

# Exponentially-fitted methods applied to fourth-order boundary value problems

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# Outline

## Introduction

An example

Exponential fitting

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The problem

Exponentially-fitted methods

Parameter selection

## Numerical examples

First example

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# Introduction

In the past 15 years, our research group has constructed modified versions of well-known

- linear multistep methods
- Runge-Kutta methods

Aim : build methods which perform very good when the solution has a known exponential or trigonometric behaviour.

## Linear multistep methods

A well known method to solve

$$y'' = f(y) \quad y(a) = y_a \quad y'(a) = y'_a$$

is the **Numerov method** (order 4)

$$y_{n+1} - 2y_n + y_{n-1} = \frac{1}{12} h^2 (f(y_{n-1}) + 10f(y_n) + f(y_{n+1}))$$

**Construction :**

impose  $\mathcal{L}[z(t); h] = 0$  for  $z(t) \in \mathcal{S} = \{1, t, t^2, t^3, t^4\}$  where

$$\begin{aligned} \mathcal{L}[z(t); h] := & z(t+h) + \alpha_0 z(t) + \alpha_{-1} z(t-h) \\ & - h^2 (\beta_1 z''(t+h) + \beta_0 z''(t) + \beta_{-1} z''(t-h)) \end{aligned}$$

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## A model problem

Consider the initial value problem

$$y'' + \omega^2 y = g(y) \quad y(a) = y_a \quad y'(a) = y'_a.$$

If  $|g(y)| \ll |\omega^2 y|$  then

$$y(t) \approx \alpha \cos(\omega t + \phi)$$

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$$\begin{aligned} \lambda &= \frac{1}{4 \sin^2 \frac{\theta}{2}} - \frac{1}{\theta^2} & \theta &:= \omega h \\ &= \frac{1}{12} + \frac{1}{240} \theta^2 + \frac{1}{6048} \theta^4 + \dots \end{aligned}$$

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## EF methods

Generalisation : to determine the coefficients of a method, we impose conditions on a linear functional. These conditions are related to the **fitting space  $\mathcal{S}$**  which contains

- polynomials :

$$\{t^q | q = 0, \dots, K\}$$

- exponential or trigonometric functions, multiplied with powers of  $t$  :

$$\{t^q \exp(\pm \mu t) | q = 0, \dots, P\}$$

or, with  $\omega = i \mu$ ,

$$\{t^q \cos(\omega t), t^q \sin(\omega t) | q = 0, \dots, P\}$$

EF method can be characterized by the couple  $(K, P)$

Classical method :  $P = -1$

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## Examples

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$M = 2$	$M = 4$	$M = 6$	$M = 8$	$M = 10$
$(1, -1)$	$(3, -1)$	$(5, -1)$	$(7, -1)$	$(9, -1)$
$(-1, 1)$	$(1, 0)$	$(3, 0)$	$(5, 0)$	$(7, 0)$
	$(-1, 1)$	$(1, 1)$	$(3, 1)$	$(5, 1)$
		$(-1, 2)$	$(1, 2)$	$(3, 2)$
			$(-1, 3)$	$(1, 3)$
				$(-1, 4)$

$$(1, 2) \implies \mathcal{S} = \left\{ 1, t, \exp(\pm\mu t), t \exp(\pm\mu t), t^2 \exp(\pm\mu t) \right\}$$

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L. Ixaru and G. Vanden Berghe

*Exponential fitting*

Kluwer Academic Publishers, Dordrecht, 2004

$$\eta_{-1}(Z) = \begin{cases} \cos(|Z|^{1/2}) & \text{if } Z < 0 \\ \cosh(Z^{1/2}) & \text{if } Z \geq 0 \end{cases}$$

$$\eta_0(Z) = \begin{cases} \sin(|Z|^{1/2})/|Z|^{1/2} & \text{if } Z < 0 \\ 1 & \text{if } Z = 0 \\ \sinh(Z^{1/2})/Z^{1/2} & \text{if } Z > 0 \end{cases} \quad Z := (\mu h)^2 = -(\omega h)^2$$

$$\eta_n(Z) := \frac{1}{Z}[\eta_{n-2}(Z) - (2n-1)\eta_{n-1}(Z)], \quad n = 1, 2, 3, \dots$$

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## Choice of $\omega$

- local optimization  
based on local truncation error (lte)  
 $\omega$  is step-dependent
- global optimization  
Preservation of geometric properties (periodicity, energy, ...)  
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## Fourth-order boundary value problem

$$y^{(4)} = F(t, y) \quad a \leq t \leq b$$

$$y(a) = A_1 \quad y''(a) = A_2$$

$$y(b) = B_1 \quad y''(b) = B_2$$

- special case :  $y^{(4)} + f(t)y = g(t)$
- mathematical modeling of viscoelastic and inelastic flows, deformation of beams, plate deflection theory, ...
- work by Doedel, Usmani, Agarwal, Cherruault et al., Van Daele et al., ...
- finite differences, B-splines, ...

