Exponentially fitted Runge-Kutta methods

Marnix Van Daele, G. Vanden Berghe,
H. Van de Vijver
Marnix.VanDaele@UGent.be

Vakgroep Toegepaste Wiskunde en Informatica
Universiteit Gent

Runge-Kutta methods

$$
\begin{aligned}
y_{n+1} &= y_n + h\sum_{i=1}^{s} b_i f(x_n + c_i h, Y_i) \\
Y_i &= y_n + h\sum_{j=1}^{s} a_{ij} f(x_n + c_j h, Y_j) \qquad i = 1, \ldots, s
\end{aligned}
$$

$$
\begin{array}{c|cccc}
c_1 & a_{11} & a_{12} & \ldots & a_{1s} \\
c_2 & a_{21} & a_{22} & \ldots & a_{2s} \\
 & & \ldots & & \\
c_s & a_{s1} & a_{s2} & \ldots & a_{ss} \\
\hline
 & b_1 & b_2 & \ldots & b_s
\end{array}
$$

Construction of RK methods

By introducing linear functionals :

$$\mathcal{L}_i[y(x);h;\mathbf{a}] = y(x+c_i h) - y(x) - h\sum_{j=1}^{s} a_{ij}\, y'(x+c_j h)$$

$$i = 1, 2, \ldots, s$$

$$\mathcal{L}[y(x);h;\mathbf{b}] = y(x+h) - y(x) - h\sum_{i=1}^{s} b_i\, y'(x+c_i h)$$

Power functions : $\begin{cases} \mathcal{L}_i[x^j;h;\mathbf{a}] = 0 & j = 1, \ldots, M \\ \mathcal{L}_[x^j;h;\mathbf{b}] = 0 & j = 1, \ldots, M' \end{cases}$

Consistency : $\mathcal{L}_i[1;h;\mathbf{a}] \equiv 0 \qquad \mathcal{L}[1;h;\mathbf{b}] \equiv 0$

$M \leq s$ $\qquad$ $M = s$: collocation methods $\qquad$ (stage order)

$s \leq M' \leq 2\,s$ $\qquad$ $M' = 2\,s$: Gauss methods $\qquad$ (order)

Construction of Exponential Fitted RK methods

$$\begin{aligned}\mathcal{L}_i[y(x);h;\mathbf{a}] &= y(x+c_ih) - y(x) - h\sum_{j=1}^{s} a_{ij}\, y'(x+c_jh) \\ & \quad i=1,2,\ldots,s \\ \mathcal{L}[y(x);h;\mathbf{b}] &= y(x+h) - y(x) - h\sum_{i=1}^{s} b_i\, y'(x+c_ih)\end{aligned}$$

Consistency : $\mathcal{L}_i[1;h;\mathbf{a}] \equiv 0 \qquad \mathcal{L}[1;h;\mathbf{b}] \equiv 0$

Exponential functions : $\exp(\pm\omega x), x\exp(\pm\omega x), x^2\exp(\pm\omega x), \ldots$

$$\begin{cases} \mathcal{L}_i[x^j\ \exp(\pm\omega x);h;\mathbf{a}] = 0 & j=0,\ \ldots,\ P \\ \mathcal{L}_[x^j\ \exp(\pm\omega x);h;\mathbf{b}] = 0 & j=0,\ \ldots,\ P' \end{cases}$$

Construction of
Exponential Fitted RK methods

$$\mathcal{L}_i[y(x);h;\mathbf{a}] \quad = \quad y(x+c_ih) - y(x) - h\sum_{j=1}^{s} a_{ij}\, y'(x+c_jh)$$

$$i = 1, 2, \ldots, s$$

$$\mathcal{L}[y(x);h;\mathbf{b}] \quad = \quad y(x+h) - y(x) - h\sum_{i=1}^{s} b_i\, y'(x+c_ih)$$

Consistency : $\mathcal{L}_i[1;h;\mathbf{a}] \equiv 0 \qquad \mathcal{L}[1;h;\mathbf{b}] \equiv 0$

Exponential functions : $\exp(\pm\omega x), x\exp(\pm\omega x), x^2\exp(\pm\omega x), \ldots$

$$\begin{cases} \mathcal{L}_i[x^j\ \exp(\pm\omega x);h;\mathbf{a}] = 0 & j = 0,\ \ldots,\ P \\ \mathcal{L}_[x^j\ \exp(\pm\omega x);h;\mathbf{b}] = 0 & j = 0,\ \ldots,\ P' \end{cases}$$

$\omega \rightarrow i\,\omega$: trigonometric functions : $x^j\ \sin\omega\, x$ and $x^j\ \cos\omega\, x$

