Exponentially-fitted methods and their stability functions

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Tenth International Conference of Numerical Analysis and Applied Mathematics, Kos, 2012
Outline

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- EFRK methods

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- Ways to construct EFRK methods
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Collaboration with Liviu Ixaru

- 14 joint papers in the period 1995-2007
- **Veerle Ledoux**: development of **Matslise**
  title Pd.D. : Study of special algorithms for solving Sturm-Liouville and Schrödinger equations

- [L. Ixaru and G. Vanden Berghe](#)
  *Exponential fitting*
Introduction

In the past years, our research group has constructed modified versions of well-known
- linear multistep methods
- Runge-Kutta methods
- …

Aim: build methods which perform very good when the solution has a known exponential of trigonometric behaviour.
Exponentially-fitted Runge-Kutta methods

The most general form of an exponentially-fitted Runge-Kutta (EFRK) method for solving

\[ y' = f(x, y) \]

is

\[ y_{n+1} = \gamma y_n + h \sum_{i=1}^{s} b_i f(x_n + c_i h, Y_i) \]

whereby

\[ Y_i = \gamma_i y_n + h \sum_{j=1}^{s} a_{ij} f(x_n + c_j h, Y_j), \quad i = 1, \ldots, s. \]
## EFRK methods

**Generalised Butcher tableau**

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<th>$a_{11}$</th>
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<tr>
<td>$\gamma$</td>
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Construction of EFRK methods

Linear functionals

\[
\begin{align*}
\mathcal{L}_i[y(x); h] &= y(x + c_i h) - \gamma_i y(x) - h \sum_{j=1}^{s} a_{ij} y'(x + c_j h) \\
& \quad i = 1, \ldots, s \\
\mathcal{L}[y(x); h] &= y(x + h) - \gamma y(x) - h \sum_{i=1}^{s} b_i y'(x + c_i h).
\end{align*}
\]

A fitting space \( S \) is introduced such that \( \forall u \in S \)

\[
\begin{align*}
\mathcal{L}_i[u(x); h] &= 0 \quad i = 1, \ldots, s \\
\mathcal{L}[u(x); h] &= 0
\end{align*}
\]
Construction of EFRK methods

Collocation

A function $P(x) \in S$ is constructed such that

$$
\begin{aligned}
P(x_n) &= y_n \\
P(x_n + c_i h)' &= f(x_n + c_i h, P(x_n + c_i h)) & \quad i = 1, \ldots, s
\end{aligned}
$$

The method is defined by $y_{n+1} := P(x_n + h)$
Construction of EFRK methods

The fitting space $S$

- Vanden Berghe et al.
  
  $$S = \left\{ x^q e^{\pm \omega x} \mid q = 0, 1, \ldots, P \right\} \cup \left\{ x^q \mid q = 0, 1, \ldots, K \right\}$$

- Calvo et al.
  
  $$S = \left\{ e^{\pm q \omega x} \mid q = 1, \ldots, P + 1 \right\} \cup \left\{ x^q \mid q = 0, 1, \ldots, K \right\}$$

The coefficients of the method then depend upon $z_0 := \omega h$

- Both Vanden Berghe and Calve consider special cases of:

  $$S = \left\{ e^{\omega q x} \mid q = 1, \ldots, s + 1 \right\}$$

  $$z_0 := (\omega_1 h, \omega_2 h, \ldots, \omega_{s+1} h)$$
The stability function of EFRK methods

The stability function $R(z, z_0)$ of an EFRK method is obtained by applying the EFRK method to

$$y' = \lambda y.$$ 

One obtains

$$y_{n+1} = R(z, z_0) y_n$$

whereby

$$R(z, z_0) = \gamma + z b^T (I - z A)^{-1} \Gamma$$

is a rational function in $z := \lambda h$ with coefficients that depend upon $z_0 := \omega h$. When $z_0 \to 0$ one obtains

$$R(z) = 1 + z b^T (I - z A)^{-1} e_s = e^z + O(z^{p+1})$$

where $e_s$ is the vector of length $s$ with unit entries and $s \leq p \leq 2s$. 