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## The formulae

$$t_j = a + jh, j = 0, 1, \dots, N+1 \quad N \geq 3 \quad h := \frac{b-a}{N+1}$$

- central formula for  $j = 2, \dots, N-1$

$$y_{j-2} + a_1 y_{j-1} + a_0 y_j + a_1 y_{j+1} + y_{j+2} = h^4 (b_2 F_{j-2} + b_1 F_{j-1} + b_0 F_j + b_1 F_{j+1} + b_2 F_{j+2})$$

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$$y_0 + \alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_3 = \gamma h^2 y_0'' + h^4 (\beta_0 F_0 + \beta_1 F_1 + \beta_2 F_2 + \beta_3 F_3 + \beta_4 F_4 + \beta_5 F_5)$$

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$$\mathcal{L}[y] := y(t-2h) + a_1 y(t-h) + a_0 y(t) + a_1 y(t+h) + y(t+2h) \\ - h^4 \left( b_2 y^{(4)}(t-2h) + b_1 y^{(4)}(t-h) + b_0 y^{(4)}(t) + b_1 y^{(4)}(t+h) + b_2 y^{(4)}(t+2h) \right)$$

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$M = 10$ :

$$y_{p-2} - 4y_{p-1} + 6y_p - 4y_{p+1} + y_{p+2} = \\ \frac{h^4}{720} \left( -y_{p-2}^{(4)} + 124y_{p-1}^{(4)} + 474y_p^{(4)} + 124y_{p+1}^{(4)} - y_{p+2}^{(4)} \right)$$

$$\mathcal{L}[y](t) = \frac{1}{3024} h^{10} y^{(10)}(t) + \mathcal{O}(h^{12})$$

## Central formula

$$\mathcal{L}[y] := y(t-2h) + a_1 y(t-h) + a_0 y(t) + a_1 y(t+h) + y(t+2h) \\ - h^4 \left( b_2 y^{(4)}(t-2h) + b_1 y^{(4)}(t-h) + b_0 y^{(4)}(t) + b_1 y^{(4)}(t+h) + b_2 y^{(4)}(t+2h) \right)$$

$$P = -1: \mathcal{L}[y] = 0 \text{ for } y \in \mathcal{S} = \{1, t, t^2, \dots, t^{M-1}\}$$

$M = 8$  and  $b_2 = 0$ :

$$y_{p-2} - 4y_{p-1} + 6y_p - 4y_{p+1} + y_{p+2} = \frac{h^4}{6} \left( y_{p-1}^{(4)} + 4y_p^{(4)} + y_{p+1}^{(4)} \right)$$

$$\mathcal{L}[y](t) = -\frac{1}{720} h^8 y^{(8)}(t) + \mathcal{O}(h^{10})$$

## Central formula

$$\mathcal{L}[y] := y(t-2h) + a_1 y(t-h) + a_0 y(t) + a_1 y(t+h) + y(t+2h) \\ - h^4 \left( b_2 y^{(4)}(t-2h) + b_1 y^{(4)}(t-h) + b_0 y^{(4)}(t) + b_1 y^{(4)}(t+h) + b_2 y^{(4)}(t+2h) \right)$$

$$P = -1 : \mathcal{L}[y] = 0 \text{ for } y \in \mathcal{S} = \{1, t, t^2, \dots, t^{M-1}\}$$

$$M = 6 \text{ and } b_1 = b_2 = 0 :$$

$$y_{p-2} - 4y_{p-1} + 6y_p - 4y_{p+1} + y_{p+2} = h^4 y_p^{(4)}$$

$$\mathcal{L}[y](t) = \frac{1}{6} h^6 y^{(6)}(t) + \mathcal{O}(h^8)$$

## EF Central formula

$$\mathcal{L}[y] := y(t-2h) + a_1 y(t-h) + a_0 y(t) + a_1 y(t+h) + y(t+2h) \\ - h^4 \left( b_2 y^{(4)}(t-2h) + b_1 y^{(4)}(t-h) + b_0 y^{(4)}(t) + b_1 y^{(4)}(t+h) + b_2 y^{(4)}(t+2h) \right)$$

$$P = 0: \mathcal{L}[y] = 0 \text{ for } y \in \mathcal{S} = \left\{ \cos(\omega t), \sin(\omega t), 1, t, t^2, \dots, t^{M-3} \right\}$$