Mixed-type RK methods

$$\mathcal{L}_i[y(x);h;\mathbf{a}] = y(x+c_i h) - y(x) - h\sum_{j=1}^{s} a_{ij}\, y'(x+c_j h) \quad i=1,2,\ldots,s$$

$$\mathcal{L}[y(x);h;\mathbf{b}] = y(x+h) - y(x) - h\sum_{i=1}^{s} b_i\, y'(x+c_i h)$$

$$\begin{cases} \mathcal{L}_i[x^j;h;\mathbf{a}] = 0 & j=1,\,\ldots,\,K \\ \mathcal{L}[x^j;h;\mathbf{b}] = 0 & j=1,\,\ldots,\,K' \end{cases}$$

$$\begin{cases} \mathcal{L}_i[x^j\ \exp(\pm\omega x);h;\mathbf{a}] = 0 & j=0,\,\ldots,\,P \\ \mathcal{L}[x^j\ \exp(\pm\omega x);h;\mathbf{b}] = 0 & j=0,\,\ldots,\,P' \end{cases}$$

Reference set :

$$\{1,\,x,\,\ldots,\,x^K, \exp(\pm\omega x), x\exp(\pm\omega x), \ldots, x^P\exp(\pm\omega x)\}$$

$$K + 2\,(P+1) = M \qquad M \le s$$

$$K' + 2\,(P'+1) = M' \qquad s \le M' \le 2\,s$$

Two stage collocation methods

We discuss the construction of two stage methods of collocation type, i.e. methods for which $M' \geq M = s = 2$.

- classical methods
  reference set for internal stages : $\{1, \, x, \, x^2\}$
- exponential fitted methods
  reference set for internal stages : $\{1, \, \exp(\omega \, x), \, \exp(-\omega \, x)\}$

In both cases, the reference set for the final stage is a superset of the reference set of the internal stages.

Classical RK methods

- $M' = 2$: arbitrary c_i's : order 2

$$\begin{array}{c|cc} c_1 & \dfrac{c_1(c_1-2c_2)}{2(c_1-c_2)} & \dfrac{c_1^2}{2(c_1-c_2)} \\ c_2 & \dfrac{c_2^2}{2(c_2-c_1)} & \dfrac{c_2(2c_1-c_2)}{2(c_1-c_2)} \\ \hline & \dfrac{(2c_2-1)}{2(c_2-c_1)} & \dfrac{(2c_1-1)}{2(c_2-c_1)} \end{array}$$

LobattoIIIA : $c_1 = 0$ and $c_2 = 1$

- $M' = 3$: add x^3 to reference set of final stage $c_1 = \dfrac{(3c_2-2)}{3(2c_2-1)}$

RadauIIA : $c_1 = \dfrac{1}{3}$ and $c_2 = 1$

- $M' = 4$: also add x^4 to reference set of final stage

Gauss : $c_1 = \dfrac{3-\sqrt{3}}{6}$ and $c_2 = \dfrac{3+\sqrt{3}}{6}$

Plte of classical RK methods

If a RK method has order p and stage order q

$$plte = \frac{h^{p+1}}{(p+1)!} \sum_{r(t)=p+1} \alpha(t)[1-\gamma(t)\psi(t)]F(t)$$

$$= h^{p+1}\left(a_{p+1}\, y^{(p+1)} + a_p\, y^{(p)} + \ldots + a_{q+1}\, y^{(q+1)}\right)$$

$$plte(LobattoIIIA) = -\frac{h^3}{12} y^{(3)}$$

$$plte(RadauIIA) = -\frac{h^4}{216}(y^{(4)} - 4 f_y y^{(3)})$$

$$plte(Gauss) = \frac{h^5}{4320}(y^{(5)} - 5 f_y y^{(4)} + 10(f_y^2 - (f_{xy} + f_{yy}\, f))y^{(3)})$$

EFRK methods $M' = 2$

$$\{1,\ \exp(\omega\, x),\ \exp(-\omega\, x)\}$$

$$\begin{array}{c|cc}
c_1 & \frac{\cosh(c_2-c_1)\,\nu-\cosh c_2\,\nu}{\nu\ \sinh(c_1-c_2)\,\nu} & \frac{\cosh c_1\,\nu-1}{\nu\ \sinh(c_1-c_2)\,\nu} \\
c_2 & \frac{1-\cosh c_2\,\nu}{\nu\ \sinh(c_1-c_2)\,\nu} & \frac{\cosh c_1\,\nu-\cosh(c_2-c_1)\,\nu}{\nu\ \sinh(c_1-c_2)\,\nu} \\
\hline
 & \frac{\cosh(1-c_2)\,\nu-\cosh c_2\,\nu}{\nu\ \sinh(c_1-c_2)\,\nu} & \frac{\cosh c_1\,\nu-\cosh(1-c_1)\,\nu}{\nu\ \sinh(c_1-c_2)\,\nu}
\end{array}
\qquad \nu = \omega\, h$$

