Exponentially-fitted methods and their stability functions

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Outline

Collaboration with Liviu Ixaru

- 14 joint papers in the period 1995-2007
- Veerle Ledoux : development of Matslise
  title Pd.D. : Study of special algoritms for solving Sturm-Liouville and Schrödinger equations
- L. Ixaru and G. Vanden Berghe
  Exponential fitting
  Kluwer Academic Publishers, Dordrecht, 2004

Introduction

In the past years, our research group has constructed modified versions of well-known

- linear multistep methods
- Runge-Kutta methods
- ...

Aim : build methods which perform very good when the solution has a known exponential of trigonometric behaviour.

Exponentially-fitted Runge-Kutta methods

The most general form of an exponentially-fitted Runge-Kutta (EFRK) method for solving

$$y' = f(x, y)$$

is

$$y_{n+1} = \gamma\, y_n + h \sum_{i=1}^{s} b_i\, f(x_n + c_i\, h, Y_i)$$

whereby

$$Y_i = \gamma_i\, y_n + h \sum_{j=1}^{s} a_{ij}\, f(x_n + c_j\, h, Y_j)\,, \qquad i = 1, \ldots, s\,.$$

EFRK methods

Generalised Butcher tableau

$$
\begin{array}{c|c|cccc}
c_1 & \gamma_1 & a_{11} & \dots & a_{1s} \\
c_2 & \gamma_2 & a_{21} & \dots & a_{2s} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
c_s & \gamma_s & a_{s1} & \dots & a_{ss} \\
\hline
 & \gamma & b_1 & \dots & b_s
\end{array}
$$

$$
\begin{array}{c|c|c}
c & \Gamma & A \\
\hline
 & \gamma & b^T
\end{array}
$$

Construction of EFRK methods

Linear functionals

$$\begin{cases} \mathcal{L}_i[y(x);h] = y(x+c_i\,h) - \gamma_i\,y(x) - h\displaystyle\sum_{j=1}^{s} a_{ij}\,y'(x+c_j\,h) \\ \qquad i=1,\ldots,s \\ \mathcal{L}[y(x);h] = y(x+h) - \gamma\,y(x) - h\displaystyle\sum_{i=1}^{s} b_i\,y'(x+c_i\,h)\,. \end{cases}$$

A fitting space $\mathcal{S}$ is introduced such that $\forall u \in \mathcal{S}$

$$\begin{cases} \mathcal{L}_i[u(x);h] = 0 \qquad i=1,\ldots,s \\ \mathcal{L}[u(x);h] = 0 \end{cases}$$

Construction of EFRK methods

Collocation

A function $P(x) \in \mathcal{S}$ is constructed such that

$$\begin{cases} P(x_n) = y_n \\ P(x_n + c_i h)' = f(x_n + c_i h, P(x_n + c_i h)) \qquad i = 1, \ldots, s \end{cases}$$

The method is defined by $y_{n+1} := P(x_n + h)$

Construction of EFRK methods

The fitting space $\mathcal{S}$

- Vanden Berghe et al.

$$\mathcal{S} = \{x^q \, e^{\pm\omega x} | q = 0, 1, \ldots, P\} \cup \{x^q | q = 0, 1, \ldots, K\}$$

- Calvo et al.

$$\mathcal{S} = \{e^{\pm q\omega x} | q = 1, \ldots, P+1\} \cup \{x^q | q = 0, 1, \ldots, K\}$$

The coefficients of the method then depend upon $z_0 := \omega \, h$

- Both Vanden Berghe and Calve consider special cases of :

$$\mathcal{S} = \{e^{\omega_q x} | q = 1, \ldots, s+1\}$$

$$z_0 := (\omega_1 \, h, \omega_2 \, h, \ldots, \omega_{s+1} \, h)$$

The stability function of EFRK methods

The stability function $R(z, z_0)$ of an EFRK method is obtained by applying the EFRK method to

$$y' = \lambda y .$$

One obtains

$$y_{n+1} = R(z, z_0)\, y_n$$

whereby

$$R(z, z_0) = \gamma + z\, b^T\, (I - z\, A)^{-1}\, \Gamma$$

is a rational function in $z := \lambda h$ with coefficients that depend upon $z_0 := \omega h$. When $z_0 \to 0$ one obtains

$$R(z) = 1 + z\, b^T\, (I - z\, A)^{-1}\, e_s = \mathrm{e}^z + \mathcal{O}(z^{p+1})$$

where e_s is the vector of length s with unit entries and $s \leq p \leq 2\, s$.