## EF Central formula

$$\mathcal{L}[y] := y(t-2h) + a_1 y(t-h) + a_0 y(t) + a_1 y(t+h) + y(t+2h) \\ - h^4 \left( b_2 y^{(4)}(t-2h) + b_1 y^{(4)}(t-h) + b_0 y^{(4)}(t) + b_1 y^{(4)}(t+h) + b_2 y^{(4)}(t+2h) \right)$$

$$P = 0: \mathcal{L}[y] = 0 \text{ for } y \in \mathcal{S} = \left\{ \cos(\omega t), \sin(\omega t), 1, t, t^2, \dots, t^{M-3} \right\}$$

$M = 10$ :

$$y_{p-2} - 4y_{p-1} + 6y_p - 4y_{p+1} + y_{p+2} = \\ h^4 \left( b_2 y_{p-2}^{(4)} + b_1 y_{p-1}^{(4)} + b_0 y_p^{(4)} + b_1 y_{p+1}^{(4)} + b_2 y_{p+2}^{(4)} \right)$$

$$b_0 = \frac{4 \cos^2 \theta - 2 - 11 \cos \theta}{6 (\cos \theta - 1)^2} + \frac{6}{\theta^4} \quad b_1 = \frac{\cos^2 \theta + 5}{6 (\cos \theta - 1)^2} - \frac{4}{\theta^4} \quad b_2 = -\frac{\cos \theta + 2}{12 (\cos \theta - 1)^2} + \frac{1}{\theta^4}$$

$$\mathcal{L}[y](t) = \frac{1}{3024} h^{10} \left( y^{(10)}(t) + \omega^2 y^{(8)}(t) \right) + \mathcal{O}(h^{12})$$

## EF Central formula

$$\mathcal{L}[y] := y(t-2h) + a_1 y(t-h) + a_0 y(t) + a_1 y(t+h) + y(t+2h) \\ - h^4 \left( b_2 y^{(4)}(t-2h) + b_1 y^{(4)}(t-h) + b_0 y^{(4)}(t) + b_1 y^{(4)}(t+h) + b_2 y^{(4)}(t+2h) \right)$$

$$P = 0: \mathcal{L}[y] = 0 \text{ for } y \in \mathcal{S} = \left\{ \cos(\omega t), \sin(\omega t), 1, t, t^2, \dots, t^{M-3} \right\}$$

$$M = 8 \text{ and } b_2 = 0:$$

$$y_{p-2} - 4y_{p-1} + 6y_p - 4y_{p+1} + y_{p+2} = h^4 \left( b_1 y_{p-1}^{(4)} + b_0 y_p^{(4)} + b_1 y_{p+1}^{(4)} \right)$$

$$b_0 = \frac{\cos \theta}{\cos \theta - 1} + \frac{4(1 - \cos \theta)}{\theta^4} \quad b_1 = \frac{1}{2(1 - \cos \theta)} + \frac{2(\cos \theta - 1)}{\theta^4}$$

$$\mathcal{L}[y](t) = -\frac{1}{720} h^8 \left( y^{(8)}(t) + \omega^2 y^{(6)}(t) \right) + \mathcal{O}(h^{10})$$

## EF Central formula

$$\mathcal{L}[y] := y(t-2h) + a_1 y(t-h) + a_0 y(t) + a_1 y(t+h) + y(t+2h) \\ - h^4 \left( b_2 y^{(4)}(t-2h) + b_1 y^{(4)}(t-h) + b_0 y^{(4)}(t) + b_1 y^{(4)}(t+h) + b_2 y^{(4)}(t+2h) \right)$$

$$P = 0: \mathcal{L}[y] = 0 \text{ for } y \in \mathcal{S} = \left\{ \cos(\omega t), \sin(\omega t), 1, t, t^2, \dots, t^{M-3} \right\}$$

$$M = 6 \text{ and } b_1 = b_2 = 0:$$

$$y_{p-2} - 4y_{p-1} + 6y_p - 4y_{p+1} + y_{p+2} = \frac{\sin^4(\theta/2)}{(\theta/2)^4} h^4 y_p^{(4)}$$

$$\mathcal{L}[y](t) = \frac{1}{6} h^6 (y^{(6)}(t) + \omega^2 y^{(4)}(t)) + \mathcal{O}(h^8)$$