EFRK methods $M' = 2$

$$\{1,\ \exp(\omega\, x),\ \exp(-\omega\, x)\}$$

$$\begin{array}{c|cc} c_1 & \dfrac{\cosh(c_2-c_1)\,\nu-\cosh c_2\,\nu}{\nu\ \sinh(c_1-c_2)\,\nu} & \dfrac{\cosh c_1\,\nu-1}{\nu\ \sinh(c_1-c_2)\,\nu} \\ c_2 & \dfrac{1-\cosh c_2\,\nu}{\nu\ \sinh(c_1-c_2)\,\nu} & \dfrac{\cosh c_1\,\nu-\cosh(c_2-c_1)\,\nu}{\nu\ \sinh(c_1-c_2)\,\nu} \\ \hline & \dfrac{\cosh(1-c_2)\,\nu-\cosh c_2\,\nu}{\nu\ \sinh(c_1-c_2)\,\nu} & \dfrac{\cosh c_1\,\nu-\cosh(1-c_1)\,\nu}{\nu\ \sinh(c_1-c_2)\,\nu} \end{array} \qquad \nu = \omega\, h$$

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & \dfrac{\cosh\nu-1}{\nu\ \sinh\nu} & \dfrac{\cosh\nu-1}{\nu\ \sinh\nu} \\ \hline & \dfrac{\cosh\nu-1}{\nu\ \sinh\nu} & \dfrac{\cosh\nu-1}{\nu\ \sinh\nu} \end{array}$$

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c_2 & \dfrac{1-\cosh c_2\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu} & \dfrac{\cosh c_1\,\nu-\cosh(c_2-c_1)\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu} \\
\hline
 & \dfrac{\cosh(1-c_2)\,\nu-\cosh c_2\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu} & \dfrac{\cosh c_1\,\nu-\cosh(1-c_1)\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu}
\end{array} \qquad \nu = \omega\, h$$

$$\begin{array}{c|cc}
0 & 0 & 0 \\
1 & \dfrac{\cosh\nu-1}{\nu\,\sinh\nu} & \dfrac{\cosh\nu-1}{\nu\,\sinh\nu} \\
\hline
 & \dfrac{\cosh\nu-1}{\nu\,\sinh\nu} & \dfrac{\cosh\nu-1}{\nu\,\sinh\nu}
\end{array} \qquad \frac{\cosh\nu-1}{\nu\,\sinh\nu} = \frac{1}{2} - \frac{1}{24}\nu^2 + \frac{1}{240}\nu^4 + \mathcal{O}(\nu^6)$$

EFRK methods $M' = 2$

$$\{1,\ \exp(\omega\, x),\ \exp(-\omega\, x)\}$$

$$\begin{array}{c|cc} c_1 & \frac{\cosh(c_2-c_1)\,\nu-\cosh c_2\,\nu}{\nu\ \sinh(c_1-c_2)\,\nu} & \frac{\cosh c_1\,\nu-1}{\nu\ \sinh(c_1-c_2)\,\nu} \\ c_2 & \frac{1-\cosh c_2\,\nu}{\nu\ \sinh(c_1-c_2)\,\nu} & \frac{\cosh c_1\,\nu-\cosh(c_2-c_1)\,\nu}{\nu\ \sinh(c_1-c_2)\,\nu} \\ \hline & \frac{\cosh(1-c_2)\,\nu-\cosh c_2\,\nu}{\nu\ \sinh(c_1-c_2)\,\nu} & \frac{\cosh c_1\,\nu-\cosh(1-c_1)\,\nu}{\nu\ \sinh(c_1-c_2)\,\nu} \end{array} \qquad \nu = \omega\, h$$

