Exponentially-fitted methods applied to fourth-order boundary value problems

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Outline

Introduction

In the past 15 years, our research group has constructed modified versions of well-known

- linear multistep methods
- Runge-Kutta methods

Aim : build methods which perform very good when the solution has a known exponential of trigonometric behaviour.

Linear multistep methods

A well known method to solve

$$y'' = f(y) \qquad y(a) = y_a \qquad y'(a) = y'_a$$

is the Numerov method (order 4)

$$y_{n+1} - 2\,y_n + y_{n-1} = \frac{1}{12}\,h^2\,\left(f(y_{n-1}) + 10\,f(y_n) + f(y_{n+1})\right)$$

Construction :
impose $\mathcal{L}[z(t);h] = 0$ for $z(t) \in \mathcal{S} = \{1,\, t,\, t^2,\, t^3,\, t^4\}$ where

$$\begin{aligned}\mathcal{L}[z(t);h] := {} & z(t+h) + \alpha_0\, z(t) + \alpha_{-1}\, z(t-h) \\ & -h^2\left(\beta_1\, z''(t+h) + \beta_0\, z''(t) + \beta_{-1}\, z''(t-h)\right)\end{aligned}$$

A model problem

Consider the initial value problem

$$y'' + \omega^2 y = g(y) \qquad y(a) = y_a \qquad y(a) = y'_a \,.$$

If $|g(y)| \ll |\omega^2 y|$ then

$$y(t) \approx \alpha \cos(\omega t + \phi)$$

To mimic this oscillatory behaviour, one could replace polynomials by trigonometric (in the complex case : exponential) functions.

EF Numerov method

Construction : impose $\mathcal{L}[z(t);h] = 0$ for $z(t) \in \mathcal{S}$ with

$$\mathcal{S} = \{1,\, t,\, t^2,\, \sin(\omega\, t),\, \cos(\omega\, t)\}$$

$$\begin{aligned}\mathcal{L}[z(t);h] &:= z(t+h) + \alpha_0\, z(t) + \alpha_{-1}\, z(t-h) \\ &\quad -h^2\left(\beta_1\, z''(t+h) + \beta_0\, z''(t) + \beta_{-1}\, z''(t-h)\right)\end{aligned}$$

$$y_{n+1} - 2\, y_n + y_{n-1} = h^2\ (\lambda f(y_{n-1}) + (1-2\,\lambda)\, f(y_n) + \lambda\, f(y_{n+1}))$$

$$\begin{aligned}\lambda &= \frac{1}{4\,\sin^2\frac{\theta}{2}} - \frac{1}{\theta^2} \qquad\qquad \theta := \omega\, h \\ &= \frac{1}{12} + \frac{1}{240}\theta^2 + \frac{1}{6048}\theta^4 + \dots\end{aligned}$$

EF methods

Generalisation : to determine the coefficients of a method, we impose conditions on a linear functional. These conditions are related to the fitting space $\mathcal{S}$ which contains

- polynomials :

$$\{t^q | q = 0, \ldots, K\}$$

- exponential or trigonometric functions, multiplied with powers of t :

$$\{t^q \exp(\pm\mu t) | q = 0, \ldots, P\}$$

or, with $\omega = i\,\mu$,

$$\{t^q \cos(\omega t), t^q \sin(\omega t) | q = 0, \ldots, P\}$$

EF method can be characterized by the couple (K, P)
Classical method : $P = -1$
number of basis functions : $M = 2P + K + 3$

Examples

$$M = 2P + K + 3$$

(K, P)				
$M = 2$	$M = 4$	$M = 6$	$M = 8$	$M = 10$
$(1, -1)$	$(3, -1)$	$(5, -1)$	$(7, -1)$	$(9, -1)$
$(-1, 1)$	$(1, 0)$	$(3, 0)$	$(5, 0)$	$(7, 0)$
	$(-1, 1)$	$(1, 1)$	$(3, 1)$	$(5, 1)$
		$(-1, 2)$	$(1, 2)$	$(3, 2)$
			$(-1, 3)$	$(1, 3)$
				$(-1, 4)$

$$(1, 2) \Longrightarrow \mathcal{S} = \left\{1,\ t,\ \exp(\pm\mu t),\ t\ \exp(\pm\mu t),\ t^2\ \exp(\pm\mu t)\right\}$$