Example : 1-stage methods

- method 1 : $\mathcal{S}_{2,0}(\omega) = \text{Span}\{1, x\}$

$$\begin{array}{c|c|c} c_1 & 1 & c_1 \\ \hline & 1 & 1 \end{array}$$

- method 2 : $\mathcal{S}_{1,1}(\omega) = \text{Span}\{1, e^{\omega x}\}$

$$\begin{array}{c|c|c} c_1 & 1 & \frac{1-e^{-c_1 z_0}}{z_0} \\ \hline & 1 & \frac{e^{(1-c_1) z_0}-e^{-c_1 z_0}}{z_0} \end{array}$$

- method 3 : $\mathcal{S}_{0,2}(\omega) = \text{Span}\{e^{\omega x}, x\, e^{\omega x}\}$

$$\begin{array}{c|c|c} c_1 & \frac{e^{c_1 z_0}}{1+c_1 z_0} & \frac{c_1}{1+c_1 z_0} \\ \hline & \frac{1-(1-c_1) z_0}{1+c_1 z_0} e^{z_0} & \frac{e^{(1-c_1) z_0}}{1+c_1 z_0} \end{array}$$

Example : 1-stage methods

- method 1 : $\mathcal{S}_{2,0}(\omega) = \text{Span}\{1, x\}$

$$R^{c_1}_{2,0}(z) = \frac{1 + (1 - c_1)\, z}{1 - c_1\, z}$$

- method 2 : $\mathcal{S}_{1,1}(\omega) = \text{Span}\{1, e^{\omega x}\}$

$$R^{c_1}_{1,1}(z, z_0) = \frac{1 + \frac{e^{(1-c_1)\, z_0} - 1}{z_0} z}{1 - \frac{1 - e^{-c_1\, z_0}}{z_0} z}$$

- method 3 : $\mathcal{S}_{0,2}(\omega) = \text{Span}\{e^{\omega x}, x\, e^{\omega x}\}$

$$R^{c_1}_{0,2}(z, z_0) = e^{z_0}\, \frac{1 + (1 - c_1)\, (z - z_0)}{1 - c_1\, (z - z_0)}$$

Example : 1-stage methods

method 1 : $R^{c_1}_{2,0}(z) = \frac{1+(1-c_1)\,z}{1-c_1\,z}$

method 3 : $R^{c_1}_{0,2}(z,z_0) = e^{z_0}\,\dfrac{1+(1-c_1)\,(z-z_0)}{1-c_1\,(z-z_0)}$

$$R^{c_1}_{0,2}(z,z_0) = e^{z_0} R^{c_1}_{2,0}(z-z_0)$$

$$\left|\frac{R^{c_1}_{0,2}(z,z_0)}{e^{z}}\right| = \left|\frac{R^{c_1}_{2,0}(z-z_0)}{e^{z-z_0}}\right|$$

It follows that the orders stars are (apart from a shift over a distance z_0) equal to each other.

General property for s-stage methods

Suppose a method $M_{k,l}$ is a EFRK method with fitting space

$$\mathcal{S}_{k,l}(\omega) = \text{Span}\{1, x, \ldots, x^{k-1}, e^{\omega x}, x\,e^{\omega x}, \ldots, x^{l-1}e^{\omega x}\}$$

$$y' = \lambda y \Longrightarrow y_{n+1} = R_{k,l}(z, z_0)\, y_n$$

Lawson : $u(x) = e^{-\omega x} y(x) \Longrightarrow u' = (\lambda - \omega)\, u$

$$\begin{aligned} y \in \mathcal{S}_{k,l}(\omega) \quad &\Longrightarrow \quad u \in \mathcal{S}_{l,k}(-\omega) \\ &\Longrightarrow \quad u_{n+1} = R_{l,k}(z - z_0, -z_0)\, u_n \\ &\Longrightarrow \quad y_{n+1} = e^{z_0} R_{l,k}(z - z_0, -z_0)\, y_n \end{aligned}$$