Example: 1-stage methods

- **method 1**: \( S_{2,0}(\omega) = \text{Span}\{1, x\} \)

\[
\begin{array}{c|cc|c}
0 & 1 & 0 \\
\hline
1 & 1 & 1
\end{array}
\]

- **method 2**: \( S_{1,1}(\omega) = \text{Span}\{1, e^{\omega x}\} \)

\[
\begin{array}{c|cc|c}
0 & 1 & 0 \\
\hline
1 & 1 & 1
\end{array}
\]

- **method 3**: \( S_{0,2}(\omega) = \text{Span}\{e^{\omega x}, x e^{\omega x}\} \)

\[
\begin{array}{c|cc|c}
0 & 1 & 0 \\
\hline
1 & 1 & 1
\end{array}
\]
Example: 1-stage methods

- **method 1**: $S_{2,0}(\omega) = \text{Span}\{1, x\}$
  \[ R_{2,0}^{c_1}(z) = \frac{1 + (1 - c_1) z}{1 - c_1 z} \]

- **method 2**: $S_{1,1}(\omega) = \text{Span}\{1, e^{\omega x}\}$
  \[ R_{1,1}^{c_1}(z, z_0) = \frac{1 + e^{(1-c_1)z_0-1} z}{1 - e^{-c_1 z_0} z} \]

- **method 3**: $S_{0,2}(\omega) = \text{Span}\{e^{\omega x}, x e^{\omega x}\}$
  \[ R_{0,2}^{c_1}(z, z_0) = e^{z_0} \frac{1 + (1 - c_1) (z - z_0)}{1 - c_1 (z - z_0)} \]
Example: 1-stage methods

method 1: \[ R_{2,0}^{c_1}(z) = \frac{1+(1-c_1)z}{1-c_1 z} \]

method 3: \[ R_{0,2}^{c_1}(z, z_0) = e^{z_0} \frac{1 + (1 - c_1)(z - z_0)}{1 - c_1 (z - z_0)} \]

\[ R_{0,2}^{c_1}(z, z_0) = e^{z_0} R_{2,0}^{c_1}(z - z_0) \]

\[ \left| \frac{R_{0,2}^{c_1}(z, z_0)}{e^z} \right| = \left| \frac{R_{2,0}^{c_1}(z - z_0)}{e^{z-z_0}} \right| \]

It follows that the orders stars are (apart from a shift over a distance \(z_0\)) equal to each other.
General property for $s$-stage methods

Suppose a method $M_{k,l}$ is an EFRK method with fitting space

$$S_{k,l}(\omega) = \text{Span}\{1, x, \ldots, x^{k-1}, e^{\omega x}, xe^{\omega x}, \ldots, x^{l-1}e^{\omega x}\}$$

$$y' = \lambda y \implies y_{n+1} = R_{k,l}(z, z_0) y_n$$

Lawson:

$$u(x) = e^{-\omega x} y(x) \implies u' = (\lambda - \omega) u$$

$$y \in S_{k,l}(\omega) \implies u \in S_{l,k}(-\omega)$$

$$\implies u_{n+1} = R_{l,k}(z - z_0, -z_0) u_n$$

$$\implies y_{n+1} = e^{z_0} R_{l,k}(z - z_0, -z_0) y_n$$
General property for $s$-stage methods

$$R_{k,l}(z, z_0) = e^{z_0} R_{l,k}(z - z_0, -z_0)$$

For the corresponding order star, this then means

$$\left| \frac{R_{k,l}(z, z_0)}{e^z} \right| = \left| \frac{R_{l,k}(z - z_0, -z_0)}{e^{z-z_0}} \right|$$
Example: 1-stage methods

stability function
\((x, y) \in [-5, 5]^2\)

order star
\((x, y) \in [-5, 5]^2\)

\[ R_{i,j}(x, -1) - e^x \]
\[ x \in [-5, 5] \]
\[ y \in [-0.1, 0.1] \]

\[ R_{2,0}^{c_1}, \quad R_{1,1}^{c_1}, \quad R_{0,2}^{c_1} \]
\[ z_0 = -1 \] and \( c_1 = 0.5 \)
Questions:

1. Can we determine the explicit form of the stability function $R(z, z_0)$ of an EFRK without computing the method?
2. Which conditions does one have to impose to a rational function to obtain the stability function of an EFRK?
3. How to construct an EFRK with a given stability function?
The stability function of a classical RK

1. Can we determine the explicit form of the stability function \( R(z) \) of a classical RK without computing the method? sometimes

2. Which conditions does one have to impose to a rational function to obtain the stability function of a RK?
   order \( p \) \( \iff \) \( R^{(i)}(0) = 1, i = 0, 1, \ldots, p \)

3. How to construct a RK with a given stability function?
   linear functionals, collocation
The stability function of an EFRK

Questions:

1. Can we determine the explicit form of the stability function $R(z, z_0)$ of an EFRK without computing the method? Which conditions does one have to impose to a rational function to obtain the stability function of an EFRK?