## EF Central formula

$$\mathcal{L}[y] := y(t-2h) + a_1 y(t-h) + a_0 y(t) + a_1 y(t+h) + y(t+2h) \\ - h^4 \left( b_2 y^{(4)}(t-2h) + b_1 y^{(4)}(t-h) + b_0 y^{(4)}(t) + b_1 y^{(4)}(t+h) + b_2 y^{(4)}(t+2h) \right)$$

$$P = 1: \mathcal{L}[y] = 0 \text{ for } y \in \mathcal{S} = \left\{ \cos(\omega t), \sin(\omega t), t \cos(\omega t), t \sin(\omega t), 1, t, t^2, \dots, t^{M-5} \right\}$$

## EF Central formula

$$\mathcal{L}[y] := y(t-2h) + a_1 y(t-h) + a_0 y(t) + a_1 y(t+h) + y(t+2h) \\ - h^4 \left( b_2 y^{(4)}(t-2h) + b_1 y^{(4)}(t-h) + b_0 y^{(4)}(t) + b_1 y^{(4)}(t+h) + b_2 y^{(4)}(t+2h) \right)$$

$$P = 1: \mathcal{L}[y] = 0 \text{ for } y \in \mathcal{S} = \{ \cos(\omega t), \sin(\omega t), t \cos(\omega t), t \sin(\omega t), 1, t, t^2, \dots, t^{M-5} \}$$

$$M = 6 \text{ and } b_1 = b_2 = 0:$$

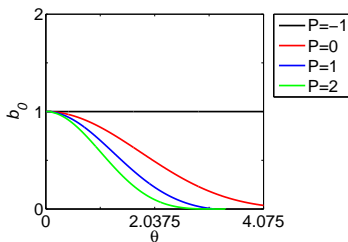
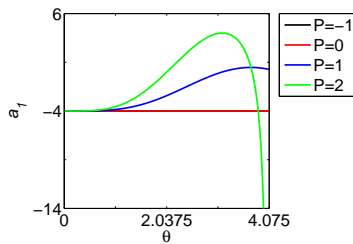
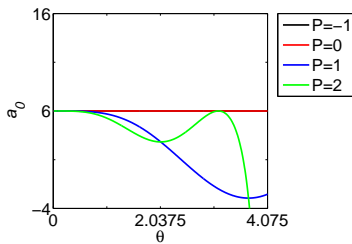
$$y_{p-2} + a_1 y_{p-1} + a_0 y_p + a_1 y_{p+1} + y_{p+2} = b_0 h^4 y_p^{(4)}$$

$$a_0 = 2 \frac{-8 \sin^2 \theta + \theta (4 \cos \theta - 1) \sin \theta - 4 \cos \theta + 4}{\theta \sin \theta + 4 \cos \theta - 4} \quad a_1 = -4 \frac{\sin \theta (\theta \cos \theta - 2 \sin \theta)}{\theta \sin \theta + 4 \cos \theta - 4}$$

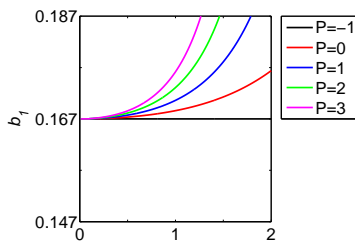
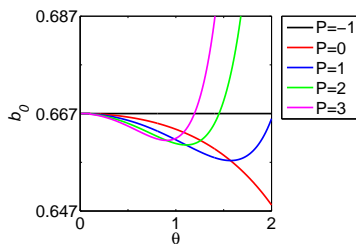
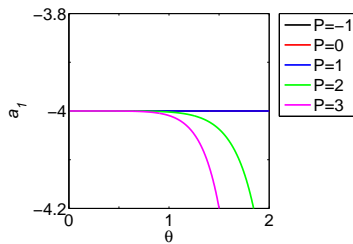
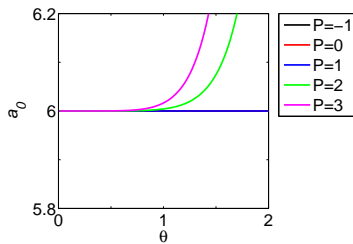
$$b_0 = 4 \frac{\sin \theta (\sin^2 \theta - 2 + 2 \cos \theta)}{\theta^3 (\theta \sin \theta + 4 \cos \theta - 4)}$$

$$\mathcal{L}[y](t) = \frac{1}{6} h^6 (y^{(6)}(t) + 2\omega^2 y^{(4)}(t) + \omega^4 y^{(2)}(t)) + \mathcal{O}(h^8)$$