General property for s-stage methods

$$R_{k,l}(z, z_0) = e^{z_0} R_{l,k}(z - z_0, -z_0)$$

For the corresponding order star, this then means

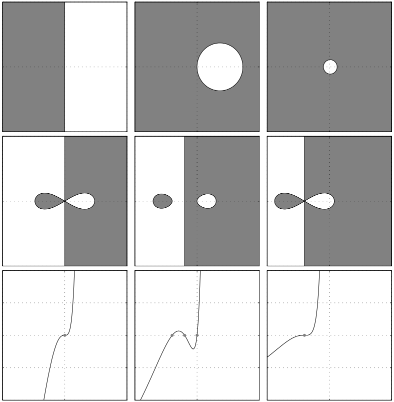
$$\left| \frac{R_{k,l}(z, z_0)}{e^z} \right| = \left| \frac{R_{l,k}(z - z_0, -z_0)}{e^{z - z_0}} \right|$$

Example : 1-stage methods

stability function
$(x, y) \in [-5, 5]^2$

order star
$(x, y) \in [-5, 5]^2$

$R_{i,j}(x, -1) - e^x$
$x \in [-5, 5]$
$y \in [-0.1, 0.1]$



$R^{c_1}_{2,0}$ $R^{c_1}_{1,1}$ $R^{c_1}_{0,2}$
$z_0 = -1$ and $c_1 = 0.5$

The stability function of an EFRK

Questions :

1. Can we determine the explicit form of the stability function $R(z, z_0)$ of an EFRK without computing the method ?
2. Which conditions does one have to impose to a rational function to obtain the stability function of an EFRK ?
3. How to construct an EFRK with a given stability function ?

The stability function of a classical RK

1. Can we determine the explicit form of the stability function $R(z)$ of a classical RK without computing the method?
   sometimes
2. Which conditions does one have to impose to a rational function to obtain the stability function of a RK ?
   order $p \iff R^{(i)}(0) = 1, i = 0, 1, \ldots, p$
3. How to construct a RK with a given stability function ?
   linear functionals, collocation

The stability function of an EFRK

Questions :

1. Can we determine the explicit form of the stability function $R(z, z_0)$ of an EFRK without computing the method? Which conditions does one have to impose to a rational function to obtain the stability function of an EFRK?
2. How to construct an EFRK with a given stability function ?

Which conditions to impose

Suppose an EFRK method is fitted to $e^{\omega x}$
Consider the test equation $y' = \lambda y$. This leads to

$$y_{n+1} = R(z, z_0)\, y_n$$

whereby $z = \lambda h$ and $z_0 = \omega h$.
If $\lambda = \omega$, then

$$y_{n+1} = R(z_0, z_0)\, y_n = y(x_{n+1}) = e^{z_0}\, y_n\,,$$

so

$$R(z_0, z_0) = e^{z_0} \quad \text{or} \quad R(z, \omega h)\big|_{z=\omega h} = e^{\omega h}$$

Which conditions to impose?

Generalisation:

For an EFRK method that is fitted to the functions $e^{\omega_q x}$, $q = 0, 1, \ldots, P$ the conditions that should be imposed, can be written down as

$$R(z, \omega_q h)\big|_{z=\omega_q h} = e^{\omega_q h} \qquad q = 0, 1, \ldots, P$$

Which conditions to impose?

Special case : what if two parameters coincide?

Suppose $R(z_0, \{z_0, z_0'\}) = e^{z_0}$ and $R(z_0', \{z_0, z_0'\}) = e^{z_0'}$.