2. How to construct an EFRK with a given stability function?
Suppose an EFRK method is fitted to $e^{\omega x}$

Consider the test equation $y' = \lambda y$. This leads to

$$y_{n+1} = R(z, z_0) \ y_n$$

whereby $z = \lambda h$ and $z_0 = \omega h$.

If $\lambda = \omega$, then

$$y_{n+1} = R(z_0, z_0) \ y_n = y(x_{n+1}) = e^{z_0} \ y_n,$$

so

$$R(z_0, z_0) = e^{z_0} \quad \text{or} \quad R(z, \omega h)\big|_{z=\omega h} = e^{\omega h}$$
Which conditions to impose?

Generalisation:

For an EFRK method that is fitted to the functions $e^{\omega_q x}$, $q = 0, 1, \ldots, P$ the conditions that should be imposed, can be written down as

$$R(z, \omega_q h) \bigg|_{z=\omega_q h} = e^{\omega_q h} \quad q = 0, 1, \ldots, P$$
Which conditions to impose?

Special case: what if two parameters coincide?

Suppose \( R(z_0, \{z_0, z'_0\}) = e^{z_0} \) and \( R(z'_0, \{z_0, z'_0\}) = e^{z'_0} \).

Then, when \( z'_0 \to z_0 \), one obtains

\[
\frac{\partial}{\partial z} R(z, z_0) \bigg|_{z=z_0} = \lim_{z'_0 \to z_0} \frac{R(z_0, \{z_0, z'_0\}) - R(z'_0, \{z_0, z'_0\})}{z_0 - z'_0} = \lim_{z'_0 \to z_0} \frac{e^{z_0} - e^{z'_0}}{z_0 - z'_0} = e^{z_0}
\]

\[
R(z, z_0) \bigg|_{z=z_0} = \frac{\partial}{\partial z} R(z, z_0) \bigg|_{z=z_0} = e^{z_0}
\]
Which conditions to impose?

• Suppose $\omega_0 = \omega_1 = \ldots = \omega_P = \omega$:

• For an EFRK method that is fitted to the functions $x^q e^{\omega x}$, $q = 0, 1, \ldots, P$ the conditions that should be imposed, can be written down as

$$\frac{\partial^q}{\partial q} R(z, z_0) \bigg|_{z=z_0} = e^{z_0} \quad q = 0, 1, \ldots, P$$

• Special case: $\omega = 0$ (as in the classical case):

$$\frac{\partial^q}{\partial q} R(z, 0) \bigg|_{z=0} = 1 \quad q = 0, 1, \ldots, P$$

i.e $R(z) = e^z + \mathcal{O}(z^{P+1})$
Which conditions to impose?

In particular, an EFRK method that is fitted to the space of functions \( \{1, x, \ldots, x^{p_1}\} \cup \{x^q e^{\omega x} | q = 0, 1, \ldots, p_2\} \), has to satisfy:

\[
\begin{align*}
\frac{\partial^q}{\partial q z} R(z, \{z_0, 0\}) \bigg|_{z=0} &= 1 & q = 0, 1, \ldots, p_1 \\
\frac{\partial^q}{\partial q z} R(z, \{z_0, 0\}) \bigg|_{z=z_0} &= e^{z_0} & q = 0, 1, \ldots, p_2.
\end{align*}
\]
Example : 1-stage methods