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & \frac{\cosh\nu-1}{\nu\ \sinh\nu} & \frac{\cosh\nu-1}{\nu\ \sinh\nu} \\ \hline & \frac{\cosh\nu-1}{\nu\ \sinh\nu} & \frac{\cosh\nu-1}{\nu\ \sinh\nu} \end{array} \qquad \frac{\cosh\nu-1}{\nu\ \sinh\nu} = \frac{1}{2} - \frac{1}{24}\nu^2 + \frac{1}{240}\nu^4 + \mathcal{O}(\nu^6)$$

$$plte(LobattoIIIA, exp) = -\frac{h^3}{12}(-\omega^2 y' + y^{(3)})$$

EFRK methods $M' = 2$

$$\{1,\, \exp(\omega\, x),\, \exp(-\omega\, x)\}$$

$$\begin{array}{c|cc} c_1 & \frac{\cosh(c_2-c_1)\,\nu-\cosh c_2\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu} & \frac{\cosh c_1\,\nu-1}{\nu\,\sinh(c_1-c_2)\,\nu} \\ c_2 & \frac{1-\cosh c_2\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu} & \frac{\cosh c_1\,\nu-\cosh(c_2-c_1)\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu} \\ \hline & \frac{\cosh(1-c_2)\,\nu-\cosh c_2\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu} & \frac{\cosh c_1\,\nu-\cosh(1-c_1)\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu} \end{array} \qquad \nu = \omega\, h$$

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & \frac{\cosh\nu-1}{\nu\,\sinh\nu} & \frac{\cosh\nu-1}{\nu\,\sinh\nu} \\ \hline & \frac{\cosh\nu-1}{\nu\,\sinh\nu} & \frac{\cosh\nu-1}{\nu\,\sinh\nu} \end{array} \qquad \frac{\cosh\nu-1}{\nu\,\sinh\nu} = \frac{1}{2} - \frac{1}{24}\nu^2 + \frac{1}{240}\nu^4 + \mathcal{O}(\nu^6)$$

$$plte(LobattoIIIA, exp) = -\frac{h^3}{12}(-\omega^2 y' + y^{(3)})$$

$\{1,\, \exp(\omega\, x),\, \exp(-\omega\, x)\}$ are the independent solutions of

$$-\omega^2 y' + y^{(3)} = 0$$

EFRK methods

How to obtain higher order methods ?

$$
\begin{array}{c|cc}
c_1 & \dfrac{\cosh(c_2-c_1)\,\nu-\cosh c_2\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu} & \dfrac{\cosh c_1\,\nu-1}{\nu\,\sinh(c_1-c_2)\,\nu} \\
c_2 & \dfrac{1-\cosh c_2\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu} & \dfrac{\cosh c_1\,\nu-\cosh(c_2-c_1)\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu} \\
\hline
 & \dfrac{\cosh(1-c_2)\,\nu-\cosh c_2\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu} & \dfrac{\cosh c_1\,\nu-\cosh(1-c_1)\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu}
\end{array}
$$

EFRK methods

How to obtain higher order methods ?

$$
\begin{array}{c|cc}
c_1 & \dfrac{\cosh(c_2-c_1)\,\nu-\cosh c_2\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu} & \dfrac{\cosh c_1\,\nu-1}{\nu\,\sinh(c_1-c_2)\,\nu} \\
c_2 & \dfrac{1-\cosh c_2\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu} & \dfrac{\cosh c_1\,\nu-\cosh(c_2-c_1)\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu} \\
\hline
 & \dfrac{\cosh(1-c_2)\,\nu-\cosh c_2\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu} & \dfrac{\cosh c_1\,\nu-\cosh(1-c_1)\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu}
\end{array}
$$

- CASE 1 : choose the classical c_i values of the Radau and Gauss-methods
- CASE 2 : increase M' to 3 or 4

EFRK methods

How to obtain higher order methods ?

RadauIIA methods

- case 1 : fixed c-values
- case 2 : ω-dependent c-values

Case 1

$$
\begin{array}{c|cc}
\frac{1}{3} & \dfrac{\cosh\nu - \cosh\frac{2}{3}\nu}{\nu\sinh\frac{2}{3}\nu} & \dfrac{1-\cosh\frac{1}{3}\nu}{\nu\sinh\frac{2}{3}\nu} \\
1 & \dfrac{\cosh\nu - 1}{\nu\sinh\frac{2}{3}\nu} & \dfrac{\cosh\frac{2}{3}\nu - \cosh\frac{1}{3}\nu}{\nu\sinh\frac{2}{3}\nu} \\
\hline
 & \dfrac{1-\cosh\frac{2}{3}\nu}{\nu\sinh\frac{2}{3}\nu} & \dfrac{\cosh\frac{2}{3}\nu - \cosh\frac{1}{3}\nu}{\nu\sinh\frac{2}{3}\nu}
\end{array}
$$