Then, when $z_0' \to z_0$, one obtains

$$
\begin{aligned}
\frac{\partial}{\partial z} R(z, z_0)\Big|_{z=z_0} &= \lim_{z_0' \to z_0} \frac{R(z_0, \{z_0, z_0'\}) - R(z_0', \{z_0, z_0'\})}{z_0 - z_0'} \\
&= \lim_{z_0' \to z_0} \frac{e^{z_0} - e^{z_0'}}{z_0 - z_0'} \\
&= e^{z_0}
\end{aligned}
$$

$$
R(z, z_0)\Big|_{z=z_0} = \frac{\partial}{\partial z} R(z, z_0)\Big|_{z=z_0} = e^{z_0}
$$

Which conditions to impose?

- Suppose $\omega_0 = \omega_1 = \ldots = \omega_P = \omega$:
- For an EFRK method that is fitted to the functions $x^q\, \mathrm{e}^{\omega x}$, $q = 0, 1, \ldots, P$ the conditions that should be imposed, can be written down as

$$\frac{\partial^q}{\partial^q z} R(z, z_0)\Big|_{z=z_0} = \mathrm{e}^{z_0} \qquad q = 0, 1, \ldots, P$$

- Special case : $\omega = 0$ (as in the classical case):

$$\frac{\partial^q}{\partial^q z} R(z, 0)\Big|_{z=0} = 1 \qquad q = 0, 1, \ldots, P$$

i.e $R(z) = \mathrm{e}^z + \mathcal{O}(z^{P+1})$

Which conditions to impose?

In particular, an EFRK method that is fitted to the space of functions $\{1, x, \ldots, x^{P_1}\} \cup \{x^q\, e^{\omega x} | q = 0, 1, \ldots, P_2\}$, has to satisfy:

$$\begin{cases} \frac{\partial^q}{\partial^q z} R(z, \{z_0, 0\})\Big|_{z=0} = 1 & q = 0, 1, \ldots, P_1 \\ \frac{\partial^q}{\partial^q z} R(z, \{z_0, 0\})\Big|_{z=z_0} = e^{z_0} & q = 0, 1, \ldots, P_2 \,. \end{cases}$$

Example : 1-stage methods

- For a one-stage method

$$R(z,z_0) = \frac{a_0 + a_1\, z}{1 + b_1\, z}\ ,$$

  where a_0, a_1 and b_1 can depend upon z_0
- 3 coefficients, so we can impose 3 conditions
- If $\mathcal{S} = \{1, x, \ldots, x^{i-1}\} \cup \{e^{\omega x}, x\, e^{\omega x}, \ldots, x^{j-1}\, e^{\omega x}\}$ then the conditions are

$$\begin{cases} \frac{\partial^q}{\partial^q z} R(z,z_0)\big|_{z=0} = 1 & q = 0, \ldots, i-1 \\ \frac{\partial^q}{\partial^q z} R(z,z_0)\big|_{z=z_0} = e^{z_0} & q = 0 \ldots, j-1 \end{cases}$$

- Four different functions $R_{i,j}(z,z_0)$

Example : 1-stage methods

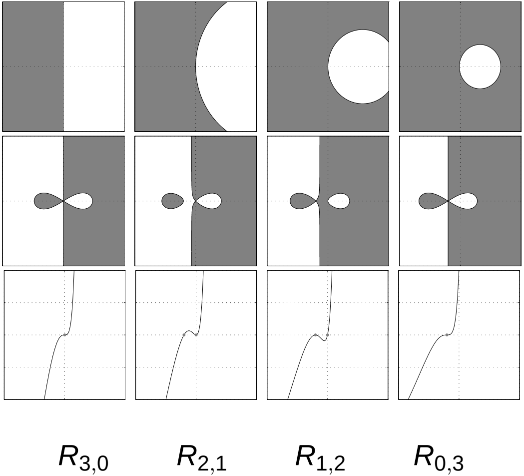
- $R_{3,0}(z,z_0) = \dfrac{1+\frac{z}{2}}{1-\frac{z}{2}}$
- $R_{2,1}(z,z_0) = \dfrac{1+\frac{1-e^{z_0}+e^{z_0}z_0}{z_0\,(e^{z_0}-1)}z}{1+\frac{1+z_0-e^{z_0}}{z_0\,(e^{z_0}-1)}z}$
- $R_{1,2}(z,z_0) = \dfrac{1-\frac{1+z_0-e^{z_0}}{z_0^2}z}{1-\frac{e^{-z_0}-1+z_0}{z_0^2}z}$
- $R_{0,3}(z,z_0) = e^{z_0}\,\dfrac{1+\frac{z-z_0}{2}}{1-\frac{z-z_0}{2}}$