Exponential Fitting

L. Ixaru and G. Vanden Berghe
Exponential fitting
Kluwer Academic Publishers, Dordrecht, 2004

$$\eta_{-1}(Z) = \begin{cases} \cos(|Z|^{1/2}) & \text{if } Z < 0 \\ \cosh(Z^{1/2}) & \text{if } Z \geq 0 \end{cases}$$

$$\eta_0(Z) = \begin{cases} \sin(|Z|^{1/2})/|Z|^{1/2} & \text{if } Z < 0 \\ 1 & \text{if } Z = 0 \\ \sinh(Z^{1/2})/Z^{1/2} & \text{if } Z > 0 \end{cases}$$

$$Z := (\mu h)^2 = -(\omega h)^2$$

$$\eta_n(Z) := \frac{1}{Z}[\eta_{n-2}(Z) - (2n-1)\eta_{n-1}(Z)], \quad n = 1, 2, 3, \ldots$$

$$\eta_n'(Z) = \frac{1}{2}\eta_{n+1}(Z), \quad n = 1, 2, 3, \ldots$$

Choice of ω

- local optimization
  based on local truncation error (lte)
  ω is step-dependent

- global optimization
  Preservation of geometric properties (periodicity, energy, . . .)
  ω is constant over the interval of integration

Fourth-order boundary value problem

$$y^{(4)} = F(t, y) \qquad a \leq t \leq b$$

$$\begin{array}{ll} y(a) = A_1 & y''(a) = A_2 \\ y(b) = B_1 & y''(b) = B_2 \end{array}$$

- special case : $y^{(4)} + f(t)y = g(t)$
- mathematical modeling of viscoelastic and inelastic flows, deformation of beams, plate deflection theory, ...
- work by Doedel, Usmani, Agarwal, Cherruault et al., Van Daele et al., ...
- finite differences, B-splines, ...

The formulae

$$t_j = a + j\,h,\ j = 0,\ 1,\ \ldots\ N+1 \qquad N \geq 3 \qquad h := \frac{b-a}{N+1}$$

- central formula for $j = 2, \ldots, N-1$

$$y_{j-2} + a_1\,y_{j-1} + a_0\,y_j + a_1\,y_{j+1} + y_{j+2} = h^4\,\left(b_2\,F_{j-2} + b_1\,F_{j-1} + b_0\,F_j + b_1\,F_{j+1} + b_2\,F_{j+2}\right)$$

whereby y_j is approximate value of $y(t_j)$ and $F_j := F(t_j,\, y_j)$.

- begin formula

$$y_0 + \alpha_1\,y_1 + \alpha_2\,y_2 + a_3\,y_3 = \gamma\,h^2\,y_0'' + h^4\,(\beta_0\,F_0 + \beta_1\,F_1 + \beta_2\,F_2 + \beta_3\,F_3 + \beta_4\,F_4 + \beta_5\,F_5)$$

- end formula

Central formula

$$\mathcal{L}[y] := y(t-2h) + a_1\, y(t-h) + a_0\, y(t) + a_1\, y(t+h) + y(t+2h)$$
$$-h^4 \left(b_2\, y^{(4)}(t-2h) + b_1\, y^{(4)}(t-h) + b_0\, y^{(4)}(t) + b_1\, y^{(4)}(t+h) + b_2\, y^{(4)}(t+2h) \right)$$

$$P = -1 : \;\; \mathcal{L}[y] = 0 \;\text{ for }\; y \in \mathcal{S} = \left\{1,\, t,\, t^2,\, \dots,\, t^{M-1}\right\}$$

$M = 10$:

$$y_{p-2} - 4\, y_{p-1} + 6\, y_p - 4\, y_{p+1} + y_{p+2} =$$
$$\frac{h^4}{720}\left(-y_{p-2}^{(4)} + 124\, y_{p-1}^{(4)} + 474\, y_p^{(4)} + 124\, y_{p+1}^{(4)} - y_{p+2}^{(4)} \right)$$

$$\mathcal{L}[y](t) = \frac{1}{3024}\, h^{10}\, y^{(10)}(t) + \mathcal{O}(h^{12})$$

$M = 8$ and $b_2 = 0$:

EF Central formula

$$\mathcal{L}[y] := y(t-2h) + a_1 y(t-h) + a_0 y(t) + a_1 y(t+h) + y(t+2h)$$
$$-h^4 \left(b_2 y^{(4)}(t-2h) + b_1 y^{(4)}(t-h) + b_0 y^{(4)}(t) + b_1 y^{(4)}(t+h) + b_2 y^{(4)}(t+2h) \right)$$

