\, \exp(\pm\mu t), \ldots, t^P \exp(\pm\mu t)\}$

A method can be characterized by the couple (K, P)

Here, we consider a generalisation of both classes :

- possibility 3 : $\{\exp(\pm\mu_0 t), \exp(\pm\mu_1 t), \ldots, \exp(\pm\mu_P t)\}$

Generalisations

To determine the coefficients of a method, we impose conditions on a linear functional. These conditions are related to the fitting space $\mathcal{S}$ which contains $\{1, t, t^2, \ldots, t^K\}$ and

- possibility 1 (Calvo et al.) : trigonometric polynomials $\{\exp(\pm\mu t), \exp(\pm 2\mu t), \ldots, \exp(\pm(P+1)\mu t)\}$
- possibility 2 (Ixaru, Vanden Berghe, V.D., ...) : exponential-fitting $\{\exp(\pm\mu t), t\ \exp(\pm\mu t), \ldots, t^P \exp(\pm\mu t)\}$

A method can be characterized by the couple (K, P)

Here, we consider a generalisation of both classes :

- possibility 3 : $\{\exp(\pm\mu_0 t), \exp(\pm\mu_1 t), \ldots, \exp(\pm\mu_P t)\}$

Generalisations

To determine the coefficients of a method, we impose conditions on a linear functional. These conditions are related to the fitting space $\mathcal{S}$ which contains $\{1, t, t^2, \ldots, t^K\}$ and

- possibility 1 (Calvo et al.) : trigonometric polynomials $\{\exp(\pm\mu t), \exp(\pm 2\mu t), \ldots, \exp(\pm(P+1)\mu t)\}$
- possibility 2 (Ixaru, Vanden Berghe, V.D., ...) : exponential-fitting $\{\exp(\pm\mu t), t\,\exp(\pm\mu t), \ldots, t^P\exp(\pm\mu t)\}$

A method can be characterized by the couple (K, P)

Here, we consider a generalisation of both classes :

- possibility 3 : $\{\exp(\pm\mu_0 t), \exp(\pm\mu_1 t), \ldots, \exp(\pm\mu_P t)\}$

Generalisations

To determine the coefficients of a method, we impose conditions on a linear functional. These conditions are related to the fitting space $\mathcal{S}$ which contains $\{1, t, t^2, \ldots, t^K\}$ and

- possibility 1 (Calvo et al.) : trigonometric polynomials $\{\exp(\pm\mu t), \exp(\pm 2\mu t), \ldots, \exp(\pm(P+1)\mu t)\}$
- possibility 2 (Ixaru, Vanden Berghe, V.D., ...) : exponential-fitting $\{\exp(\pm\mu t), t\ \exp(\pm\mu t), \ldots, t^P \exp(\pm\mu t)\}$

A method can be characterized by the couple (K, P)

Here, we consider a generalisation of both classes :

- possibility 3 : $\{\exp(\pm\mu_0 t), \exp(\pm\mu_1 t), \ \ldots, \exp(\pm\mu_P t)\}$

Motivation

work by Hollevoet, V.D. and Vanden Berghe

- "On the leading error term of exponentially fitted Numerov methods", ICNAAM 2008
- "The optimal exponentially-fitted Numerov method for solving two-point boundary value methods", J. CAM 2009

EF-approach of Ixaru and Vanden Berghe :

$$\mathcal{L}[z(t);h] := z(t+h) + a_0\, z(t) + a_{-1}\, z(t-h)$$
$$-h^2\left(b_1\, z''(t+h) + b_0\, z''(t) + b_{-1}\, z''(t-h)\right)$$

$z(t) \in \mathcal{S}_{K,P}(\mu) =$
$$\{1, t, t^2, \ldots, t^K\} \cup \{\exp(\pm\mu\, t), t\ \exp(\pm\mu\, t), \ldots, t^P \exp(\pm\mu\, t)\}$$

Motivation

work by Hollevoet, V.D. and Vanden Berghe

- “On the leading error term of exponentially fitted Numerov methods”, ICNAAM 2008
- “The optimal exponentially-fitted Numerov method for solving two-point boundary value methods”, J. CAM 2009

EF-approach of Ixaru and Vanden Berghe :

$$\begin{aligned}\mathcal{L}[z(t);h] := {} & z(t+h) + a_0\,z(t) + a_{-1}\,z(t-h) \\ & -h^2\left(b_1\,z''(t+h) + b_0\,z''(t) + b_{-1}\,z''(t-h)\right)\end{aligned}$$

$z(t) \in \mathcal{S}_{K,P}(\mu) =$

$$\{1, t, t^2, \ldots, t^K\} \cup \{\exp(\pm\mu\,t), t\,\exp(\pm\mu\,t), \ldots, t^P\exp(\pm\mu\,t)\}$$

Motivation

work by Hollevoet, V.D. and Vanden Berghe

- "On the leading error term of exponentially fitted Numerov methods", ICNAAM 2008
- "The optimal exponentially-fitted Numerov method for solving two-point boundary value methods", J. CAM 2009

EF-approach of Ixaru and Vanden Berghe :

$$\begin{aligned}\mathcal{L}[z(t);h] := {} & z(t+h) + a_0\, z(t) + a_{-1}\, z(t-h) \\ & -h^2\left(b_1\, z''(t+h) + b_0\, z''(t) + b_{-1}\, z''(t-h)\right)\end{aligned}$$

$$\begin{aligned} z(t) \in \mathcal{S}_{K,P}(\mu) = {} & \\ & \{1, t, t^2, \ldots, t^K\} \cup \{\exp(\pm\mu\, t),\ t\ \exp(\pm\mu\, t),\ \ldots,\ t^P \exp(\pm\mu\, t)\}\end{aligned}$$

Motivation

μ is determined from the lte :

$$h^6 \phi_P(Z) D^{K+1} (D^2 - \mu^2)^{P+1} y(t_j) + \mathcal{O}(h^8) \qquad \phi_P(Z) = -\frac{1}{240} + \mathcal{O}(Z)$$

At $t = t_j$, $\mu^2 := \mu_j^2$ such that

$$E_{P,j} := D^{K+1} (D^2 - \mu_j^2)^{P+1} y(t_j) = 0$$

- $P = 0 : y^{(6)}(t_j) - \mu_j^2 y^{(4)}(t_j) = 0 \Longrightarrow \mu_j^2 \in \mathbb{R}$
- $P = 1 : y^{(6)}(t_j) - 2\mu^2 y^{(4)}(t_j) + \mu^4 y^{(2)}(t_j) = 0$ may only have complex roots μ_j^2, such that $y_j \in \mathbb{C}$.

  To solve this problem, we propose the new type of EF methods : EF multiparameter methods

Motivation

μ is determined from the lte :

$$h^6 \phi_P(Z) D^{K+1} (D^2 - \mu^2)^{P+1} y(t_j) + \mathcal{O}(h^8) \qquad \phi_P(Z) = -\frac{1}{240} + \mathcal{O}(Z)$$

At $t = t_j$, $\mu^2 := \mu_j^2$ such that

$$E_{P,j} := D^{K+1} (D^2 - \mu_j^2)^{P+1} y(t_j) = 0$$

- $P = 0 : y^{(6)}(t_j) - \mu_j^2 y^{(4)}(t_j) = 0 \Longrightarrow \mu_j^2 \in \mathbb{R}$
- $P = 1 : y^{(6)}(t_j) - 2\mu^2 y^{(4)}(t_j) + \mu^4 y^{(2)}(t_j) = 0$ may only have complex roots μ_j^2, such that $y_j \in \mathbb{C}$.

To solve this problem, we propose the new type of EF methods : EF multiparameter methods

Motivation

μ is determined from the lte :

$$h^6\,\phi_P(Z)\,D^{K+1}\,(D^2-\mu^2)^{P+1}\,y(t_j)+\mathcal{O}(h^8) \qquad \phi_P(Z) = -\frac{1}{240}+\mathcal{O}(Z)$$

At $t = t_j$, $\mu^2 := \mu_j^2$ such that

$$E_{P,j} := D^{K+1}\,(D^2-\mu_j^2)^{P+1}\,y(t_j) = 0$$

- $P = 0 : y^{(6)}(t_j) - \mu_j^2\,y^{(4)}(t_j) = 0 \Longrightarrow \mu_j^2 \in \mathbb{R}$
- $P = 1 : y^{(6)}(t_j) - 2\,\mu^2\,y^{(4)}(t_j) + \mu^4\,y^{(2)}(t_j) = 0$ may only have complex roots μ_j^2, such that $y_j \in \mathbb{C}$.