Example : 1-stage methods

stability function
$(x, y) \in [-5, 5]^2$

order star
$(x, y) \in [-5, 5]^2$

$R_{i,j}(x, -1) - e^x$
$x \in [-5, 5]$
$y \in [-0.1, 0.1]$



$z_0 = -1$

The stability function of an EFRK

Questions :

1. Can we determine the explicit form of the stability function $R(z, z_0)$ of an EFRK without computing the method? Which conditions does one have to impose to a rational function to obtain the stability function of an EFRK?
2. How to construct an EFRK with a given stability function?

The stability function of an EFRK

Questions :

1. Can we determine the explicit form of the stability function $R(z, z_0)$ of an EFRK without computing the method? Which conditions does one have to impose to a rational function to obtain the stability function of an EFRK?
2. How to construct an EFRK with a given stability function?
   - integration factor methods
   - exponential collocation
   - linear functionals

1. Integrating factor methods

We start from the equation

$$y' = f(x, y)$$

which we rewrite as

$$y' - \omega y = f(x, y) - \omega y = \tilde{f}(x, y)$$

Lawson : if $u(x) = e^{-\omega x} y(x)$ then

$$u' = g(x, u)$$

where

$$g(x, u) = e^{-\omega x} \tilde{f}(x, e^{\omega x} u)$$

1. Integrating factor methods

Apply any Runge-Kutta method defined by (A, b, c) to $u' = g(x, u)$:

$$u_{n+1} = u_n + h \sum_{j=1}^{s} b_j K_j$$

where $K_i = g(x_n + c_i h,\, u_n + h \sum_{j=1}^{s} a_{ij} K_j)$

Expressed in terms of y and $\tilde{f}$, this then gives

$$y_{n+1} = e^{\omega h} y_n + h \sum_{i=1}^{s} b_i e^{\omega (1-c_i) h} k_i$$

$$k_i = \tilde{f}(x_n + c_i h,\, e^{\omega c_i h} y_n + h \sum_{j=1}^{s} a_{ij}\, e^{\omega (c_i - c_j h} k_j) \qquad i = 1, \ldots, s$$

Example : 1-stage method

Applying

$$\begin{array}{c|c|c} c_1 & 1 & c_1 \\ \hline & 1 & 1 \end{array}$$

to $u' = g(x, u)$, and expressed in terms of y and $\tilde{f}$ gives

$$Y_1 = e^{c_1 \omega h} y_n + h\, c_1 \tilde{f}(x_n + c_1 h, Y_1)$$
$$y_{n+1} = e^{\omega h} y_n + h\, e^{(1-c_1)\omega h} \tilde{f}(x_n + c_1 h, Y_1)$$

For $y' = f(x, y)$ this method is identical to method 3 defined by

$$\begin{array}{c|c|c} c_1 & \frac{e^{c_1 z_0}}{1+c_1 z_0} & \frac{c_1}{1+c_1 z_0} \\ \hline & \frac{1-(1-c_1) z_0}{1+c_1 z_0} e^{z_0} & \frac{\exp((1-c_1) z_0)}{1+c_1 z_0} \end{array} \qquad z_0 := \omega h$$