$P = 0$: $\mathcal{L}[y] = 0$ for $y \in \mathcal{S} = \left\{ \cos(\omega t), \sin(\omega t), 1, t, t^2, \ldots, t^{M-3} \right\}$

$M = 10$:

$$y_{p-2} - 4y_{p-1} + 6y_p - 4y_{p+1} + y_{p+2} =$$
$$h^4 \left(b_2 y^{(4)}_{p-2} + b_1 y^{(4)}_{p-1} + b_0 y^{(4)}_p + b_1 y^{(4)}_{p+1} + b_2 y^{(4)}_{p+2} \right)$$

$$b_0 = \frac{4\cos^2\theta - 2 - 11\cos\theta}{6(\cos\theta - 1)^2} + \frac{6}{\theta^4} \quad b_1 = \frac{\cos^2\theta + 5}{6(\cos\theta - 1)^2} - \frac{4}{\theta^4} \quad b_2 = -\frac{\cos\theta + 2}{12(\cos\theta - 1)^2} + \frac{1}{\theta^4}$$

$$\mathcal{L}[y](t) = \frac{1}{3024} h^{10} \left(y^{(10)}(t) + \omega^2 y^{(8)}(t) \right) + \mathcal{O}(h^{12})$$

EF Central formula

$$\mathcal{L}[y] := y(t-2h) + a_1\, y(t-h) + a_0\, y(t) + a_1\, y(t+h) + y(t+2h)$$
$$-h^4\,\left(b_2\, y^{(4)}(t-2h) + b_1\, y^{(4)}(t-h) + b_0\, y^{(4)}(t) + b_1\, y^{(4)}(t+h) + b_2\, y^{(4)}(t+2h)\right)$$

$P=1$: $\mathcal{L}[y] = 0$ for $y \in \mathcal{S} = \left\{\cos(\omega t),\ \sin(\omega t),\ t\,\cos(\omega t),\ t\,\sin(\omega t),\ 1,\ t,\ t^2,\ \ldots,\ t^{M-5}\right\}$

$M = 6$ and $b_1 = b_2 = 0$:

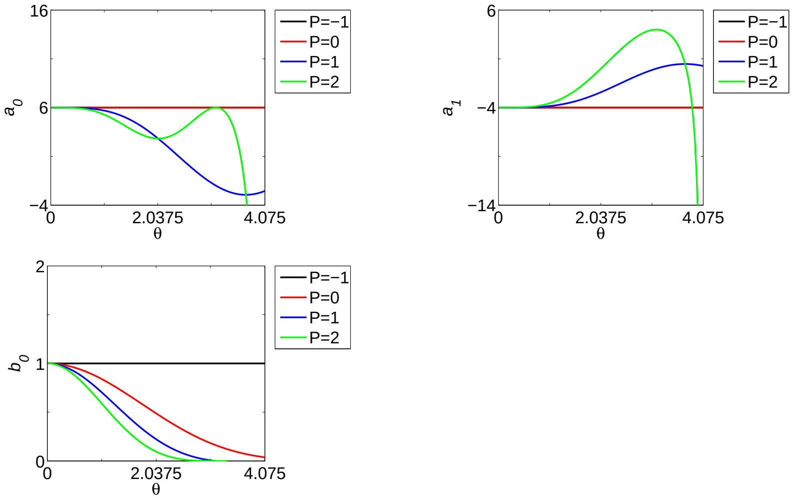
$$y_{p-2} + a_1\, y_{p-1} + a_0\, y_p + a_1\, y_{p+1} + y_{p+2} = b_0\, h^4\, y_p^{(4)}$$

$$a_0 = 2\,\frac{-8\,\sin^2\theta + \theta\,(4\,\cos\theta - 1)\sin\theta - 4\,\cos\theta + 4}{\theta\,\sin\theta + 4\,\cos\theta - 4} \qquad a_1 = -4\,\frac{\sin\theta\,(\theta\,\cos\theta - 2\,\sin\theta)}{\theta\,\sin\theta + 4\,\cos\theta - 4}$$