  To solve this problem, we propose the new type of EF methods : EF multiparameter methods

Motivation

μ is determined from the lte :

$$h^6\,\phi_P(Z)\,D^{K+1}\,(D^2-\mu^2)^{P+1}\,y(t_j)+\mathcal{O}(h^8) \qquad \phi_P(Z)=-\frac{1}{240}+\mathcal{O}(Z)$$

At $t = t_j$, $\mu^2 := \mu_j^2$ such that

$$E_{P,j} := D^{K+1}\,(D^2-\mu_j^2)^{P+1}\,y(t_j)=0$$

- $P = 0 : y^{(6)}(t_j) - \mu_j^2\,y^{(4)}(t_j) = 0 \Longrightarrow \mu_j^2 \in \mathbb{R}$
- $P = 1 : y^{(6)}(t_j) - 2\,\mu^2\,y^{(4)}(t_j) + \mu^4\,y^{(2)}(t_j) = 0$ may only have complex roots μ_j^2, such that $y_j \in \mathbb{C}$.

To solve this problem, we propose the new type of EF methods : EF multiparameter methods

Motivation

μ is determined from the lte :

$$h^6\,\phi_P(Z)\,D^{K+1}\,(D^2-\mu^2)^{P+1}\,y(t_j)+\mathcal{O}(h^8) \qquad \phi_P(Z)=-\frac{1}{240}+\mathcal{O}(Z)$$

At $t=t_j$, $\mu^2:=\mu_j^2$ such that

$$E_{P,j}:=D^{K+1}\,(D^2-\mu_j^2)^{P+1}\,y(t_j)=0$$

- $P=0: y^{(6)}(t_j)-\mu_j^2\,y^{(4)}(t_j)=0 \Longrightarrow \mu_j^2\in\mathbb{R}$
- $P=1: y^{(6)}(t_j)-2\,\mu^2\,y^{(4)}(t_j)+\mu^4\,y^{(2)}(t_j)=0$ may only have complex roots μ_j^2, such that $y_j\in\mathbb{C}$.

To solve this problem, we propose the new type of EF methods : EF multiparameter methods

Motivation

μ is determined from the lte :

$$h^6\,\phi_P(Z)\,D^{K+1}\,(D^2-\mu^2)^{P+1}\,y(t_j)+\mathcal{O}(h^8) \qquad \phi_P(Z) = -\frac{1}{240}+\mathcal{O}(Z)$$

At $t = t_j$, $\mu^2 := \mu_j^2$ such that

$$E_{P,j} := D^{K+1}\,(D^2-\mu_j^2)^{P+1}\,y(t_j) = 0$$

- $P = 0 : y^{(6)}(t_j) - \mu_j^2\,y^{(4)}(t_j) = 0 \Longrightarrow \mu_j^2 \in \mathbb{R}$
- $P = 1 : y^{(6)}(t_j) - 2\,\mu^2\,y^{(4)}(t_j) + \mu^4\,y^{(2)}(t_j) = 0$ may only have complex roots μ_j^2, such that $y_j \in \mathbb{C}$.

To solve this problem, we propose the new type of EF methods : EF multiparameter methods

Motivation

μ is determined from the lte :

$$h^6\,\phi_P(Z)\,D^{K+1}\,(D^2-\mu^2)^{P+1}\,y(t_j)+\mathcal{O}(h^8) \qquad \phi_P(Z)=-\frac{1}{240}+\mathcal{O}(Z)$$

At $t = t_j$, $\mu^2 := \mu_j^2$ such that

$$E_{P,j} := D^{K+1}\,(D^2-\mu_j^2)^{P+1}\,y(t_j) = 0$$

- $P = 0 : y^{(6)}(t_j) - \mu_j^2\,y^{(4)}(t_j) = 0 \Longrightarrow \mu_j^2 \in \mathbb{R}$
- $P = 1 : y^{(6)}(t_j) - 2\,\mu^2\,y^{(4)}(t_j) + \mu^4\,y^{(2)}(t_j) = 0$ may only have complex roots μ_j^2, such that $y_j \in \mathbb{C}$.

  To solve this problem, we propose the new type of EF methods : EF multiparameter methods

Aim

The construction of symmetric, symplectic EF multiparameter Runge-Kutta methods Gauss-type methods

Previous work on

- EF symplectic RK-like methods by Van de Vyver (2006)
- EF symmetric, symplectic RK methods by Calvo et al. (2008-2010)
- EF symmetric, symplectic RK-like methods by Vanden Berghe - V.D. (2010)

Aim

The construction of symmetric, symplectic EF multiparameter Runge-Kutta methods Gauss-type methods

Previous work on

- EF symplectic RK-like methods by Van de Vyver (2006)
- EF symmetric, symplectic RK methods by Calvo et al. (2008-2010)
- EF symmetric, symplectic RK-like methods by Vanden Berghe - V.D. (2010)

General approach

Associate linear functionals to the internal stages

$$\mathcal{L}_i[y(x); h; \mathbf{a}] = y(x + c_i h) - y(x) - h \sum_{j=1}^{s} a_{ij} y'(x + c_j h)$$

where $i = 1, \ldots, s$ and the final stage

$$\mathcal{L}[y(x); h; \mathbf{b}] = y(x + h) - y(x) - h \sum_{j=1}^{s} b_i y'(x + c_i h)$$

and impose $\begin{cases} \mathcal{L}_i[y(x); h; \mathbf{a}] = 0 & \text{for } y(x) \in S_{int} \\ \mathcal{L}[y(x); h; \mathbf{b}] = 0 & \text{for } y(x) \in S_{fin} \end{cases}$

also taking into account the symplecticity and symmetry conditions.

General approach

Associate linear functionals to the internal stages

$$\mathcal{L}_i[y(x); h; \mathrm{a}] = y(x + c_i h) - y(x) - h \sum_{j=1}^{s} a_{ij}\, y'(x + c_j h)$$

where $i = 1, \ldots, s$ and the final stage

$$\mathcal{L}[y(x); h; \mathrm{b}] = y(x + h) - y(x) - h \sum_{j=1}^{s} b_i\, y'(x + c_i h)$$

and impose $\begin{cases} \mathcal{L}_i[y(x); h; \mathrm{a}] = 0 & \text{for } \; y(x) \in S_{int} \\ \mathcal{L}[y(x); h; \mathrm{b}] = 0 & \text{for } \; y(x) \in S_{fin} \end{cases}$

also taking into account the symplecticity and symmetry conditions.

General approach

Associate linear functionals to the internal stages

$$\mathcal{L}_i[y(x);h;\mathrm{a}] = y(x+c_i h) - y(x) - h\sum_{j=1}^{s} a_{ij}\, y'(x+c_j h)$$

where $i = 1, \ldots, s$ and the final stage

$$\mathcal{L}[y(x);h;\mathrm{b}] = y(x+h) - y(x) - h\sum_{j=1}^{s} b_i\, y'(x+c_i h)$$

and impose $\begin{cases} \mathcal{L}_i[y(x);h;\mathrm{a}] = 0 & \text{for } y(x) \in S_{int} \\ \mathcal{L}[y(x);h;\mathrm{b}] = 0 & \text{for } y(x) \in S_{fin} \end{cases}$
also taking into account the symplecticity and symmetry conditions.

Van de Vyver's approach

In order to construct a symplectic EF version of the Gauss $s = 2$ method with fixed knots $c_1 = \frac{3-\sqrt{3}}{6}$ and $c_2 = \frac{3+\sqrt{3}}{6}$ and

$S_{int} = \{\exp(\mu x), \exp(-\mu x)\}$ $\quad$ $S_{fin} = \{1, x, \exp(\mu x), \exp(-\mu x)\}$

Van de Vyver considers modified RK-methods

$$\mathcal{L}_i[y(x); h; \mathrm{a}] = y(x + c_i h) - \gamma_i y(x) - h \sum_{j=1}^{s} a_{ij} y'(x + c_j h)$$

where $i = 1, \ldots, s$ and the final stage

$$\mathcal{L}[y(x); h; \mathrm{b}] = y(x + h) - y(x) - h \sum_{j=1}^{s} b_i y'(x + c_i h)$$

The concept of modified RK methods is also used by Vanden Berghe and V.D.

Van de Vyver's approach

In order to construct a symplectic EF version of the Gauss $s = 2$ method with fixed knots $c_1 = \frac{3-\sqrt{3}}{6}$ and $c_2 = \frac{3+\sqrt{3}}{6}$ and

$S_{int} = \{\exp(\mu x), \exp(-\mu x)\}$ $\qquad$ $S_{fin} = \{1, x, \exp(\mu x), \exp(-\mu x)\}$

Van de Vyver considers modified RK-methods

$$\mathcal{L}_i[y(x); h; \mathrm{a}] = y(x + c_i h) - \gamma_i y(x) - h \sum_{j=1}^{s} a_{ij} y'(x + c_j h)$$

where $i = 1, \ldots, s$ and the final stage

$$\mathcal{L}[y(x); h; \mathrm{b}] = y(x + h) - y(x) - h \sum_{j=1}^{s} b_i y'(x + c_i h)$$

The concept of modified RK methods is also used by Vanden Berghe and V.D.