Case 1

$$
\begin{array}{c|cc}
\frac{1}{3} & \frac{\cosh\nu - \cosh\frac{2}{3}\nu}{\nu\,\sinh\frac{2}{3}\nu} & \frac{1-\cosh\frac{1}{3}\nu}{\nu\,\sinh\frac{2}{3}\nu} \\
1 & \frac{\cosh\nu - 1}{\nu\,\sinh\frac{2}{3}\nu} & \frac{\cosh\frac{2}{3}\nu - \cosh\frac{1}{3}\nu}{\nu\,\sinh\frac{2}{3}\nu} \\
\hline
 & \frac{1-\cosh\frac{2}{3}\nu}{\nu\,\sinh\frac{2}{3}\nu} & \frac{\cosh\frac{2}{3}\nu - \cosh\frac{1}{3}\nu}{\nu\,\sinh\frac{2}{3}\nu}
\end{array}
$$

$$
\begin{aligned}
a_{11} &= \frac{5}{12} + \frac{25}{1296}\nu^2 - \frac{5}{23328}\nu^4 + \mathcal{O}(\nu^6) \\
a_{12} &= -\frac{1}{12} + \frac{7}{1296}\nu^2 - \frac{31}{116640}\nu^4 + \mathcal{O}(\nu^6) \\
a_{21} = b_1 &= \frac{3}{4} + \frac{1}{144}\nu^2 + \frac{13}{38880}\nu^4 + \mathcal{O}(\nu^6) \\
a_{22} = b_2 &= \frac{1}{4} - \frac{1}{144}\nu^2 + \frac{11}{38880}\nu^4 + \mathcal{O}(\nu^6)
\end{aligned}
$$

Case 1

$$
\begin{aligned}
\sum_i b_i &= 1 \qquad\qquad\qquad + \frac{1}{1620}\nu^4 + \mathcal{O}(\nu^6)\\
\sum_i b_i c_i &= \frac{1}{2} - \frac{1}{216}\nu^2 + \frac{23}{58320}\nu^4 + \mathcal{O}(\nu^6)\\
\sum_{i,j} b_i a_{ij} &= \frac{1}{2} + \quad \frac{1}{72}\nu^2 + \frac{7}{19440}\nu^4 + \mathcal{O}(\nu^6)\\
\sum_i b_i c_i^2 &= \frac{1}{3} - \frac{1}{162}\nu^2 + \frac{7}{21870}\nu^4 + \mathcal{O}(\nu^6)\\
\sum_{i,j,k} b_i\, a_{ij}\, a_{ik} &= \frac{1}{3} + \frac{1}{162}\,\nu^2 + \frac{7}{7290}\nu^4 + \mathcal{O}(\nu^6)\\
\sum_{i,j} b_i\, a_{ij}\, c_i &= \frac{1}{3} \qquad\qquad\qquad - \frac{1}{2430}\,\nu^4 + \mathcal{O}(\nu^6)\\
\sum_{i,j,k} b_i\, a_{ij}\, a_{jk} &= \frac{1}{6} + \frac{11}{648}\nu^2 + \frac{13}{58320}\nu^4 + \mathcal{O}(\nu^6)\\
\sum_{i,j} b_i a_{ij} c_j &= \frac{1}{6} + \frac{1}{216}\nu^2 + \frac{7}{58320}\nu^4 + \mathcal{O}(\nu^6)
\end{aligned}
$$

$$\nu = \omega\, h$$

Case 1

$$
\begin{array}{ll}
h & \sum_i b_i = 1 \qquad\qquad + \frac{1}{1620}\nu^4 + \mathcal{O}(\nu^6) \\
h^2 & \sum_i b_i c_i = \frac{1}{2} - \frac{1}{216}\nu^2 + \frac{23}{58320}\nu^4 + \mathcal{O}(\nu^6) \\
h^2 & \sum_{i,j} b_i a_{ij} = \frac{1}{2} + \frac{1}{72}\nu^2 + \frac{7}{19440}\nu^4 + \mathcal{O}(\nu^6) \\
h^3 & \sum_i b_i c_i^2 = \frac{1}{3} - \frac{1}{162}\nu^2 + \frac{7}{21870}\nu^4 + \mathcal{O}(\nu^6) \\
h^3 & \sum_{i,j,k} b_i\, a_{ij}\, a_{ik} = \frac{1}{3} + \frac{1}{162}\nu^2 + \frac{7}{7290}\nu^4 + \mathcal{O}(\nu^6) \\
h^3 & \sum_{i,j} b_i\, a_{ij}\, c_i = \frac{1}{3} \qquad\qquad - \frac{1}{2430}\nu^4 + \mathcal{O}(\nu^6) \\
h^3 & \sum_{i,j,k} b_i\, a_{ij}\, a_{jk} = \frac{1}{6} + \frac{11}{648}\nu^2 + \frac{13}{58320}\nu^4 + \mathcal{O}(\nu^6) \\
h^3 & \sum_{i,j} b_i a_{ij} c_j = \frac{1}{6} + \frac{1}{216}\nu^2 + \frac{7}{58320}\nu^4 + \mathcal{O}(\nu^6)
\end{array}
$$