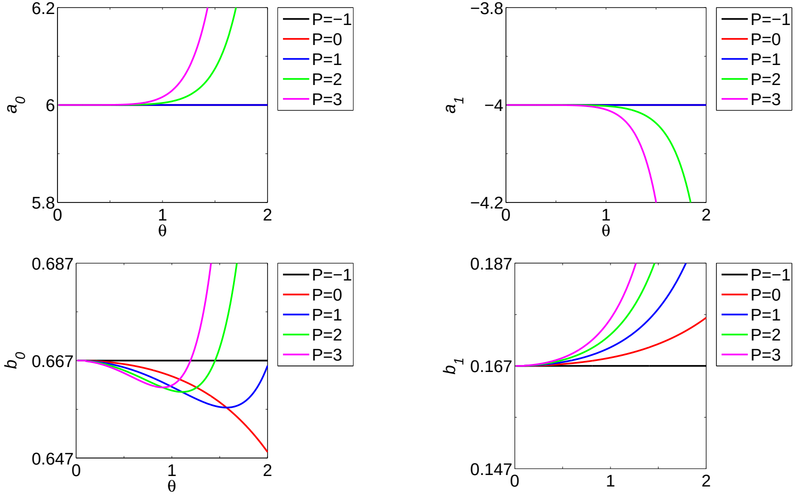
$$b_0 = 4\,\frac{\sin\theta\left(\sin^2\theta - 2 + 2\,\cos\theta\right)}{\theta^3\,(\theta\,\sin\theta + 4\,\cos\theta - 4)}$$

$$\mathcal{L}[y](t) = \frac{1}{6}\,h^6\,(y^{(6)}(t) + 2\,\omega^2\,y^{(4)}(t) + \omega^4\,y^{(2)}(t)) + \mathcal{O}(h^8)$$

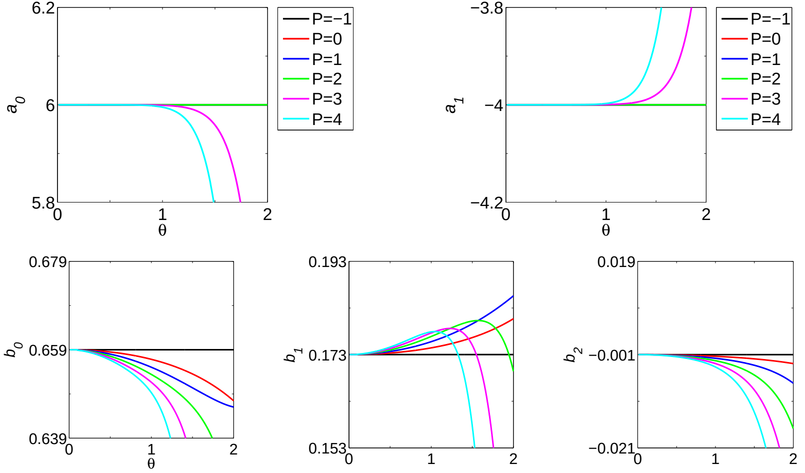
Coefficients of Central formula $M = 6$



Coefficients of Central formula $M = 8$



Coefficients of Central formula $M = 10$



Central formula : coefficients

E.g. b_0 in case $M = 6$

In closed form ...

- $P = -1$:

$$b_0 = 1$$

- $P = 0$:

$$b_0 = 4\,\frac{(\cos\theta - 1)^2}{\theta^4}$$

- $P = 1$:

$$b_0 = -4\,\frac{\sin\theta\,(\cos\theta - 1)^2}{\theta^3\,(4\,\cos\theta - 4 + \theta\,\sin\theta)}$$

- $P = 2$:

$$b_0 = -2\,\frac{\sin^3\theta}{\theta^2\,(\theta\,\cos\theta - 3\,\sin\theta)}$$

Central formula : coefficients

E.g. b_0 in case $M = 6$

As a series . . .