Van de Vyver's approach

In order to construct a symplectic EF version of the Gauss $s=2$ method with fixed knots $c_1 = \frac{3-\sqrt{3}}{6}$ and $c_2 = \frac{3+\sqrt{3}}{6}$ and

$S_{int} = \{\exp(\mu x), \exp(-\mu x)\}$ $\qquad$ $S_{fin} = \{1, x, \exp(\mu x), \exp(-\mu x)\}$

Van de Vyver considers modified RK-methods

$$\mathcal{L}_i[y(x); h; \mathrm{a}] = y(x + c_i h) - \gamma_i y(x) - h \sum_{j=1}^{s} a_{ij} y'(x + c_j h)$$

where $i = 1, \ldots, s$ and the final stage

$$\mathcal{L}[y(x); h; \mathrm{b}] = y(x + h) - y(x) - h \sum_{j=1}^{s} b_i y'(x + c_i h)$$

The concept of modified RK methods is also used by Vanden Berghe and V.D.

Extra conditions

A modified Runge-Kutta method is called symplectic iff

$$\frac{b_j}{\gamma_j} a_{ji} + \frac{b_i}{\gamma_i} a_{ij} - b_i b_j = 0 \qquad 1 \leq i, j \leq s\,.$$

A modified Runge-Kutta method is called symmetric iff

$$c_i = 1 - c_{s+1-i} \qquad b_i = b_{s+1-i} \qquad a_{i,j} = \gamma_i\, b_j - a_{s+1-i,s+1-j}$$

$$\gamma_i = \gamma_{s+1-i}$$

for all $1 \leq i, j \leq s$.

Extra conditions

A modified Runge-Kutta method is called symplectic iff

$$\frac{b_j}{\gamma_j}\, a_{ji} + \frac{b_i}{\gamma_i}\, a_{ij} - b_i\, b_j = 0 \qquad 1 \leq i, j \leq s\,.$$

A modified Runge-Kutta method is called symmetric iff

$$c_i = 1 - c_{s+1-i} \qquad b_i = b_{s+1-i} \qquad a_{i,j} = \gamma_i\, b_j - a_{s+1-i,s+1-j}$$

$$\gamma_i = \gamma_{s+1-i}$$

for all $1 \leq i, j \leq s$.

The case $s = 2$

We consider a 2-stage modified Runge-Kutta method

$$\begin{array}{c|c|cc} c_1 & \gamma_1 & a_{11} & a_{12} \\ c_2 & \gamma_2 & a_{21} & a_{22} \\ \hline & & b_1 & b_2 \end{array}$$

$$\text{Symmetry}: c_1 = \frac{1}{2} - \theta \qquad c_2 = \frac{1}{2} + \theta \qquad b_1 = b_2$$

$$a_{11} + a_{22} = \gamma_1 b_1 \qquad a_{21} + a_{12} = \gamma_2 b_1$$

$$\text{Symplecticity}: a_{11} = \frac{\gamma_1 b_1}{2} \quad \frac{a_{12}}{\gamma_1} + \frac{a_{21}}{\gamma_2} = b1 \quad a_{22} = \frac{\gamma_2 b_2}{2}$$

The case $s = 2$

We consider a 2-stage modified Runge-Kutta method

$$\begin{array}{c|c|cc} c_1 & \gamma_1 & a_{11} & a_{12} \\ c_2 & \gamma_2 & a_{21} & a_{22} \\ \hline & & b_1 & b_2 \end{array}$$

$$\text{Symmetry} : c_1 = \frac{1}{2} - \theta \qquad c_2 = \frac{1}{2} + \theta \qquad b_1 = b_2$$

$$a_{11} + a_{22} = \gamma_1 \, b_1 \qquad a_{21} + a_{12} = \gamma_2 \, b_1$$

$$\text{Symplecticity} : a_{11} = \frac{\gamma_1 \, b_1}{2} \quad \frac{a_{12}}{\gamma_1} + \frac{a_{21}}{\gamma_2} = b1 \quad a_{22} = \frac{\gamma_2 \, b_2}{2}$$

The case $s = 2$

We consider a 2-stage modified Runge-Kutta method

$$\begin{array}{c|c|cc} c_1 & \gamma_1 & a_{11} & a_{12} \\ c_2 & \gamma_2 & a_{21} & a_{22} \\ \hline & & b_1 & b_2 \end{array}$$

Symmetry : $c_1 = \frac{1}{2} - \theta \qquad c_2 = \frac{1}{2} + \theta \qquad b_1 = b_2$

$$a_{11} + a_{22} = \gamma_1 \, b_1 \qquad a_{21} + a_{12} = \gamma_2 \, b_1$$

Symplecticity : $a_{11} = \frac{\gamma_1 \, b_1}{2} \quad \frac{a_{12}}{\gamma_1} + \frac{a_{21}}{\gamma_2} = b1 \quad a_{22} = \frac{\gamma_2 \, b_2}{2}$

The case $s = 2$

A symmetric, symplectic modified EF Runge-Kutta method has the form

$$
\begin{array}{c|c|cc}
\frac{1}{2} - \theta & \gamma_1 & \frac{\gamma_1 b_1}{2} & \frac{\gamma_1 b_1}{2} + \lambda \\
\frac{1}{2} + \theta & \gamma_1 & \frac{\gamma_1 b_1}{2} - \lambda & \frac{\gamma_1 b_1}{2} \\
\hline
 & & b_1 & b_1
\end{array}
$$

Four parameters: b_1, γ_1, λ and θ

The case $s = 2$

A symmetric, symplectic modified EF Runge-Kutta method has the form

$$\begin{array}{c|c|cc} \frac{1}{2}-\theta & \gamma_1 & \frac{\gamma_1 b_1}{2} & \frac{\gamma_1 b_1}{2}+\lambda \\ \frac{1}{2}+\theta & \gamma_1 & \frac{\gamma_1 b_1}{2}-\lambda & \frac{\gamma_1 b_1}{2} \\ \hline & & b_1 & b_1 \end{array}$$

Four parameters: b_1, γ_1, λ and θ

The case $s = 2$

We consider the construction of a method for which

$$S_{int} = \{\exp(\mu x), \exp(-\mu x)\}$$

and

$$S_{fin} = \{\exp(\mu x),\ \exp(-\mu x),\ \exp(\mu_2 x),\ \exp(-\mu_2 x)\}$$

Special cases :

- $\mu_2 = 2\mu$ (Calvo)
- $\mu_2 \to \mu$ (Vanden Berghe)

First we impose

$$S_{int} = \{\exp(\mu x), \exp(-\mu x)\} \qquad S_{fin} = \{\exp(\mu x), \exp(-\mu x)\}$$

The case $s = 2$

We consider the construction of a method for which

$$S_{int} = \{\exp(\mu x), \exp(-\mu x)\}$$

and

$$S_{fin} = \{\exp(\mu x),\ \exp(-\mu x),\ \exp(\mu_2 x),\ \exp(-\mu_2 x)\}$$

Special cases :

- $\mu_2 = 2\mu$ (Calvo)
- $\mu_2 \to \mu$ (Vanden Berghe)

First we impose

$$S_{int} = \{\exp(\mu x), \exp(-\mu x)\} \qquad S_{fin} = \{\exp(\mu x), \exp(-\mu x)\}$$

The case $s = 2$

We consider the construction of a method for which

$$S_{int} = \{\exp(\mu x), \exp(-\mu x)\}$$

and

$$S_{fin} = \{\exp(\mu x),\ \exp(-\mu x),\ \exp(\mu_2 x),\ \exp(-\mu_2 x)\}$$

Special cases :

- $\mu_2 = 2\mu$ (Calvo)
- $\mu_2 \to \mu$ (Vanden Berghe)

First we impose

$$S_{int} = \{\exp(\mu x), \exp(-\mu x)\} \qquad S_{fin} = \{\exp(\mu x), \exp(-\mu x)\}$$

...the case $s = 2$ ...

Imposing

$$S_{int} = \{\exp(\mu x), \exp(-\mu x)\} \qquad S_{fin} = \{\exp(\mu x), \exp(-\mu x)\}$$

leads to formula's also obtained by Vanden Berghe et al.

$$b_1 = \frac{1}{2} \frac{\sinh(z/2)}{\cosh(z\theta)\ (z/2)} = b_2$$

$$\gamma_1 = 2\,\frac{\cosh(z\theta)}{\cosh(z/2)} - \frac{1}{\cosh(z/2)\cosh(z\theta)} = \gamma_2$$

$$\lambda = -\frac{\sinh(z\theta)}{\cosh(z\theta)\, z}$$

$$z := \mu h$$

Following Ixaru :

$$b_1 = \frac{1}{2} \frac{\eta(Z/4)}{\xi(Z\,\theta^2)} = b_2 \qquad Z := z^2$$

...the case $s = 2$ ...