Stability function of IF methods

$$y' = \lambda y$$

Lawson's transformation : $u(x) = e^{-\omega x} y(x)$

$$u' = (\lambda - \omega) u$$

Suppose a purely polynomial method M with stability function $R_M(z)$ is applied, then

$$u_{n+1} = R_M(z - z_0) u_n$$

Re-expressed in terms of the y-variable, this gives

$$y_{n+1} = e^{z_0} R_M(z - z_0) y_n$$

$$R(z, z_0) = e^{z_0} R_M(z - z_0)$$

2. Exponential collocation methods

$$y' - \omega y = f(x, y) - \omega y = \tilde{f}(x, y)$$

A function $P(x) \in \mathcal{S}$ is constructed such that for $Q(x) := e^{\omega x}(e^{-\omega x} P(x))'$

$$\begin{cases} P(x_n) = y_n \\ Q(x_n + c_i h) = \tilde{f}(x_n + c_i h, P(x_n + c_i h)) \qquad i = 1, \ldots, s \end{cases}$$

The method is defined by $y_{n+1} := P(x_n + h)$.

2. Exponential collocation methods

$$(e^{-\omega x}P(x))' = e^{-\omega x}Q(x)$$

$$\int_{x_n}^{x_n+t\,h} d(e^{-\omega x}P(x)) = \int_{x_n}^{x_n+t\,h} e^{-\omega x}Q(x)dx$$

$$P(x_n+t\,h) = e^{t\,\omega\,h}P(x_n) + h\int_0^t e^{\omega\,(t-\tau)h}Q(x_n+\tau\,h)d\tau$$

$$Q(x_n+\tau\,h) = \sum_{j=1}^{s} l_j(\tau)k_j \qquad k_i := \tilde{f}(x_n+c_i\,h, P(x_n+c_i\,h))$$

2. Exponential collocation methods

$$P(x_n + t\,h) = \mathrm{e}^{t\,\omega\,h} P(x_n) + h \int_0^t \mathrm{e}^{\omega\,(t-\tau)h} Q(x_n + \tau\,h)\mathrm{d}\tau$$

$$Q(x_n + \tau\,h) = \sum_{j=1}^{s} l_j(\tau) k_j \qquad k_i := \tilde{f}(x_n + c_i\,h, P(x_n + c_i\,h))$$

$$\begin{cases} P(x_n + c_i\,h) = \mathrm{e}^{c_i\,\omega\,h} P(x_n) + h \sum_{j=1}^{s} a_{ij}\,k_j \quad i = 1, \ldots, s \\ P(x_n + h) = \mathrm{e}^{\omega\,h} P(x_n) + h \sum_{j=1}^{s} b_j\,k_j \end{cases}$$

$$a_{ij} := \int_0^{c_i} \mathrm{e}^{\omega\,(c_i-\tau)h} l_j(\tau)\mathrm{d}\tau \qquad \text{and} \qquad b_j := \int_0^1 \mathrm{e}^{\omega\,(1-\tau)h} l_j(\tau)\mathrm{d}\tau$$

2. Exponential collocation methods

The exponential collocation method for the problem

$$y' - \omega y = \tilde{f}(x, y)$$

is thus given by

$$y_{n+1} = e^{\omega h} y_n + h \sum_{i=1}^{s} b_i k_i$$

with

$$k_i = \tilde{f}(x_n + c_i h, e^{c_i \omega h} y_n + h \sum_{j=1}^{s} a_{ij} k_j) \quad i = 1, \ldots, s$$

Example 1: polynomial interpolation

Suppose $\mathcal{S}_Q = \Pi_{s-1}$ (space of polynomials of degree $\leq s-1$)

$$P' - \omega P = Q(x) \Longrightarrow P(x) = \alpha\, e^{\omega x} + \tilde{Q}(x)$$

where α is a constant and $\tilde{Q} \in \Pi_{s-1}$.

$$P(x) \in \Pi_{s-1} \cup \text{Span}\{e^{\omega x}\}$$

$$\tau^q = \sum_{j=1}^{s} l_j(\tau) c_j^q \qquad q = 0, \ldots, s-1$$

$$\int_0^{c_i} e^{\omega (c_i - \tau) h} \tau^q d\tau = \sum_{j=1}^{s} \int_0^{c_i} e^{\omega (c_i - \tau) h} l_j(\tau) d\tau c_j^q = \sum_{j=1}^{s} a_{ij} c_j^q$$