# Coefficients of Central formula $M = 6$

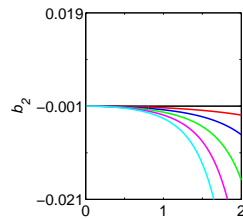
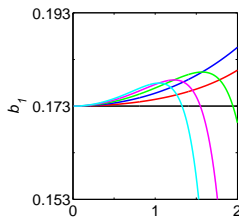
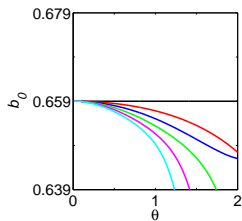
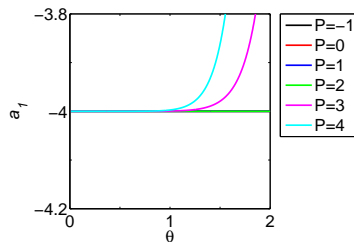
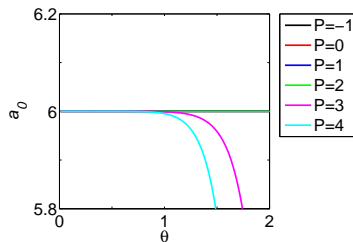


# Coefficients of Central formula $M = 8$





# Coefficients of Central formula $M = 10$





## Central formula : coefficients

E.g.  $b_0$  in case  $M = 6$

In closed form ...

- $P = -1$  :

$$b_0 = 1$$

- $P = 0$  :

$$b_0 = 4 \frac{(\cos \theta - 1)^2}{\theta^4}$$

- $P = 1$  :

$$b_0 = -4 \frac{\sin \theta (\cos \theta - 1)^2}{\theta^3 (4 \cos \theta - 4 + \theta \sin \theta)}$$

- $P = 2$  :

$$b_0 = -2 \frac{\sin^3 \theta}{\theta^2 (\theta \cos \theta - 3 \sin \theta)}$$

## Central formula : coefficients

E.g.  $b_0$  in case  $M = 6$

As a series ...

- $P = -1$  :

$$b_0 = 1$$

- $P = 0$  :

$$b_0 = 1 - \frac{1}{6}\theta^2 + \frac{1}{80}\theta^4 + \mathcal{O}(\theta^6)$$

- $P = 1$  :

$$b_0 = 1 - \frac{1}{3}\theta^2 + \frac{37}{720}\theta^4 + \mathcal{O}(\theta^6)$$

- $P = 2$  :

$$b_0 = 1 - \frac{1}{2}\theta^2 + \frac{7}{60}\theta^4 + \mathcal{O}(\theta^6)$$

## Central formula : local truncation error

$$\text{lte} = \mathcal{L}[y](t)$$

As an infinite series :

$$\text{lte} = h^M C_M D^{K+1} (D^2 + \omega^2)^{P+1} y(t) + \mathcal{O}(h^{M+2})$$

In closed form : (Coleman and Ixaru)

$$\text{lte} = h^M \Phi_{K,P}(Z) D^{K+1} (D^2 + \omega^2)^{P+1} y(\xi)$$

$$Z \in \text{some interval} \quad \Phi_{K,P}(0) \neq 0 \quad \xi \in (t - 2h, t + 2h)$$

## Central formula : local truncation error

$$\text{lte} = \mathcal{L}[y](t)$$

As an **inifinite series** :

$$\text{lte} = h^M C_M D^{K+1} (D^2 + \omega^2)^{P+1} y(t) + \mathcal{O}(h^{M+2})$$

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## Central formula : local truncation error

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## Local truncation error

$$\text{lte} = h^M C_M D^{K+1} (D^2 + \omega^2)^{P+1} y(t) + \mathcal{O}(h^{M+2}),$$

$$\text{At } t_j : D^{(K+1)} (D^2 + \omega_j^2)^{(P+1)} y(t) \Big|_{t=t_j} = 0 \quad j = 2, \dots, N-1$$

- $P = 0 :$

$$y^{(K+3)}(t_j) + y^{(K+1)}(t_j) \omega_j^2 = 0$$

- $P = 1 :$

$$y^{(K+5)}(t_j) + 2 y^{(K+3)}(t_j) \omega_j^2 + y^{(K+1)}(t_j) \omega_j^4 = 0$$

- $P = 2 :$

$$y^{(K+7)}(t_j) + 3 y^{(K+5)}(t_j) \omega_j^2 + 3 y^{(K+3)}(t_j) \omega_j^4 + y^{(K+1)}(t_j) \omega_j^6 = 0$$