Imposing

$$S_{int} = \{\exp(\mu x), \exp(-\mu x)\} \qquad S_{fin} = \{\exp(\mu x), \exp(-\mu x)\}$$

leads to formula's also obtained by Vanden Berghe et al.

$$b_1 = \frac{1}{2} \frac{\sinh(z/2)}{\cosh(z\theta)\ (z/2)} = b_2$$

$$\gamma_1 = 2\frac{\cosh(z\theta)}{\cosh(z/2)} - \frac{1}{\cosh(z/2)\cosh(z\theta)} = \gamma_2$$

$$\lambda = -\frac{\sinh(z\theta)}{\cosh(z\theta)\, z}$$

$$z := \mu h$$

Following Ixaru :

$$b_1 = \frac{1}{2} \frac{\eta(Z/4)}{\xi(Z\,\theta^2)} = b_2 \qquad Z := z^2$$

...the case $s = 2$ ...

Next we impose

$$S_{fin} = \{\exp(\mu x), \exp(-\mu x)\} \cup \{\exp(\mu_2 x), \exp(-\mu_2 x)\}$$

$$b_1 = \frac{1}{2}\frac{\sinh(z_2/2)}{\cosh(z_2\theta)\ (z_2/2)} = \frac{1}{2}\frac{\sinh(z/2)}{\cosh(z\theta)\ (z/2)}$$

This leads to a formula for θ : $F(z) = F(z_2)$ where

$$F(u) = \frac{\sinh(u/2)}{\cosh(u\theta)\ (u/2)}$$

In general, an iterative procedure is needed to determine θ.

. . . the case $s = 2$ . . .

Next we impose

$$S_{fin} = \{\exp(\mu x), \exp(-\mu x)\} \cup \{\exp(\mu_2 x), \exp(-\mu_2 x)\}$$

$$b_1 = \frac{1}{2} \frac{\sinh(z_2/2)}{\cosh(z_2\theta)\ (z_2/2)} = \frac{1}{2} \frac{\sinh(z/2)}{\cosh(z\theta)\ (z/2)}$$

This leads to a formula for θ : $F(z) = F(z_2)$ where

$$F(u) = \frac{\sinh(u/2)}{\cosh(u\theta)\ (u/2)}$$

In general, an iterative procedure is needed to determine θ.

...the case $s = 2$ ...

Next we impose

$$S_{fin} = \{\exp(\mu x), \exp(-\mu x)\} \cup \{\exp(\mu_2 x), \exp(-\mu_2 x)\}$$

$$b_1 = \frac{1}{2} \frac{\sinh(z_2/2)}{\cosh(z_2\theta)\ (z_2/2)} = \frac{1}{2} \frac{\sinh(z/2)}{\cosh(z\theta)\ (z/2)}$$

This leads to a formula for θ : $F(z) = F(z_2)$ where

$$F(u) = \frac{\sinh(u/2)}{\cosh(u\theta)\ (u/2)}$$

In general, an iterative procedure is needed to determine θ.

...the case $s = 2$ ...

Next we impose

$$S_{fin} = \{\exp(\mu x), \exp(-\mu x)\} \cup \{\exp(\mu_2 x), \exp(-\mu_2 x)\}$$

$$b_1 = \frac{1}{2}\frac{\sinh(z_2/2)}{\cosh(z_2\theta)\ (z_2/2)} = \frac{1}{2}\frac{\sinh(z/2)}{\cosh(z\theta)\ (z/2)}$$

This leads to a formula for θ : $F(z) = F(z_2)$ where

$$F(u) = \frac{\sinh(u/2)}{\cosh(u\theta)\ (u/2)}$$

In general, an iterative procedure is needed to determine θ.

...the case $s = 2$ ...

Next we impose

$$S_{fin} = \{\exp(\mu x), \exp(-\mu x)\} \cup \{\exp(\mu_2 x), \exp(-\mu_2 x)\}$$

$$b_1 = \frac{1}{2} \frac{\sinh(z_2/2)}{\cosh(z_2\theta)\ (z_2/2)} = \frac{1}{2} \frac{\sinh(z/2)}{\cosh(z\theta)\ (z/2)}$$

This leads to a formula for θ : $F(z) = F(z_2)$ where

$$F(u) = \frac{\sinh(u/2)}{\cosh(u\theta)\ (u/2)}$$

In general, an iterative procedure is needed to determine θ.

. . . the case $s = 2$ . . .

$$F(z) = F(z_2) \text{ where } F(u) = \frac{\sinh(u/2)}{\cosh(u\theta)\,(u/2)}$$

Special cases :

- $z_2 = 2z : \theta = \frac{1}{z}\text{acosh}\left(\frac{\cosh(z/2) + \sqrt{8 + \cosh^2(z/2)}}{4}\right)$

  For this value of θ : $\gamma_1 = \gamma_2 = 1$

  This is the EFRK method of Calvo et al.

. . . the case $s = 2$ . . .

$$F(z) = F(z_2) \text{ where } F(u) = \frac{\sinh(u/2)}{\cosh(u\theta)\,(u/2)}$$

Special cases :

- $z_2 = 2z : \theta = \frac{1}{z}\text{acosh}\left(\frac{\cosh(z/2) + \sqrt{8 + \cosh^2(z/2)}}{4}\right)$

  For this value of θ : $\gamma_1 = \gamma_2 = 1$

  This is the EFRK method of Calvo et al.

. . . the case $s = 2$ . . .

$$F(z) = F(z_2) \text{ where } F(u) = \frac{\sinh(u/2)}{\cosh(u\theta)\ (u/2)}$$

Special cases :

- $z_2 = 2z$: $\theta = \frac{1}{z}\text{acosh}\left(\frac{\cosh(z/2) + \sqrt{8 + \cosh^2(z/2)}}{4}\right)$

  For this value of θ : $\gamma_1 = \gamma_2 = 1$

  This is the EFRK method of Calvo et al.

...the case $s = 2$ ...

$$F(z) = F(z_2) \text{ where } F(u) = \frac{\sinh(u/2)}{\cosh(u\theta)\ (u/2)}$$

Special cases :

- $z_2 = 2z : \theta = \frac{1}{z} \text{acosh}\left(\frac{\cosh(z/2) + \sqrt{8 + \cosh^2(z/2)}}{4}\right)$

  For this value of θ : $\gamma_1 = \gamma_2 = 1$

  This is the EFRK method of Calvo et al.

... the case $s = 2$ ...

$$F(z) = F(z_2) \text{ where } F(u) = \frac{\sinh(u/2)}{\cosh(u\theta)\ (u/2)}$$

Special cases :

- $z_2 = 2z : \theta = \frac{1}{z}\text{acosh}\left(\frac{\cosh(z/2) + \sqrt{8 + \cosh^2(z/2)}}{4}\right)$

  For this value of θ : $\gamma_1 = \gamma_2 = 1$

  This is the EFRK method of Calvo et al.

... the case $s = 2$ ...

$$F(z) = F(z_2) \text{ where } F(u) = \frac{\sinh(u/2)}{\cosh(u\theta)\,(u/2)}$$

Special cases :

- $z_2 = z : F'(z) = 0$

$$\Longrightarrow \theta = \frac{1}{z}\frac{\cosh(z\theta)}{\sinh(z\theta)}\left(\frac{\cosh(z/2)}{\sinh(z/2)/(z/2)} - 1\right)$$

  This is the method of Vanden Berghe et al. with

$$S_{fin} = \{\exp(\mu x), \exp(-\mu x)\} \cup \{x\ \exp(\mu x), x\ \exp(-\mu x)\}$$

- $z_2 = 0 : F(z) = 1 \Longrightarrow \theta = \frac{1}{z}\text{acosh}\left(\frac{\sinh(z/2)}{(z/2)}\right)$

  This is the method of Vanden Berghe et al. with

$$S_{fin} = \{\exp(\mu x), \exp(-\mu x)\} \cup \{1, x\}$$

. . . the case $s = 2$ . . .

$$F(z) = F(z_2) \text{ where } F(u) = \frac{\sinh(u/2)}{\cosh(u\theta)\ (u/2)}$$

Special cases :

- $z_2 = z$: $F'(z) = 0$

$$\Longrightarrow \theta = \frac{1}{z}\frac{\cosh(z\theta)}{\sinh(z\theta)}\left(\frac{\cosh(z/2)}{\sinh(z/2)/(z/2)} - 1\right)$$

  This is the method of Vanden Berghe et al. with

$$S_{fin} = \{\exp(\mu x), \exp(-\mu x)\} \cup \{x\ \exp(\mu x), x\ \exp(-\mu x)\}$$

- $z_2 = 0 : F(z) = 1 \Longrightarrow \theta = \frac{1}{z}\text{acosh}\left(\frac{\sinh(z/2)}{(z/2)}\right)$

  This is the method of Vanden Berghe et al. with

$$S_{fin} = \{\exp(\mu x), \exp(-\mu x)\} \cup \{1, x\}$$

...the case $s=2$ ...