- $P = -1$:

$$b_0 = 1$$

- $P = 0$:

$$b_0 = 1 - \frac{1}{6}\theta^2 + \frac{1}{80}\theta^4 + \mathcal{O}(\theta^6)$$

- $P = 1$:

$$b_0 = 1 - \frac{1}{3}\theta^2 + \frac{37}{720}\theta^4 + \mathcal{O}(\theta^6)$$

- $P = 2$:

$$b_0 = 1 - \frac{1}{2}\theta^2 + \frac{7}{60}\theta^4 + \mathcal{O}(\theta^6)$$

Central formula : local truncation error

$$\text{lte} = \mathcal{L}[y](t)$$

As an inifinite series :

$$\text{lte} = h^M\, C_M\, D^{K+1}\,(D^2+\omega^2)^{P+1} y(t) + \mathcal{O}(h^{M+2})$$

In closed form : (Coleman and Ixaru)

$$\text{lte} = h^M\, \Phi_{K,P}(Z)\, D^{K+1}\,(D^2+\omega^2)^{P+1} y(\xi)$$

$Z \in$ some interval $\quad \Phi_{K,P}(0) \neq 0 \quad \xi \in (t-2\,h,\, t+2\,h)$

Local truncation error

$$\text{lte} = h^M \, C_M \, D^{K+1} \, (D^2 + \omega^2)^{P+1} y(t) + \mathcal{O}(h^{M+2}) \,,$$

$$\text{At } t_j : D^{(K+1)} \, (D^2 + \omega_j^2)^{(P+1)} y(t) \Big|_{t=t_j} = 0 \qquad j = 2, \ldots, N-1$$

- $P = 0$:

$$y^{(K+3)}(t_j) + y^{(K+1)}(t_j) \, \omega_j^2 = 0$$

- $P = 1$:

$$y^{(K+5)}(t_j) + 2 \, y^{(K+3)}(t_j) \, \omega_j^2 + y^{(K+1)}(t_j) \, \omega_j^4 = 0$$

- $P = 2$:

$$y^{(K+7)}(t_j) + 3 \, y^{(K+5)}(t_j) \, \omega_j^4 + 3 \, y^{(K+3)}(t_j) \, \omega_j^4 + y^{(K+1)}(t_j) \, \omega_j^6 = 0$$

-

Local truncation error

$$\text{lte} = h^M \, C_M \, D^{K+1} \, (D^2 + \omega^2)^{P+1} y(t) + \mathcal{O}(h^{M+2}) \,,$$

$$\text{At } t_j : D^{(K+1)} \, (D^2 + \omega_j^2)^{(P+1)} y(t) \Big|_{t=t_j} = 0 \qquad j = 2, \ldots, N-1$$

ω_j^2 is solution of equation of degree $P+1$.

- Which value of P should be chosen ?
- Which root ω_j should be chosen ?

Parameter selection

$$\text{lte} = h^M C_M D^{K+1} (D^2 - \mu^2)^{P+1} y(t) + \mathcal{O}(h^{M-2})$$

Suppose $y(t)$ takes the form $t^{P_0} e^{\mu_0 t}$

Then lte$= 0$ for any EF rule with $P \geq P_0$ and $\mu_j = \mu_0$

Theorem

If $y(t) = t^{P_0} e^{\mu_0 t}$ then $\nu = \mu_0^2$ is a root of multiplicity $P - P_0 + 1$ of $D^{K+1} (D^2 - \nu)^{P+1} y(t) = 0$.

- if $P = P_0$, then $\mu = \mu_0$ will be a single root
- if $P = P_0 + 1$, then $\mu = \mu_0$ will be a double root
- if $P = P_0 + 2$, then $\mu = \mu_0$ will be a triple root
- ...

Parameter selection

Suppose $y(t)$ does not take the form $t^{P_0}\, e^{\mu_0\, t}$

Then $y(t) \notin \mathcal{S}$ for any P.

For a given value of P :

$$D^{(K+1)}\,(D^2 - \mu_j^2)^{(P+1)} y(t)\Big|_{t=t_j} = 0$$

At each point t_j, this gives $P+1$ values for μ_j^2.

Idea : keep $|\mu_j\, h|$ as small as possible.
If possible, choose $P \geq 1$ to avoid too large values for $|\mu_j|$.