-

## Local truncation error

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-

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-



## Local truncation error

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-

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$$y^{(K+5)}(t_j) + 2y^{(K+3)}(t_j) \omega_j^2 + y^{(K+1)}(t_j) \omega_j^4 = 0$$

- $P = 2 :$

$$y^{(K+7)}(t_j) + 3y^{(K+5)}(t_j) \omega_j^4 + 3y^{(K+3)}(t_j) \omega_j^4 + y^{(K+1)}(t_j) \omega_j^6 = 0$$

-

## Local truncation error

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$$\text{At } t_j : D^{(K+1)} (D^2 + \omega_j^2)^{(P+1)} y(t) \Big|_{t=t_j} = 0 \quad j = 2, \dots, N-1$$

$\omega_j^2$  is solution of equation of degree  $P + 1$ .

- Which value of  $P$  should be chosen ?
- Which root  $\omega_j$  should be chosen ?

## Local truncation error

$$l_{te} = h^M C_M D^{K+1} (D^2 + \omega^2)^{P+1} y(t) + \mathcal{O}(h^{M+2}),$$

$$\text{At } t_j : D^{(K+1)} (D^2 + \omega_j^2)^{(P+1)} y(t) \Big|_{t=t_j} = 0 \quad j = 2, \dots, N-1$$

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$\omega_j^2$  is solution of equation of degree  $P + 1$ .

- Which value of  $P$  should be chosen ?
- Which root  $\omega_j$  should be chosen ?

## Parameter selection

$$\text{lte} = h^M C_M D^{K+1} (D^2 - \mu^2)^{P+1} y(t) + \mathcal{O}(h^{M-2})$$

Suppose  $y(t)$  takes the form  $t^{P_0} e^{\mu_0 t}$

Then  $\text{lte} = 0$  for any EF rule with  $P \geq P_0$  and  $\mu_j = \mu_0$

### Theorem

*If  $y(t) = t^{P_0} e^{\mu_0 t}$  then  $\nu = \mu_0^2$  is a root of multiplicity  $P - P_0 + 1$  of  $D^{K+1} (D^2 - \nu)^{P+1} y(t) = 0$ .*

- if  $P = P_0$ , then  $\mu = \mu_0$  will be a single root
- if  $P = P_0 + 1$ , then  $\mu = \mu_0$  will be a double root
- if  $P = P_0 + 2$ , then  $\mu = \mu_0$  will be a triple root
- ...

## Parameter selection

$$\text{lte} = h^M C_M D^{K+1} (D^2 - \mu^2)^{P+1} y(t) + \mathcal{O}(h^{M-2})$$

Suppose  $y(t)$  takes the form  $t^{P_0} e^{\mu_0 t}$

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$$\text{lte} = h^M C_M D^{K+1} (D^2 - \mu^2)^{P+1} y(t) + \mathcal{O}(h^{M-2})$$

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### Theorem

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- if  $P = P_0$ , then  $\mu = \mu_0$  will be a single root
- if  $P = P_0 + 1$ , then  $\mu = \mu_0$  will be a double root
- if  $P = P_0 + 2$ , then  $\mu = \mu_0$  will be a triple root
- ...



## Parameter selection

$$l_{te} = h^M C_M D^{K+1} (D^2 - \mu^2)^{P+1} y(t) + \mathcal{O}(h^{M-2})$$

Suppose  $y(t)$  takes the form  $t^{P_0} e^{\mu_0 t}$

Then  $l_{te} = 0$  for any EF rule with  $P \geq P_0$  and  $\mu_j = \mu_0$

### Theorem

*If  $y(t) = t^{P_0} e^{\mu_0 t}$  then  $\nu = \mu_0^2$  is a root of multiplicity  $P - P_0 + 1$  of  $D^{K+1} (D^2 - \nu)^{P+1} y(t) = 0$ .*

- if  $P = P_0$ , then  $\mu = \mu_0$  will be a single root
- if  $P = P_0 + 1$ , then  $\mu = \mu_0$  will be a double root
- if  $P = P_0 + 2$ , then  $\mu = \mu_0$  will be a triple root
- ...

## Parameter selection

Suppose  $y(t)$  does not take the form  $t^{P_0} e^{\mu_0 t}$

Then  $y(t) \notin \mathcal{S}$  for any  $P$ .

For a given value of  $P$  :

$$D^{(K+1)} (D^2 - \mu_j^2)^{(P+1)} y(t) \Big|_{t=t_j} = 0$$

At each point  $t_j$ , this gives  $P + 1$  values for  $\mu_j^2$ .

Idea : keep  $|\mu_j h|$  as small as possible.