$$\nu = \omega\, h$$

Case 1

h	$\sum_i b_i = 1 + \frac{1}{1620}\nu^4 + \mathcal{O}(\nu^6)$	h^5
h^2	$\sum_i b_i c_i = \frac{1}{2} - \frac{1}{216}\nu^2 + \frac{23}{58320}\nu^4 + \mathcal{O}(\nu^6)$	h^4
h^2	$\sum_{i,j} b_i a_{ij} = \frac{1}{2} + \frac{1}{72}\nu^2 + \frac{7}{19440}\nu^4 + \mathcal{O}(\nu^6)$	h^4
h^3	$\sum_i b_i c_i^2 = \frac{1}{3} - \frac{1}{162}\nu^2 + \frac{7}{21870}\nu^4 + \mathcal{O}(\nu^6)$	h^5
h^3	$\sum_{i,j,k} b_i\, a_{ij}\, a_{ik} = \frac{1}{3} + \frac{1}{162}\nu^2 + \frac{7}{7290}\nu^4 + \mathcal{O}(\nu^6)$	h^5
h^3	$\sum_{i,j} b_i\, a_{ij}\, c_i = \frac{1}{3} - \frac{1}{2430}\nu^4 + \mathcal{O}(\nu^6)$	h^5
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h^3	$\sum_{i,j} b_i a_{ij} c_j = \frac{1}{6} + \frac{1}{216}\nu^2 + \frac{7}{58320}\nu^4 + \mathcal{O}(\nu^6)$	h^5

$$\nu = \omega\, h$$

Case 1

$$plte(RadauIIA, exp, \text{case } 1) =$$
$$-\frac{h^4}{216}\left(-\omega^2 y^{(2)} + y^{(4)} - 4f_y(-\omega^2 y' + y^{(3)})\right)$$

Case 1

$$plte(RadauIIA, exp, \text{case } 1) = \\ -\frac{h^4}{216}\left(-\omega^2 y^{(2)} + y^{(4)} - 4 f_y(-\omega^2 y' + y^{(3)})\right)$$

- internal stages :

  $\{1, \exp(\omega\, x), \exp(-\omega\, x)\}$ are the independent solutions of $-\omega^2 y' + y^{(3)} = 0$

Case 1

$$plte(RadauIIA, exp, \text{case } 1) =$$

$$-\frac{h^4}{216}\left(-\omega^2 y^{(2)} + y^{(4)} - 4 f_y(-\omega^2 y' + y^{(3)})\right)$$

- internal stages :

  $\{1, \exp(\omega\, x), \exp(-\omega\, x)\}$ are the independent solutions of $-\omega^2 y' + y^{(3)} = 0$

- final stage :

  $\{1, \exp(\omega\, x), \exp(-\omega\, x)\}$ are independent solutions of $-\omega^2 y^{(2)} + y^{(4)} = 0$ and for $y(x) = x$ there is an error $\mathcal{O}(h^5)$

Case 2 : $M' = 3$

$$
\begin{array}{c|cc}
c_1 & \frac{\cosh(c_2-c_1)\,\nu-\cosh c_2\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu} & \frac{\cosh c_1\,\nu-1}{\nu\,\sinh(c_1-c_2)\,\nu} \\
c_2 & \frac{1-\cosh c_2\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu} & \frac{\cosh c_1\,\nu-\cosh(c_2-c_1)\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu} \\
\hline
 & \frac{\cosh(1-c_2)\,\nu-\cosh c_2\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu} & \frac{\cosh c_1\,\nu-\cosh(1-c_1)\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu}
\end{array}
$$

Case 2 : $M' = 3$

$$
\begin{array}{c|cc}
c_1 & \frac{\cosh(c_2-c_1)\,\nu-\cosh c_2\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu} & \frac{\cosh c_1\,\nu-1}{\nu\,\sinh(c_1-c_2)\,\nu} \\
c_2 & \frac{1-\cosh c_2\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu} & \frac{\cosh c_1\,\nu-\cosh(c_2-c_1)\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu} \\
\hline
 & \frac{\cosh(1-c_2)\,\nu-\cosh c_2\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu} & \frac{\cosh c_1\,\nu-\cosh(1-c_1)\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu}
\end{array}
$$