Example 1 : polynomial interpolation

$$\int_0^{c_i} e^{\omega (c_i-\tau)h}\tau^q \mathrm{d}\tau = \sum_{j=1}^{s}\int_0^{c_i} e^{\omega (c_i-\tau)h} l_j(\tau)\mathrm{d}\tau c_j^q = \sum_{j=1}^{s} a_{ij}c_j^q$$

$$\sum_{j=1}^{s} a_{ij} = \frac{e^{c_i\,\omega\,h}-1}{\omega\,h} \qquad i = 1,\ldots,s$$

$$\sum_{j=1}^{s} a_{ij}c_j^q = -\frac{c_i^q}{\omega h}+\frac{q}{\omega h}\sum_{j=1}^{s} a_{ij}c_j^{q-1} \qquad q = 1,\,2,\ldots s-1; i = 1,\ldots,s$$

One-stage method

$$Y_1 = e^{c_1\,\omega\,h}\,y_n + h\,\frac{e^{c_1\,\omega\,h}-1}{\omega\,h}\tilde{f}(x_n + c_1\,h, Y_1)$$

$$y_{n+1} = e^{\omega\,h}\,y_n + h\,\frac{e^{\omega\,h}-1}{\omega\,h}\tilde{f}(x_n + c_1\,h, Y_1)$$

For $y' = f(x, y)$ this method is identical to method 2 defined by

$$\begin{array}{c|c|c} c_1 & 1 & \frac{1-e^{-c_1\,z_0}}{z_0} \\ \hline & 1 & \frac{e^{(1-c_1)\,z_0}-e^{-c_1\,z_0}}{z_0} \end{array}$$

Example 2 : exponential interpolation

Suppose $\mathcal{S}_Q = e^{\omega x}\Pi_{s-1}$ (functions of the form $e^{\omega x} p_{s-1}(x)$ where $p_{s-1}(x) \in \Pi_{s-1}$)

$$P' - \omega P = Q(x) \Longrightarrow P(x) = e^{\omega x} p_s(x)$$

with $p_s(x) \in \Pi_s$ (in fact, $p_s'(x) = p_{s-1}(x)$)

$$P(x) \in e^{\omega x}\Pi_s$$

The collocation conditions then become

$$p_s'(x) = e^{-\omega x} f(x, e^{\omega x} p_s(x)) = g(x, p_s(x))$$

This is the classical polynomial collocation method $u' = g(x, u)$
The resulting method will be the same as the IF method.

3. Linear functionals

$$\mathcal{L}_i[y(x);h] := y(x_n+c_i\,h)-\gamma_i\,y_n-h\sum_{j=1}^{s} a_{ij}(y'(x_n+c_j\,h)-\omega\,y(x_n+c_j\,h))$$

$$i=1,\ldots,s$$

$$\mathcal{L}[y(x);h] := y(x_n+h)-\gamma\,y_n-h\sum_{i=1}^{s} b_i(y'(x_n+c_i\,h)-\omega\,y(x_n+c_i\,h))$$

Require that

$$\begin{cases} \mathcal{L}_i[u(x);h]=0 & i=1,\ldots,s \\ \mathcal{L}[u(x);h]=0 \end{cases}$$

for $u(x)=e^{\omega x}$ and for each function u in the s dimensional space $\mathcal{S}_Q$

Conclusions

- We have analysed properties of stability functions of EFRK methods.
- Whereas purely polynomial methods impose conditions on the stability function $R(z)$ for $z = 0$ solely, EFRK that are fitted for parameter values $\omega_1, \omega_2, \ldots, \omega_n$ impose conditions for $z = \omega_1 h, \ldots, z = \omega_n h$ on the stability function $R(z, \{z_1, \ldots, z_n\})$.
- Nice relations exist between the different stability functions and, more in particular, between the corresponding order stars.
- The stability functions of integrating factor methods and exponential collocation methods were considered
- Exponential-fitting, integrating factor and exponential collocation can lead to the same method