First example

$$y^{(4)} - \frac{384\,t^4}{(2+t^2)^4}y = 24\,\frac{2-11\,t^2}{(2+t^2)^4}$$

$$y(-1) = \frac{1}{3} \qquad y(1) = \frac{1}{3}$$

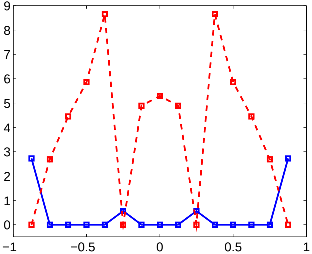
$$y''(-1) = \frac{2}{27} \qquad y''(1) = \frac{2}{27}$$

$$\text{Solution}: y(t) = \frac{1}{2+t^2}$$

μ_j for $M = 8$

$$P = 0 : y^{(8)}(t_j) - y^{(6)}(t_j)\,\mu_j^2 = 0$$

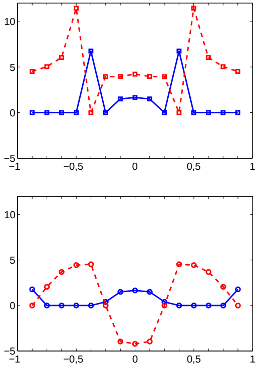
- re-express higher order derivatives in terms of y, y', y'' and y'''
- approximate y', y'' and y''' in terms of y
- an initial approximation for y can be computed with a polynomial rule



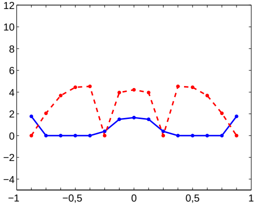
Real and imaginary part of μ_j

μ_j for $M = 8$

$$P = 1 : y^{(8)}(t_j) - 2\,y^{(6)}(t_j)\,\mu_j^2 + y^{(4)}(t_j)\,\mu_j^4 = 0$$



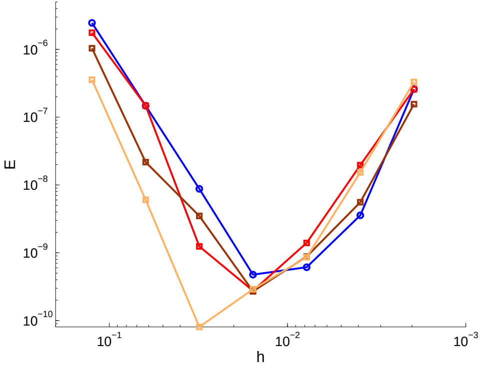
Real and imag. part of $\mu_{1,j}$ and $\mu_{2,j}$



Real and imaginary part of μ_j with smallest norm

μ_j for $M = 8$

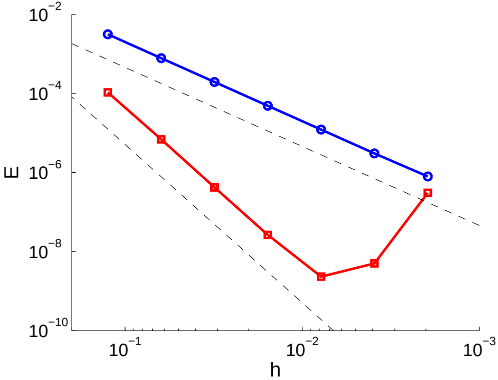
$$P = 1 : y^{(8)}(t_j) - 2\,y^{(6)}(t_j)\,\mu_j^2 + y^{(4)}(t_j)\,\mu_j^4 = 0$$



error obtained with $\mu_{1,j}$, $\mu_{2,j}$ and μ with smallest norm

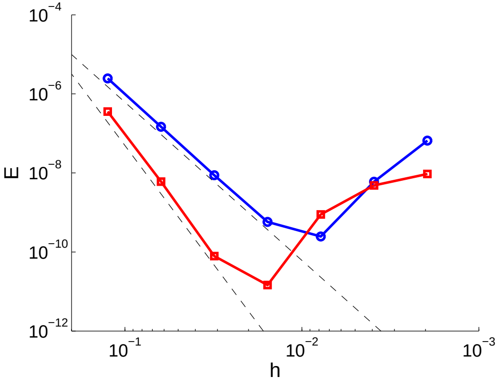
Global error

$$M = 6: \quad \begin{array}{ll} (K,P) = (5,-1) & : \text{second-order method} \\ (K,P) = (1,1) & : \text{fourth-order method} \end{array}$$



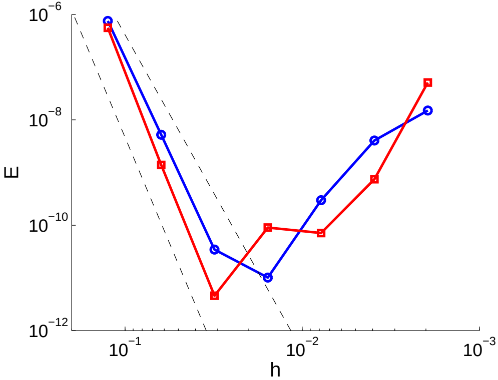
Global error

$$M = 8: \quad \begin{array}{ll} (K,P) = (7,-1) & : \text{fourth-order method} \\ (K,P) = (3,1) & : \text{sixth-order method} \end{array}$$