Add x to the reference set : $b_1 + b_2 - 1 = 0$

Case 2 : $M' = 3$

$$
\begin{array}{c|cc}
c_1 & \frac{\cosh(c_2-c_1)\,\nu-\cosh c_2\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu} & \frac{\cosh c_1\,\nu-1}{\nu\,\sinh(c_1-c_2)\,\nu} \\
c_2 & \frac{1-\cosh c_2\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu} & \frac{\cosh c_1\,\nu-\cosh(c_2-c_1)\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu} \\
\hline
 & \frac{\cosh(1-c_2)\,\nu-\cosh c_2\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu} & \frac{\cosh c_1\,\nu-\cosh(1-c_1)\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu}
\end{array}
$$

Add x to the reference set : $b_1 + b_2 - 1 = 0$

$$\cosh(1-c_2)\,\nu-\cosh c_2\,\nu+\cosh c_1\,\nu-\cosh(1-c_1)\,\nu = \nu\,\sinh(c_1-c_2)\,\nu$$

Case 2 : $M' = 3$

$$
\begin{array}{c|cc}
c_1 & \frac{\cosh(c_2-c_1)\,\nu-\cosh c_2\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu} & \frac{\cosh c_1\,\nu-1}{\nu\,\sinh(c_1-c_2)\,\nu} \\
c_2 & \frac{1-\cosh c_2\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu} & \frac{\cosh c_1\,\nu-\cosh(c_2-c_1)\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu} \\
\hline
 & \frac{\cosh(1-c_2)\,\nu-\cosh c_2\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu} & \frac{\cosh c_1\,\nu-\cosh(1-c_1)\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu}
\end{array}
$$

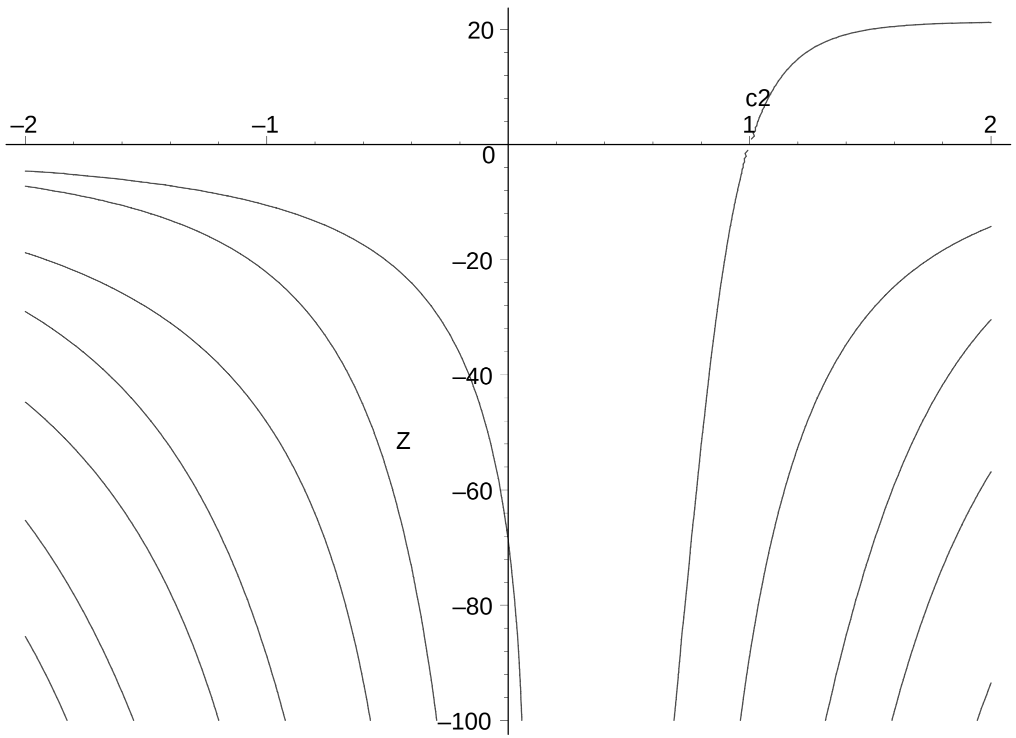
Add x to the reference set : $b_1 + b_2 - 1 = 0$

$$
\cosh(1-c_2)\,\nu-\cosh c_2\,\nu+\cosh c_1\,\nu-\cosh(1-c_1)\,\nu = \nu\,\sinh(c_1-c_2)\,\nu
$$