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Then  $y(t) \notin \mathcal{S}$  for any  $P$ .

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## First example

$$y^{(4)} - \frac{384 t^4}{(2 + t^2)^4} y = 24 \frac{2 - 11 t^2}{(2 + t^2)^4}$$

$$y(-1) = \frac{1}{3} \quad y(1) = \frac{1}{3}$$

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## $\mu_j$ for $M = 8$

$$P = 0 : y^{(8)}(t_j) - y^{(6)}(t_j) \mu_j^2 = 0$$

- re-express higher order derivatives in terms of  $y$ ,  $y'$ ,  $y''$  and  $y'''$
- approximate  $y'$ ,  $y''$  and  $y'''$  in terms of  $y$
- an initial approximation for  $y$  can be computed with a polynomial rule

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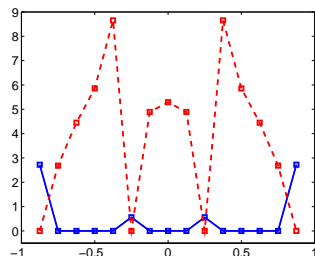
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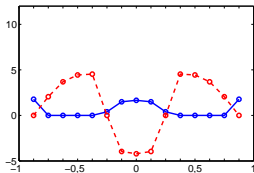
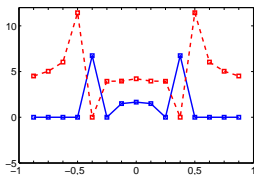
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Real and imaginary part  
of  $\mu_j$

$\mu_j$  for  $M = 8$ 

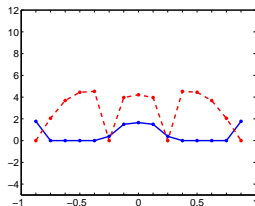
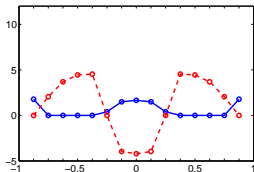
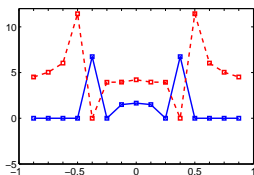
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Real and imag. part of  $\mu_{1,j}$  and  $\mu_{2,j}$

$\mu_j$  for  $M = 8$ 

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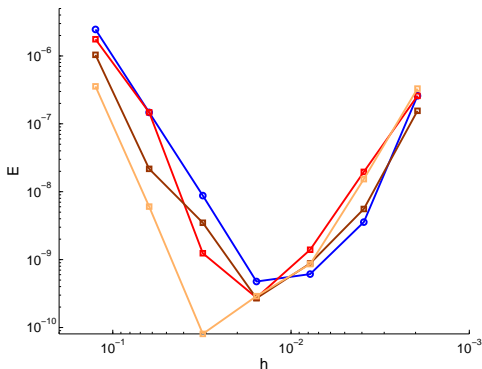


Real and imaginary  
part of  $\mu_j$  with smallest  
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Real and imag. part of  $\mu_{1,j}$  and  $\mu_{2,j}$

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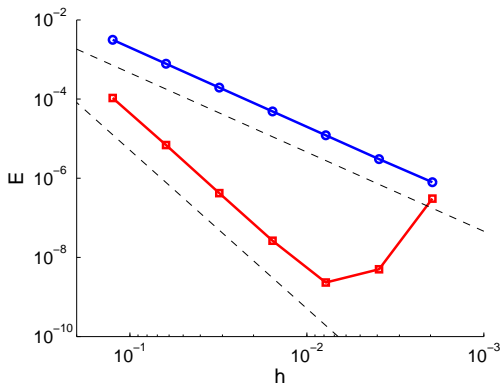


error obtained with  $\mu_{1,j}$ ,  $\mu_{2,j}$  and  $\mu$  with smallest norm



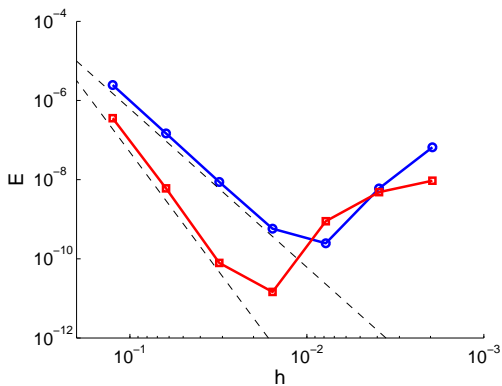
# Global error

$M = 6$  :  $(K, P) = (5, -1)$  : second-order method  
 $(K, P) = (1, 1)$  : fourth-order method