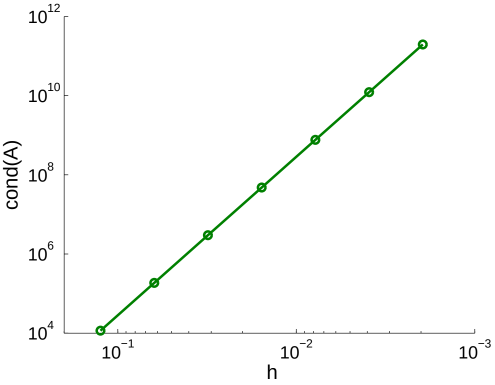


Global error

$$M = 10: \begin{array}{ll} (K,P) = (9,-1) & : \text{sixth-order method} \\ (K,P) = (5,1) & : \text{eighth-order method} \end{array}$$



Condition number



Second example

$$y^{(4)} - t = 4\,e^t$$

$$y(-1) = -1/e \qquad y(1) = e$$

$$y''(-1) = 1/e \qquad y''(1) = 3\,e$$

$$\text{Solution : } y(t) = e^t\, t$$

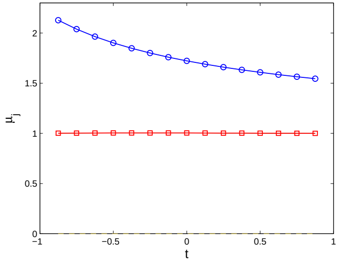
μ_j for $M = 6$

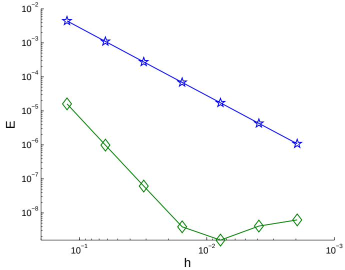
$$P = 1 : y^{(6)}(t_j) - 2\,y^{(4)}(t_j)\,\mu_j^2 + y^{(2)}(t_j)\,\mu_j^4 = 0$$

differentiating the differential equation :

$$(y^{(2)}(t_j) + 4\,e^{t_j}) - 2\,(y_j + 4\,e^{t_j})\mu_j^2 + y^{(2)}(t_j)\,\mu_j^4 = 0$$

$y^{(2)}(t_j)$ approximated by fourth-order finite difference scheme





two real roots $\mu^{(1)}$ and $\mu^{(2)}$

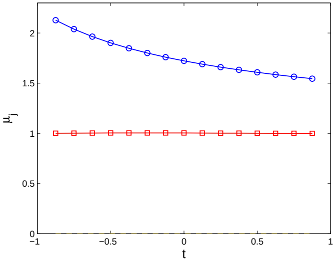
$M = 6$

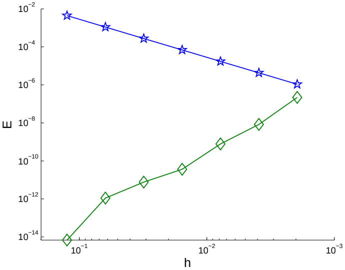
$$P = 1 : y^{(6)}(t_j) - 2\,y^{(4)}(t_j)\,\mu_j^2 + y^{(2)}(t_j)\,\mu_j^4 = 0$$

differentiating the differential equation :

$$(y^{(2)}(t_j) + 4\,e^{t_j}) - 2\,(y_j + 4\,e^{t_j})\mu_j^2 + y^{(2)}(t_j)\,\mu_j^4 = 0$$

$y^{(2)}(t_j)$ approximated by sixth-order finite difference scheme





two real roots $\mu^{(1)}$ and $\mu^{(2)}$

Conclusions

- Fourth-order boundary value problems are solved by means of parameterized exponentially-fitted methods.
- A suitable value for the parameter can be found from the roots of the leading term of the local truncation error.
- If a constant value is found, then a very accurate solution can be obtained.
- However, the methods strongly suffer from the fact that the system to be solved is ill-conditioned for small values of the mesh size.