- For a one-stage method

\[ R(z, z_0) = \frac{a_0 + a_1 z}{1 + b_1 z} , \]

where \( a_0, a_1 \) and \( b_1 \) can depend upon \( z_0 \)
- 3 coefficients, so we can impose 3 conditions
- If \( S = \{1, x, \ldots, x^{i-1}\} \cup \{e^{\omega x}, xe^{\omega x}, \ldots, x^{j-1}e^{\omega x}\} \) then the conditions are

\[
\begin{align*}
\left. \frac{\partial^q}{\partial q z} R(z, z_0) \right|_{z=0} &= 1 & q = 0, \ldots, i - 1 \\
\left. \frac{\partial^q}{\partial q z} R(z, z_0) \right|_{z=z_0} &= e^{z_0} & q = 0 \ldots, j - 1
\end{align*}
\]
- Four different functions \( R_{i,j}(z, z_0) \)
Example: 1-stage methods

- \( R_{3,0}(z, z_0) = \frac{1 + \frac{z}{2}}{1 - \frac{z}{2}} \)

- \( R_{2,1}(z, z_0) = \frac{1 + \frac{1 - e^{z_0} + e^{z_0} z_0}{z_0 (e^{z_0} - 1)} z}{1 + \frac{1 + z_0 - e^{z_0}}{z_0 (e^{z_0} - 1)} z} \)

- \( R_{1,2}(z, z_0) = \frac{1 - \frac{1 + z_0 - e^{z_0}}{z_0 (e^{z_0} - 1)} z}{1 - \frac{e^{-z_0} - 1 + z_0}{z_0^2} z} \)

- \( R_{0,3}(z, z_0) = e^{z_0} \frac{1 + \frac{z - z_0}{2}}{1 - \frac{z - z_0}{2}} \)
Example: 1-stage methods

stability function
$(x, y) \in [-5, 5]^2$

order star
$(x, y) \in [-5, 5]^2$

$R_{i,j}(x, -1) - e^x$
$x \in [-5, 5]$
$y \in [-0.1, 0.1]$

$R_{3,0}$  $R_{2,1}$  $R_{1,2}$  $R_{0,3}$

$z_0 = -1$
The stability function of an EFRK

Questions:

1. Can we determine the explicit form of the stability function $R(z, z_0)$ of an EFRK without computing the method? Which conditions does one have to impose to a rational function to obtain the stability function of an EFRK?

2. How to construct an EFRK with a given stability function?
The stability function of an EFRK

Questions:

1. Can we determine the explicit form of the stability function $R(z, z_0)$ of an EFRK without computing the method? Which conditions does one have to impose to a rational function to obtain the stability function of an EFRK?

2. How to construct an EFRK with a given stability function?
   - integration factor methods
   - exponential collocation
   - linear functionals
1. Integrating factor methods

We start from the equation

$$y' = f(x, y)$$

which we rewrite as

$$y' - \omega y = f(x, y) - \omega y = \tilde{f}(x, y)$$

**Lawson**: if $u(x) = e^{-\omega x} y(x)$ then

$$u' = g(x, u)$$

where

$$g(x, u) = e^{-\omega x} \tilde{f}(x, e^{\omega x} u)$$
1. Integrating factor methods

Apply any Runge-Kutta method defined by \((A, b, c)\) to \(u' = g(x, u)\):

\[
u_{n+1} = u_n + h \sum_{j=1}^{s} b_j K_j
\]

where \(K_i = g(x_n + c_i h, u_n + h \sum_{j=1}^{s} a_{ij} K_j)\)

Expressed in terms of \(y\) and \(\tilde{f}\), this then gives

\[
y_{n+1} = e^{\omega h} y_n + h \sum_{i=1}^{s} b_i e^{\omega (1-c_i) h} k_i
\]

\[
k_i = \tilde{f}(x_n + c_i h, e^{\omega c_i h} y_n + h \sum_{j=1}^{s} a_{ij} e^{\omega (c_i-c_j h) k_j}) \quad i = 1, \ldots, s
\]
Example: 1-stage method

Applying

\[
\begin{array}{c|c|c}
  c_1 & 1 & c_1 \\
  \hline
  1 & 1 & 1 \\
\end{array}
\]

to \( u' = g(x, u) \), and expressed in terms of \( y \) and \( \tilde{f} \) gives

\[
Y_1 = e^{c_1 \omega h} y_n + h c_1 \tilde{f}(x_n + c_1 h, Y_1)
\]
\[
y_{n+1} = e^{\omega h} y_n + h e^{(1-c_1)\omega h} \tilde{f}(x_n + c_1 h, Y_1)
\]