$F(z) = F(z_2)$ where $F(u) = \dfrac{\sinh(u/2)}{\cosh(u\theta)\ (u/2)}$

Special cases :

- $z_2 = z : F'(z) = 0$

$$\Longrightarrow \theta = \frac{1}{z}\frac{\cosh(z\theta)}{\sinh(z\theta)}\left(\frac{\cosh(z/2)}{\sinh(z/2)/(z/2)} - 1\right)$$

This is the method of Vanden Berghe et al. with

$$S_{fin} = \{\exp(\mu x), \exp(-\mu x)\} \cup \{x\ \exp(\mu x), x\ \exp(-\mu x)\}$$

- $z_2 = 0 : F(z) = 1 \Longrightarrow \theta = \frac{1}{z}\text{acosh}\left(\frac{\sinh(z/2)}{(z/2)}\right)$

This is the method of Vanden Berghe et al. with

$$S_{fin} = \{\exp(\mu x), \exp(-\mu x)\} \cup \{1, x\}$$

...the case $s = 2$ ...

$F(z) = F(z_2)$ where $F(u) = \dfrac{\sinh(u/2)}{\cosh(u\theta)\,(u/2)}$

Special cases :

- $z_2 = z$: $F'(z) = 0$

$$\Longrightarrow \theta = \frac{1}{z}\frac{\cosh(z\theta)}{\sinh(z\theta)}\left(\frac{\cosh(z/2)}{\sinh(z/2)/(z/2)} - 1\right)$$

  This is the method of Vanden Berghe et al. with

$$S_{fin} = \{\exp(\mu x), \exp(-\mu x)\} \cup \{x\ \exp(\mu x), x\ \exp(-\mu x)\}$$

- $z_2 = 0$: $F(z) = 1 \Longrightarrow \theta = \frac{1}{z}\text{acosh}\left(\frac{\sinh(z/2)}{(z/2)}\right)$

  This is the method of Vanden Berghe et al. with

$$S_{fin} = \{\exp(\mu x), \exp(-\mu x)\} \cup \{1, x\}$$

... the case $s = 2$ ...

$$F(z) = F(z_2) \text{ where } F(u) = \frac{\sinh(u/2)}{\cosh(u\theta)\ (u/2)}$$

Special cases :

- $z_2 = z : F'(z) = 0$

$$\Longrightarrow \theta = \frac{1}{z}\frac{\cosh(z\theta)}{\sinh(z\theta)}\left(\frac{\cosh(z/2)}{\sinh(z/2)/(z/2)} - 1\right)$$

  This is the method of Vanden Berghe et al. with

$$S_{fin} = \{\exp(\mu x), \exp(-\mu x)\} \cup \{x\ \exp(\mu x), x\ \exp(-\mu x)\}$$

- $z_2 = 0 : F(z) = 1 \Longrightarrow \theta = \frac{1}{z}\text{acosh}\left(\frac{\sinh(z/2)}{(z/2)}\right)$

  This is the method of Vanden Berghe et al. with

$$S_{fin} = \{\exp(\mu x), \exp(-\mu x)\} \cup \{1, x\}$$

...the case $s = 2$ ...

$F(z) = F(z_2)$ where $F(u) = \dfrac{\sinh(u/2)}{\cosh(u\theta)\ (u/2)}$

Special cases :

- $z_2 = z$: $F'(z) = 0$

$$\Longrightarrow \theta = \frac{1}{z}\frac{\cosh(z\theta)}{\sinh(z\theta)}\left(\frac{\cosh(z/2)}{\sinh(z/2)/(z/2)} - 1\right)$$

This is the method of Vanden Berghe et al. with

$$S_{fin} = \{\exp(\mu x), \exp(-\mu x)\} \cup \{x\ \exp(\mu x), x\ \exp(-\mu x)\}$$

- $z_2 = 0$: $F(z) = 1 \Longrightarrow \theta = \frac{1}{z}\text{acosh}\left(\frac{\sinh(z/2)}{(z/2)}\right)$

This is the method of Vanden Berghe et al. with

$$S_{fin} = \{\exp(\mu x), \exp(-\mu x)\} \cup \{1, x\}$$

... the case $s = 2$ ...

$$F(z) = F(z_2) \text{ where } F(u) = \frac{\sinh(u/2)}{\cosh(u\theta)\,(u/2)}$$

Special cases :

- $z_2 = z$: $F'(z) = 0$

$$\Longrightarrow \theta = \frac{1}{z}\frac{\cosh(z\theta)}{\sinh(z\theta)}\left(\frac{\cosh(z/2)}{\sinh(z/2)/(z/2)} - 1\right)$$

  This is the method of Vanden Berghe et al. with

$$S_{fin} = \{\exp(\mu x), \exp(-\mu x)\} \cup \{x\ \exp(\mu x), x\ \exp(-\mu x)\}$$

- $z_2 = 0$: $F(z) = 1 \Longrightarrow \theta = \frac{1}{z}\text{acosh}\left(\frac{\sinh(z/2)}{(z/2)}\right)$

  This is the method of Vanden Berghe et al. with

$$S_{fin} = \{\exp(\mu x), \exp(-\mu x)\} \cup \{1, x\}$$

...the case $s = 2$ ...

$$F(z) = F(z_2) \text{ where } F(u) = \frac{\sinh(u/2)}{\cosh(u\theta)\ (u/2)}$$

Special cases :

- $z_2 = z : F'(z) = 0$

$$\Longrightarrow \theta = \frac{1}{z}\frac{\cosh(z\theta)}{\sinh(z\theta)}\left(\frac{\cosh(z/2)}{\sinh(z/2)/(z/2)} - 1\right)$$

  This is the method of Vanden Berghe et al. with

$$S_{fin} = \{\exp(\mu x), \exp(-\mu x)\} \cup \{x\ \exp(\mu x), x\ \exp(-\mu x)\}$$

- $z_2 = 0 : F(z) = 1 \Longrightarrow \theta = \frac{1}{z}\text{acosh}\left(\frac{\sinh(z/2)}{(z/2)}\right)$

  This is the method of Vanden Berghe et al. with

$$S_{fin} = \{\exp(\mu x), \exp(-\mu x)\} \cup \{1, x\}$$

. . . the case $s = 2$ . . .

$$F(z) = F(z_2) \text{ where } F(u) = \frac{\sinh(u/2)}{\cosh(u\theta)\ (u/2)}$$

Special cases :

- $z_2 = z : F'(z) = 0$

$$\Longrightarrow \theta = \frac{1}{z}\frac{\cosh(z\theta)}{\sinh(z\theta)}\left(\frac{\cosh(z/2)}{\sinh(z/2)/(z/2)} - 1\right)$$

  This is the method of Vanden Berghe et al. with

$$S_{fin} = \{\exp(\mu x), \exp(-\mu x)\} \cup \{x\ \exp(\mu x), x\ \exp(-\mu x)\}$$

- $z_2 = 0 : F(z) = 1 \Longrightarrow \theta = \frac{1}{z}\text{acosh}\left(\frac{\sinh(z/2)}{(z/2)}\right)$

  This is the method of Vanden Berghe et al. with

$$S_{fin} = \{\exp(\mu x), \exp(-\mu x)\} \cup \{1, x\}$$

. . . the case $s = 2$ . . .

What if

- $z \approx 0$
- $z_2 \approx 0$
- $z \approx 0$ and $z_2 \approx 0$
- $z_2 \approx z$

. . . the case $s = 2$ . . .