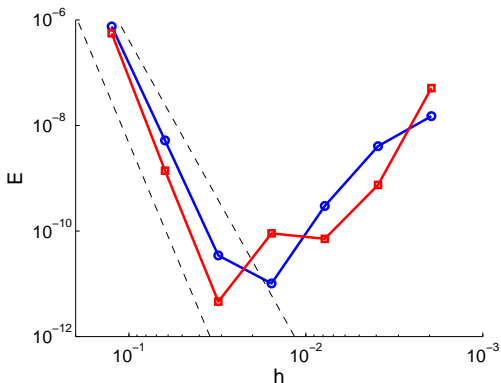
# Global error

$M = 8$  :  $(K, P) = (7, -1)$  : fourth-order method  
 $(K, P) = (3, 1)$  : sixth-order method

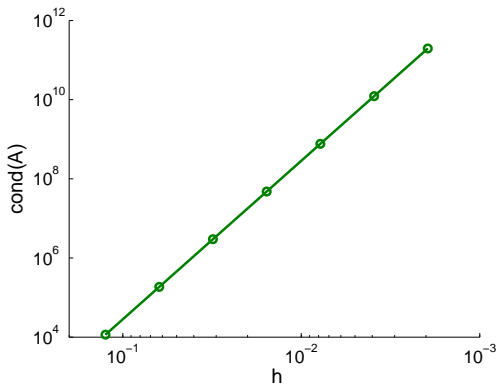


# Global error

$M = 10$  :  $(K, P) = (9, -1)$  : sixth-order method  
 $(K, P) = (5, 1)$  : eighth-order method



# Condition number



## Second example

$$y^{(4)} - t = 4e^t$$

$$y(-1) = -1/e \quad y(1) = e$$

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## $\mu_j$ for $M = 6$

$$P = 1 : y^{(6)}(t_j) - 2y^{(4)}(t_j)\mu_j^2 + y^{(2)}(t_j)\mu_j^4 = 0$$

differentiating the differential equation :

$$(y^{(2)}(t_j) + 4e^{t_j}) - 2(y_j + 4e^{t_j})\mu_j^2 + y^{(2)}(t_j)\mu_j^4 = 0$$

$y^{(2)}(t_j)$  approximated by fourth-order finite difference scheme

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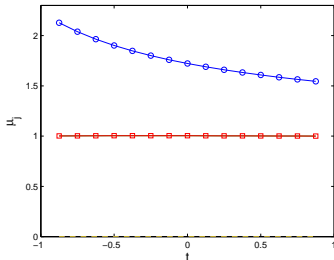
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two real roots  $\mu^{(1)}$  and  $\mu^{(2)}$

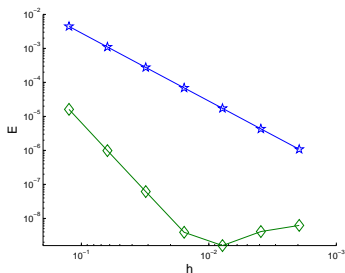
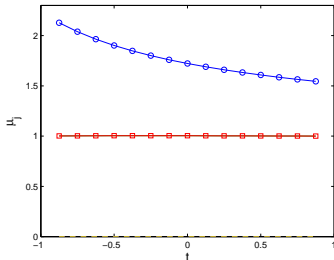
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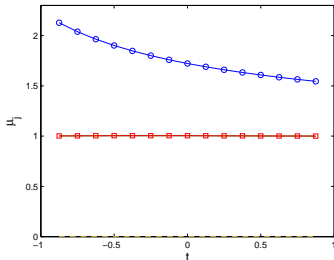
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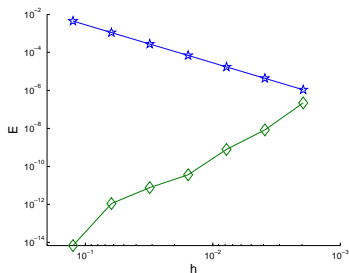
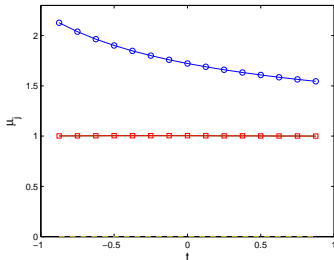
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# Conclusions

- Fourth-order boundary value problems are solved by means of parameterized exponentially-fitted methods.
- A suitable value for the parameter can be found from the roots of the leading term of the local truncation error.
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- However, the methods strongly suffer from the fact that the system to be solved is ill-conditioned for small values of the mesh size.

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