For \( y' = f(x, y) \) this method is identical to method 3 defined by

\[
\begin{array}{c|c|c|c|c}
  c_1 & e^{c_1 z_0} & \frac{c_1}{1+c_1 z_0} & \frac{c_1}{\exp((1-c_1) z_0)} & z_0 := \omega h \\
  \hline
  \frac{1-(1-c_1) z_0}{1+c_1 z_0} e^{z_0} & \frac{1}{1+c_1 z_0} & \frac{1}{1+c_1 z_0} & \frac{1}{1+c_1 z_0} & \frac{1}{1+c_1 z_0} \\
\end{array}
\]
Stability function of IF methods

\[ y' = \lambda y \]

Lawson’s transformation: \[ u(x) = e^{-\omega x} y(x) \]

\[ u' = (\lambda - \omega) u \]

Suppose a purely polynomial method \( M \) with stability function \( R_M(z) \) is applied, then

\[ u_{n+1} = R_M(z - z_0) u_n \]

Re-expressed in terms of the \( y \)-variable, this gives

\[ y_{n+1} = e^{z_0} R_M(z - z_0) y_n \]

\[ R(z, z_0) = e^{z_0} R_M(z - z_0) \]
2. Exponential collocation methods

\[ y' - \omega y = f(x, y) - \omega y = \tilde{f}(x, y) \]

A function \( P(x) \in S \) is constructed such that for

\[
Q(x) := e^{\omega x}(e^{-\omega x}P(x))'
\]

\[
\begin{align*}
P(x_n) &= y_n \\
Q(x_n + c_i h) &= \tilde{f}(x_n + c_i h, P(x_n + c_i h)) \quad i = 1, \ldots, s
\end{align*}
\]

The method is defined by \( y_{n+1} := P(x_n + h) \).
2. Exponential collocation methods

\[(e^{-\omega x} P(x))' = e^{-\omega x} Q(x)\]

\[\int_{x_n}^{x_n+th} d(e^{-\omega x} P(x)) = \int_{x_n}^{x_n+th} e^{-\omega x} Q(x) dx\]

\[P(x_n + th) = e^{t\omega h} P(x_n) + h \int_0^t e^{\omega (t-\tau)h} Q(x_n + \tau h) d\tau\]

\[Q(x_n + \tau h) = \sum_{j=1}^s l_j(\tau) k_j \quad k_i := \tilde{f}(x_n + c_i h, P(x_n + c_i h))\]
2. Exponential collocation methods

\[ P(x_n + t h) = e^{t \omega h} P(x_n) + h \int_0^t e^{\omega (t-\tau) h} Q(x_n + \tau h) d\tau \]

\[ Q(x_n + \tau h) = \sum_{j=1}^s l_j(\tau) k_j \quad k_i := \tilde{f}(x_n + c_i h, P(x_n + c_i h)) \]

\[
\begin{cases}
  P(x_n + c_i h) = e^{c_i \omega h} P(x_n) + h \sum_{j=1}^s a_{ij} k_j & i = 1, \ldots, s \\
  P(x_n + h) = e^{\omega h} P(x_n) + h \sum_{j=1}^s b_j k_j \\
\end{cases}
\]

\[ a_{ij} := \int_0^{c_i} e^{\omega (c_i-\tau) h} l_j(\tau) d\tau \quad \text{and} \quad b_j := \int_0^1 e^{\omega (1-\tau) h} l_j(\tau) d\tau \]
2. Exponential collocation methods

The exponential collocation method for the problem

\[ y' - \omega y = \tilde{f}(x, y) \]

is thus given by

\[ y_{n+1} = e^{\omega h}y_n + h \sum_{i=1}^{s} b_i k_i \]

with

\[ k_i = \tilde{f}(x_n + c_i h, e^{c_i \omega h}y_n + h \sum_{j=1}^{s} a_{ij} k_j) \quad i = 1, \ldots, s \]
Example 1: polynomial interpolation

Suppose $S_Q = \Pi_{s-1}$ (space of polynomials of degree $\leq s - 1$)

$$P' - \omega P = Q(x) \implies P(x) = \alpha e^{\omega x} + \tilde{Q}(x)$$

where $\alpha$ is a constant and $\tilde{Q} \in \Pi_{s-1}$.