If $z \to 0$ and $z_2 \to 0$:

$$
\begin{aligned}
\theta \;=\; & \frac{\sqrt{3}}{6} + \frac{\sqrt{3}}{2160}\left(z^2 + {z_2}^2\right) \\
& - \frac{\sqrt{3}}{10886400}\left(27\,z^4 - 106\,z^2{z_2}^2 + 27\,{z_2}^4\right) \\
& + \frac{\sqrt{3}}{435456000}\left(3\,{z_2}^4 - 34\,z^2{z_2}^2 + 3\,z^4\right)\left(z^2 + {z_2}^2\right) \\
& + \dots
\end{aligned}
$$

...the case $s = 2$ ...

$$F(z_2) = F(z)$$

If $z_2 - z$ is very small :

$$F(z_2) = F(z) + (z_2 - z)F'(z) + \frac{1}{2}(z_2 - z)^2 F''(z) + \dots$$

$$F'(z) + (z_2 - z)F''(z) = 0$$

...the case $s = 2$ ...

$$F(z_2) = F(z)$$

If $z_2 - z$ is very small :

$$F(z_2) = F(z) + (z_2 - z)F'(z) + \frac{1}{2}(z_2 - z)^2 F''(z) + \dots$$

$$F'(z) + (z_2 - z)F''(z) = 0$$

... the case $s = 2$ ...

$$F(z_2) = F(z)$$

If $z_2 - z$ is very small :

$$F(z_2) = F(z) + (z_2 - z)\,F'(z) + \frac{1}{2}(z_2 - z)^2\,F''(z) + \ldots$$

$$F'(z) + (z_2 - z)\,F''(z) = 0$$

The case $s = 3$

A symmetric, symplectic modified EF Runge-Kutta method has the form

$$\begin{array}{c|c|ccc}
\frac{1}{2}-\theta & \gamma_1 & \frac{\gamma_1 b_1}{2} & \frac{\gamma_1 b_2}{2}-\alpha_2 & \frac{\gamma_1 b_1}{2}-\alpha_3 \\
\frac{1}{2} & \gamma_2 & \frac{\gamma_2 b_1}{2}-\alpha_4 & \frac{\gamma_2 b_2}{2} & \frac{\gamma_2 b_1}{2}+\alpha_4 \\
\frac{1}{2}+\theta & \gamma_1 & \frac{\gamma_1 b_1}{2}+\alpha_3 & \frac{\gamma_1 b_2}{2}+\alpha_2 & \frac{\gamma_1 b_1}{2} \\
\hline
 & & b_1 & b_2 & b_1
\end{array}
\qquad \frac{b_1}{\gamma_1}\alpha_2+\frac{b_2}{\gamma_2}\alpha_4=0$$

Parameters : $b_1, b_2, \gamma_1, \gamma_2, \alpha_2, \alpha_3, \theta$

The case $s = 3$

We consider the construction of a method for which

$$S_{int} = \{1, \exp(\mu x), \exp(-\mu x)\}$$

and

$$S_{fin} = \{1, \exp(\mu x), \exp(-\mu x), \exp(\mu_2 x), \exp(-\mu_2 x)\}$$

Special cases :

- $\mu_2 = 2\mu$ (Calvo)
- $\mu_2 \to \mu$ (Vanden Berghe)

First we impose

$$S_{int} = \{1, \exp(\mu x), \exp(-\mu x)\} \qquad S_{fin} = \{1, \exp(\mu x), \exp(-\mu x)\}$$

The case $s = 3$

We consider the construction of a method for which

$$S_{int} = \{1, \exp(\mu x), \exp(-\mu x)\}$$

and

$$S_{fin} = \{1, \exp(\mu x), \exp(-\mu x), \exp(\mu_2 x), \exp(-\mu_2 x)\}$$

Special cases :

- $\mu_2 = 2\mu$ (Calvo)
- $\mu_2 \to \mu$ (Vanden Berghe)

First we impose

$$S_{int} = \{1, \exp(\mu x), \exp(-\mu x)\} \qquad S_{fin} = \{1, \exp(\mu x), \exp(-\mu x)\}$$

The case $s = 3$

We consider the construction of a method for which

$$S_{int} = \{1, \exp(\mu\, x), \exp(-\mu\, x)\}$$

and

$$S_{fin} = \{1, \exp(\mu\, x), \exp(-\mu\, x), \exp(\mu_2\, x), \exp(-\mu_2\, x)\}$$

Special cases :

- $\mu_2 = 2\,\mu$ (Calvo)
- $\mu_2 \to \mu$ (Vanden Berghe)

First we impose

$$S_{int} = \{1, \exp(\mu\, x), \exp(-\mu\, x)\} \qquad S_{fin} = \{1, \exp(\mu\, x), \exp(-\mu\, x)\}$$

. . . the case $s=3$ . . .

Imposing

$$S_{int}=\{1,\exp(\mu x),\exp(-\mu x)\} \qquad S_{fin}=\{1,\exp(\mu x),\exp(-\mu x)\}$$

leads to formula's also obtained by Calvo et al. since

$$\gamma_1=1=\gamma_2$$

$$b_1=\frac{1}{2}\frac{\frac{\sinh(z)}{z}-\frac{\sinh(z/2)}{z/2}}{\cosh(2z\theta)-\cosh(z\theta)}$$

$$b_2=\ldots \quad \alpha_2=\ldots \quad \alpha_3=\ldots$$

Following Ixaru :

$$b_1=\frac{1}{2}\frac{\eta(Z)-\eta(Z/4)}{\xi(4Z\theta^2)-\xi(Z\theta^2)}$$

... the case $s = 3$ ...

Imposing

$$S_{int} = \{1, \exp(\mu x), \exp(-\mu x)\} \qquad S_{fin} = \{1, \exp(\mu x), \exp(-\mu x)\}$$

leads to formula's also obtained by Calvo et al. since

$$\gamma_1 = 1 = \gamma_2$$

$$b_1 = \frac{1}{2} \frac{\frac{\sinh(z)}{z} - \frac{\sinh(z/2)}{z/2}}{\cosh(2\, z\, \theta) - \cosh(z\, \theta)}$$

$$b_2 = \ldots \quad \alpha_2 = \ldots \quad \alpha_3 = \ldots$$

Following Ixaru :

$$b_1 = \frac{1}{2} \frac{\eta(Z) - \eta(Z/4)}{\xi(4\, Z \theta^2) - \xi(Z\, \theta^2)}$$

...the case $s = 3$ ...

Next we impose

$$S_{fin} = \{1, \exp(\mu\, x), \exp(-\mu\, x)\} \cup \{\exp(\mu_2\, x), \exp(-\mu_2\, x)\}$$

We then obtain

$$b_1 = \frac{1}{2} \frac{\frac{\sinh(z_2/2)}{z_2/2} - \frac{\sinh(z/2)}{z/2}}{\cosh(z_2\,\theta) - \cosh(z\,\theta)} \qquad b_2 = \ldots$$

which has exactly the same form as the expression we already had :

$$b_1 = \frac{1}{2} \frac{\frac{\sinh(z)}{z} - \frac{\sinh(z/2)}{z/2}}{\cosh(2\,z\,\theta) - \cosh(z\,\theta)}$$

The first expression makes clear that the final stage by accident also integrates $\{\exp(2\,\mu\, x), \exp(-2\,\mu\, x)\}$ exactly :

$$S_{fin} = \{1, \exp(\pm\mu\, x), \exp(\pm 2\,\mu\, x), \exp(\pm\mu_2\, x)\}$$

. . . the case $s = 3$ . . .

Next we impose

$$S_{fin} = \{1, \exp(\mu x), \exp(-\mu x)\} \cup \{\exp(\mu_2 x), \exp(-\mu_2 x)\}$$

We then obtain

$$b_1 = \frac{1}{2} \frac{\frac{\sinh(z_2/2)}{z_2/2} - \frac{\sinh(z/2)}{z/2}}{\cosh(z_2\,\theta) - \cosh(z\,\theta)} \qquad b_2 = \ldots$$

which has exactly the same form as the expression we already had :

$$b_1 = \frac{1}{2} \frac{\frac{\sinh(z)}{z} - \frac{\sinh(z/2)}{z/2}}{\cosh(2\,z\,\theta) - \cosh(z\,\theta)}$$

The first expression makes clear that the final stage by accident also integrates $\{\exp(2\,\mu\,x), \exp(-2\,\mu\,x)\}$ exactly :

$$S_{fin} = \{1, \exp(\pm\mu\,x), \exp(\pm 2\,\mu\,x), \exp(\pm\mu_2\,x)\}$$

... the case $s = 3$ ...

Next we impose

$$S_{fin} = \{1, \exp(\mu x), \exp(-\mu x)\} \cup \{\exp(\mu_2 x), \exp(-\mu_2 x)\}$$

We then obtain

$$b_1 = \frac{1}{2}\frac{\frac{\sinh(z_2/2)}{z_2/2} - \frac{\sinh(z/2)}{z/2}}{\cosh(z_2\,\theta) - \cosh(z\,\theta)} \qquad b_2 = \ldots$$

which has exactly the same form as the expression we already had :

$$b_1 = \frac{1}{2}\frac{\frac{\sinh(z)}{z} - \frac{\sinh(z/2)}{z/2}}{\cosh(2\,z\,\theta) - \cosh(z\,\theta)}$$

The first expression makes clear that the final stage by accident also integrates $\{\exp(2\,\mu\,x), \exp(-2\,\mu\,x)\}$ exactly :

$$S_{fin} = \{1, \exp(\pm\mu\,x), \exp(\pm 2\,\mu\,x), \exp(\pm\mu_2\,x)\}$$

...the case $s = 3$ ...

Next we impose

$$S_{fin} = \{1, \exp(\mu x), \exp(-\mu x)\} \cup \{\exp(\mu_2 x), \exp(-\mu_2 x)\}$$

We then obtain

$$b_1 = \frac{1}{2} \frac{\frac{\sinh(z_2/2)}{z_2/2} - \frac{\sinh(z/2)}{z/2}}{\cosh(z_2\,\theta) - \cosh(z\,\theta)} \qquad b_2 = \ldots$$

which has exactly the same form as the expression we already had :

$$b_1 = \frac{1}{2} \frac{\frac{\sinh(z)}{z} - \frac{\sinh(z/2)}{z/2}}{\cosh(2\,z\,\theta) - \cosh(z\,\theta)}$$

The first expression makes clear that the final stage by accident also integrates $\{\exp(2\,\mu x), \exp(-2\,\mu x)\}$ exactly :

$$S_{fin} = \{1, \exp(\pm\mu x), \exp(\pm 2\,\mu x), \exp(\pm\mu_2 x)\}$$

...the case $s=3$ ...

Combining both results, we obtain the relation from which θ can be determined :

$$\frac{1}{2}\,\frac{\frac{\sinh(z_2/2)}{z_2/2}-\frac{\sinh(z/2)}{z/2}}{\cosh(z_2\,\theta)-\cosh(z\,\theta)}=\frac{1}{2}\,\frac{\frac{\sinh(z)}{z}-\frac{\sinh(z/2)}{z/2}}{\cosh(2\,z\,\theta)-\cosh(z\,\theta)}$$

$$G(z,z_2)=G(z,2\,z)$$

$$\text{with}\quad G(a,b):=\frac{\frac{\sinh(a/2)}{a/2}-\frac{\sinh(b/2)}{b/2}}{\cosh(a\,\theta)-\cosh(b\,\theta)}$$

In general, an iterative procedure is needed to determine θ.

...the case $s = 3$ ...

Combining both results, we obtain the relation from which θ can be determined :

$$\frac{1}{2}\,\frac{\frac{\sinh(z_2/2)}{z_2/2}-\frac{\sinh(z/2)}{z/2}}{\cosh(z_2\,\theta)-\cosh(z\,\theta)} = \frac{1}{2}\,\frac{\frac{\sinh(z)}{z}-\frac{\sinh(z/2)}{z/2}}{\cosh(2\,z\,\theta)-\cosh(z\,\theta)}$$

$$G(z, z_2) = G(z, 2\,z)$$

$$\text{with}\quad G(a,b) := \frac{\frac{\sinh(a/2)}{a/2}-\frac{\sinh(b/2)}{b/2}}{\cosh(a\,\theta)-\cosh(b\,\theta)}$$

In general, an iterative procedure is needed to determine θ.

...the case $s = 3$ ...

Combining both results, we obtain the relation from which θ can be determined :

$$\frac{1}{2}\,\frac{\dfrac{\sinh(z_2/2)}{z_2/2}-\dfrac{\sinh(z/2)}{z/2}}{\cosh(z_2\,\theta)-\cosh(z\,\theta)}=\frac{1}{2}\,\frac{\dfrac{\sinh(z)}{z}-\dfrac{\sinh(z/2)}{z/2}}{\cosh(2\,z\,\theta)-\cosh(z\,\theta)}$$

$$G(z,\,z_2)=G(z,\,2\,z)$$

$$\text{with}\;\; G(a,b):=\frac{\dfrac{\sinh(a/2)}{a/2}-\dfrac{\sinh(b/2)}{b/2}}{\cosh(a\,\theta)-\cosh(b\,\theta)}$$

In general, an iterative procedure is needed to determine θ.

. . . the case $s = 3$ . . .

Combining both results, we obtain the relation from which θ can be determined :

$$\frac{1}{2}\,\frac{\frac{\sinh(z_2/2)}{z_2/2}-\frac{\sinh(z/2)}{z/2}}{\cosh(z_2\,\theta)-\cosh(z\,\theta)}=\frac{1}{2}\,\frac{\frac{\sinh(z)}{z}-\frac{\sinh(z/2)}{z/2}}{\cosh(2\,z\,\theta)-\cosh(z\,\theta)}$$

$$G(z,\,z_2)=G(z,\,2\,z)$$

$$\text{with}\;\; G(a,b):=\frac{\frac{\sinh(a/2)}{a/2}-\frac{\sinh(b/2)}{b/2}}{\cosh(a\,\theta)-\cosh(b\,\theta)}$$

In general, an iterative procedure is needed to determine θ.

...the case $s = 3$ ...

Special case : $z_2 = 3\,z$: the method of Calvo et al.

$$\theta = \frac{2}{z} acosh(\beta_1)$$

$$\beta_1 = \frac{1}{6}\sqrt{15 + 6\,\cosh\left(z/2\right) + 3\,\sqrt{15 + 8\,\cosh\left(z/2\right) + 2\,\cosh\left(z\right)}}$$

$$\theta = \frac{\sqrt{15}}{10}\left(1 + \frac{z^2}{150} - \frac{31\,z^4}{240000} + \frac{89\,z^6}{144000000} + \ldots\right)$$

. . . the case $s = 3$ . . .

Special case : $z_2 = 3\,z$: the method of Calvo et al.

$$\theta = \frac{2}{z} acosh(\beta_1)$$

$$\beta_1 = \frac{1}{6}\sqrt{15 + 6\,\cosh\,(z/2) + 3\,\sqrt{15 + 8\,\cosh\,(z/2) + 2\,\cosh\,(z)}}$$

$$\theta = \frac{\sqrt{15}}{10}\left(1 + \frac{z^2}{150} - \frac{31\,z^4}{240000} + \frac{89\,z^6}{144000000} + \ldots\right)$$

...the case $s = 3$ ...

Special case : $z_2 = 3\,z$: the method of Calvo et al.

$$\theta = \frac{2}{z} acosh(\beta_1)$$

$$\beta_1 = \frac{1}{6}\sqrt{15 + 6\,\cosh(z/2) + 3\,\sqrt{15 + 8\,\cosh(z/2) + 2\,\cosh(z)}}$$

$$\theta = \frac{\sqrt{15}}{10}\left(1 + \frac{z^2}{150} - \frac{31\,z^4}{240000} + \frac{89\,z^6}{144000000} + \ldots\right)$$

...the case $s = 3$ ...

Special case : $z_2 = z/2$:

$$\theta = \frac{4}{z} acosh(\beta_3)$$

$$\beta_3 = \frac{1}{4}\sqrt{6 + 2\sqrt{9 + 8\,(\cosh(z/4))^2 + 8\,\cosh(z/4)}}$$

$$\theta = \frac{\sqrt{15}}{10}\left(1 + \frac{z^2}{400} - \frac{253\,z^4}{11520000} + \frac{1241\,z^6}{9216000000} - \dots\right)$$

...the case $s=3$ ...

Special case : $z_2 = z$:

$$G(z,z) = G(z,\, 2z)$$

$$G(a,b) := \frac{\frac{\sinh(a/2)}{a/2} - \frac{\sinh(b/2)}{b/2}}{\cosh(a\,\theta) - \cosh(b\,\theta)} = \frac{G_N(a,b)}{G_D(a,b)}$$

$$G(z,z) = \lim_{z_2 \to z} G(z,z_2) = \left(\frac{0}{0}\right) = \frac{\frac{\partial}{\partial z_2} G_N(z,z_2)\,\Big|_{z_2=z}}{\frac{\partial}{\partial z_2} G_D(z,z_2)\,\Big|_{z_2=z}}$$

$$= \frac{\cosh(z/2) - \frac{\sinh(z/2)}{z/2}}{z\,\theta\,\sinh(z\theta)}$$

$$S_{fin} = \{1,\ \exp(\pm\mu\, x),\ \exp(\pm 2\,\mu\, x),\ x\ \exp(\pm\mu\, x)\}$$

... the case $s = 3$ ...

Special case : $z_2 = z$:

$$G(z, z) = G(z,\, 2\,z)$$

$$G(a, b) := \frac{\frac{\sinh(a/2)}{a/2} - \frac{\sinh(b/2)}{b/2}}{\cosh(a\,\theta) - \cosh(b\,\theta)} = \frac{G_N(a, b)}{G_D(a, b)}$$

$$G(z, z) = \lim_{z_2 \to z} G(z, z_2) = \left(\frac{0}{0}\right) = \frac{\frac{\partial}{\partial z_2} G_N(z, z_2)\,\Big|_{z_2 = z}}{\frac{\partial}{\partial z_2} G_D(z, z_2)\,\Big|_{z_2 = z}}$$

$$= \frac{\cosh(z/2) - \frac{\sinh(z/2)}{z/2}}{z\,\theta\,\sinh(z\theta)}$$

$$S_{fin} = \{1,\, \exp(\pm\mu\, x),\, \exp(\pm 2\,\mu\, x),\, x\, \exp(\pm\mu\, x)\}$$

...the case $s = 3$ ...