$$P(x) \in \Pi_{s-1} \cup Span\{e^{\omega x}\}$$

$$\tau^q = \sum_{j=1}^{s} l_j(\tau)c_j^q \quad q = 0, \ldots, s - 1$$

$$\int_0^{c_i} e^{\omega(c_i - \tau) h} \tau^q d\tau = \sum_{j=1}^{s} \int_0^{c_i} e^{\omega(c_i - \tau) h} l_j(\tau) d\tau c_j^q = \sum_{j=1}^{s} a_{ij}c_j^q$$
Example 1: polynomial interpolation

\[ \int_0^{c_i} e^{\omega(c_i-\tau)h} \tau^q d\tau = \sum_{j=1}^{s} \int_0^{c_i} e^{\omega(c_i-\tau)h} l_j(\tau) d\tau c_j^q = \sum_{j=1}^{s} a_{ij} c_j^q \]

\[ \sum_{j=1}^{s} a_{ij} = \frac{e^{c_i \omega h} - 1}{\omega h} \quad i = 1, \ldots, s \]

\[ \sum_{j=1}^{s} a_{ij} c_j^q = -\frac{c_i^q}{\omega h} + \frac{q}{\omega h} \sum_{j=1}^{s} a_{ij} c_j^{q-1} \quad q = 1, 2, \ldots s-1; i = 1, \ldots, s \]
One-stage method

\[ Y_1 = e^{c_1 \omega h} y_n + h \frac{e^{c_1 \omega h} - 1}{\omega h} \tilde{f}(x_n + c_1 h, Y_1) \]

\[ y_{n+1} = e^{\omega h} y_n + h \frac{e^{\omega h} - 1}{\omega h} \tilde{f}(x_n + c_1 h, Y_1) \]

For \( y' = f(x, y) \) this method is identical to method 2 defined by

\[
\begin{array}{c|c|c}
  c_1 & 1 & \frac{1-e^{-c_1 z_0}}{z_0} \\
  & 1 & \frac{e^{(1-c_1)z_0} - e^{-c_1 z_0}}{z_0} \\
\end{array}
\]
Example 2: exponential interpolation

Suppose $S_Q = e^{\omega x} \Pi_{s-1}$ (functions of the form $e^{\omega x} p_{s-1}(x)$ where $p_{s-1}(x) \in \Pi_{s-1}$)

$$P' - \omega P = Q(x) \implies P(x) = e^{\omega x} p_s(x)$$

with $p_s(x) \in \Pi_s$ (in fact, $p'_s(x) = p_{s-1}(x)$)

$$P(x) \in e^{\omega x} \Pi_s$$

The collocation conditions then become

$$p'_s(x) = e^{-\omega x} f(x, e^{\omega x} p_s(x)) = g(x, p_s(x))$$

This is the classical polynomial collocation method $u' = g(x, u)$

The resulting method will be the same as the IF method.
3. Linear functionals

\[ \mathcal{L}_i[y(x); h] := y(x_n + c_i h) - \gamma_i y_n - h \sum_{j=1}^{s} a_{ij}(y'(x_n + c_j h) - \omega y(x_n + c_j h)) \]

\[ i = 1, \ldots, s \]

\[ \mathcal{L}[y(x); h] := y(x_n + h) - \gamma y_n - h \sum_{i=1}^{s} b_i(y'(x_n + c_i h) - \omega y(x_n + c_i h)) \]

Require that

\[ \left\{ \begin{array}{l}
\mathcal{L}_i[u(x); h] = 0 \quad i = 1, \ldots, s \\
\mathcal{L}[u(x); h] = 0
\end{array} \right. \]

for \( u(x) = e^{\omega x} \) and for each function \( u \) in the \( s \) dimensional space \( S_Q \).
Conclusions

- We have analysed properties of stability functions of EFRK methods.
- Whereas purely polynomial methods impose conditions on the stability function $R(z)$ for $z = 0$ solely, EFRK that are fitted for parameter values $\omega_1, \omega_2, \ldots, \omega_n$ impose conditions for $z = \omega_1 h, \ldots, z = \omega_n h$ on the stability function $R(z, \{z_1, \ldots, z_n\})$.
- Nice relations exist between the different stability functions and, more in particular, between the corresponding order stars.
- The stability functions of integrating factor methods and exponential collocation methods were considered.
- Exponential-fitting, integrating factor and exponential collocation can lead to the same method.