Special case : $z_2 = z$:

$$G(z, z) = G(z,\, 2z)$$

$$G(a,b) := \frac{\frac{\sinh(a/2)}{a/2} - \frac{\sinh(b/2)}{b/2}}{\cosh(a\,\theta) - \cosh(b\,\theta)} = \frac{G_N(a,b)}{G_D(a,b)}$$

$$G(z,z) = \lim_{z_2 \to z} G(z, z_2) \quad = \quad \left(\frac{0}{0}\right) = \frac{\frac{\partial}{\partial z_2} G_N(z, z_2)\,\big|_{z_2=z}}{\frac{\partial}{\partial z_2} G_D(z, z_2)\,\big|_{z_2=z}}$$

$$= \quad \frac{\cosh(z/2) - \frac{\sinh(z/2)}{z/2}}{z\,\theta\,\sinh(z\theta)}$$

$$S_{fin} = \{1,\ \exp(\pm\mu\, x),\ \exp(\pm 2\,\mu\, x),\ x\ \exp(\pm\mu\, x)\}$$

…the case $s = 3$ …

If $z \to 0$ and $z_2 \to 0$:

$$\theta = \frac{\sqrt{15}}{10} + \frac{\sqrt{15}}{21000}\left(5\,z^2 + {z_2}^2\right) - \frac{\sqrt{15}}{1058400000}\left(2295\,z^4 + 85\,z^2{z_2}^2 + 131\,{z_2}^4\right) + \frac{\sqrt{15}}{97796160000000} \times \left(1730250\,z^6 - 1653665\,z^4{z_2}^2 - 5765\,z^2{z_2}^4 + 26974\,{z_2}^6\right) + \ldots$$

Some tests for the $s = 3$ case

We have considered three problems

- Kepler's problem
- a perturbed Kepler problem
- Euler's problem

and four methods

- Classical Gauss method of order 6
- Calvo method with variable c_i-values
- Calvo method with fixed c_i-values
- my 2 parameter method

Problem 1 : Kepler's problem

$$H(p,q) = \frac{1}{2}\left(p_1^2 + p_2^2\right) - \frac{1}{\sqrt{q_1^2 + q_2^2}}$$

at $t = 0 : (q_1, q_2, p_1, p_2) = \left(1 - e, 0, 0, \sqrt{\frac{1+e}{1-e}}\right)$

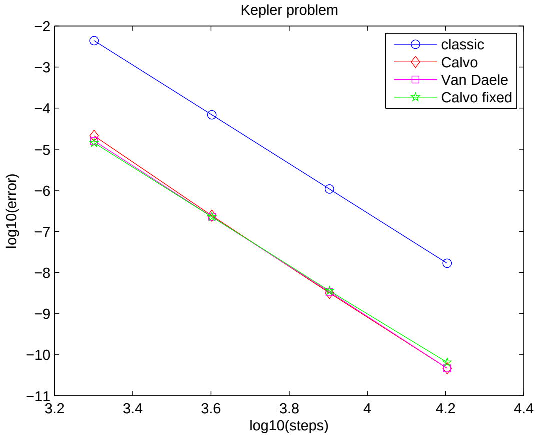
whereby $e = 0.001$

Integrated in [0, 1000] with $h = 2^{-m}$, $m = 1, \ldots, 4$.

$(q_1(t), q_2(t), p_1(t), p_2(t)) = (cos(E){-}e, \sqrt{1 - e^2}\, sin(E), q_1'(t), q_2'(t))$

whereby $t = E - e\,\sin(E)$

Problem 1 : Kepler's problem



$$z = \frac{i}{(q_1^2 + q_2^2)^{3/2}} h \qquad z_2 = z/2$$

Problem 2 : a Perturbed Kepler problem

$$H(p,q) = \frac{1}{2}\left(p_1^2 + p_2^2\right) - \frac{1}{\sqrt{q_1^2 + q_2^2}} - \frac{2\,\epsilon + \epsilon^2}{3\,\sqrt{(q_1^2 + q_2^2)^3}}$$

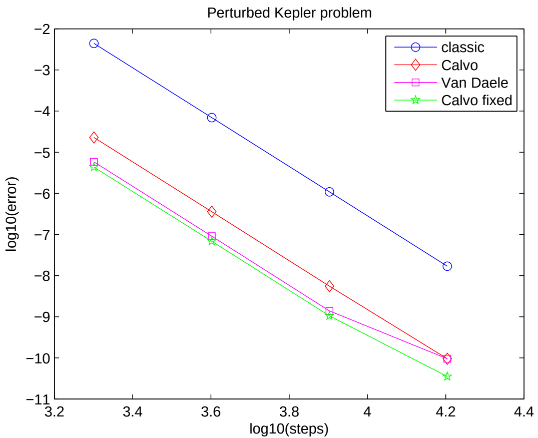
at $t = 0 : (q_1,\, q_2,\, p_1,\, p_2) = (1,\, 0,\, 0, 1 + \epsilon)$

whereby $\epsilon = 0.001$

Integrated in $[0,\, 1000]$ with $h = 2^{-m}$, $m = 1, \ldots, 4$.

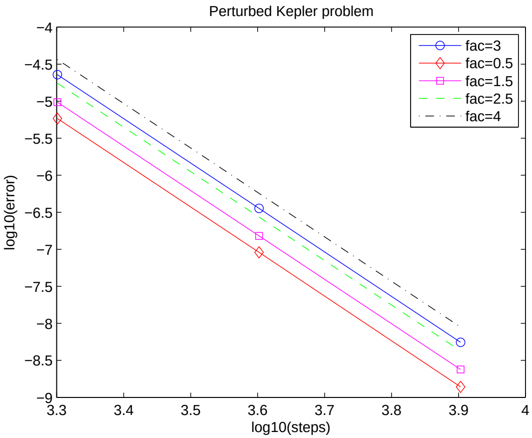
$$(q_1(t),\, q_2(t),\, p_1(t),\, p_2(t)) = (\cos((1{+}\epsilon)\,t), \sin(1{+}\epsilon)\,t),\, q_1'(t),\, q_2'(t))$$

Problem 2 : a Perturbed Kepler problem



$$z = \mathrm{i}h \qquad z_2 = z/2$$

Problem 2 : a Perturbed Kepler problem



$$z = \mathrm{i}\,h \qquad z_2 = \mathrm{fac}\, z$$

Problem 3 : Euler's problem

$$\dot{q} = ((\alpha - \beta)\, q_2\, q_3,\ (1 - \alpha)\, q_1\, q_3,\ (\beta - 1)\, q_1\, q_2)^T$$

at $t = 0 : (q_1, q_2, q_3) = (0, 1, 1)$

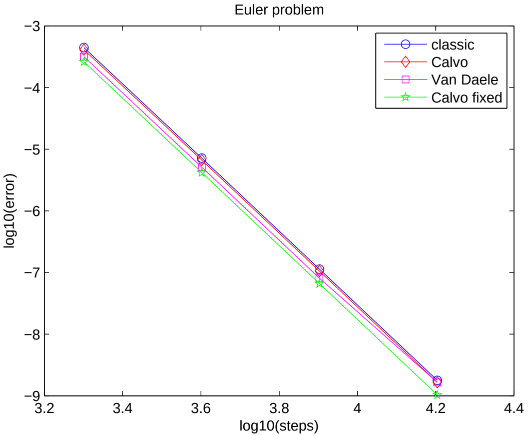
whereby $\alpha = 1 + \dfrac{1}{\sqrt{1.51}}$ and $\beta = 1 - \dfrac{0.51}{\sqrt{1.51}}$

Integrated in [0, 1000] with $h = 2^{-m}$, $m = 1, \ldots, 4$.

$$(q_1(t), q_2(t), q_3(t)) = (\sqrt{1.51}\,\mathrm{sn}(t, 0.51), \mathrm{cn}(t, 0.51), \mathrm{cn}(d, 0.51))$$

Problem is periodic with $T = 7.45056320933095$.

Problem 3 : Euler's problem



$$z = \mathrm{i}\,\frac{2\,\pi}{T}\,h \qquad z_2 = z/2$$

Conclusions

- we constructed a new family of exponentially-fitted variants of the Runge-Kutta methods of Gauss type
- these methods contain parameters $\mu_0, \mu_1, \ldots$
- special case $\mu_0 = \mu_1 = \mu_2 \ldots$ and $\mu_0 = \mu_1/2 = \mu_2/3 \ldots$ gives known families of EF methods
- open problem (needs more testing) : how to choose the parameters