- case 2a : $c_1 = \frac{1}{3}$
- case 2b : $c_2 = 1$

Case 2a : $b_1 + b_2 = 1$ and $c_1 = \frac{1}{3}$

c_1	$\frac{\cosh(c_2-c_1)\,\nu-\cosh c_2\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu}$	$\frac{\cosh c_1\,\nu-1}{\nu\,\sinh(c_1-c_2)\,\nu}$
c_2	$\frac{1-\cosh c_2\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu}$	$\frac{\cosh c_1\,\nu-\cosh(c_2-c_1)\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu}$
	$\frac{\cosh(1-c_2)\,\nu-\cosh c_2\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu}$	$\frac{\cosh c_1\,\nu-\cosh(1-c_1)\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu}$



$$Z = \nu^2 = \omega^2\,h^2$$

Case 2a : $b_1 + b_2 = 1$ and $c_1 = \frac{1}{3}$

$$
\begin{array}{c|cc}
c_1 & \dfrac{\cosh(c_2-c_1)\,\nu-\cosh c_2\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu} & \dfrac{\cosh c_1\,\nu-1}{\nu\,\sinh(c_1-c_2)\,\nu} \\
c_2 & \dfrac{1-\cosh c_2\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu} & \dfrac{\cosh c_1\,\nu-\cosh(c_2-c_1)\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu} \\
\hline
 & \dfrac{\cosh(1-c_2)\,\nu-\cosh c_2\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu} & \dfrac{\cosh c_1\,\nu-\cosh(1-c_1)\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu}
\end{array}
$$

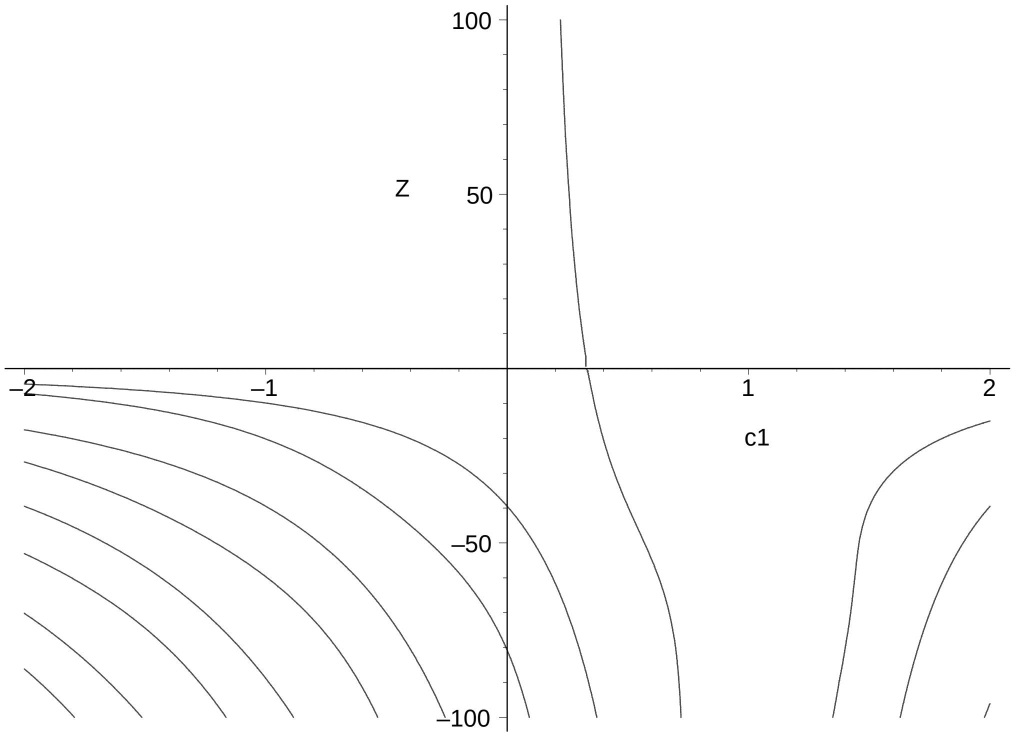
$$
c_2 = \begin{cases}
\dfrac{2}{3} + \dfrac{1}{\sqrt{Z}} \log\left(\dfrac{-G_1 + 1 + \sqrt{Z}G_1^{1/3}}{-1 + G_1 - \sqrt{Z}G_1^{2/3}}\right) & Z > 0 \\
1 & Z = 0 \\
\dfrac{2}{3} - \dfrac{i}{\sqrt{-Z}} \log\left(\dfrac{-G_2 + 1 + i\sqrt{-Z}G_2^{1/3}}{-1 + G_2 - i\sqrt{-Z}G_2^{2/3}}\right) & Z < 0
\end{cases}
$$

$$
G_1 = \exp(\sqrt{Z}) \qquad G_2 = \exp(i\,\sqrt{-Z})
$$

$$
c_2 = 1 + \frac{1}{135}Z + \frac{19}{102060}Z^2 + \mathcal{O}(Z^3)
$$

Case 2b : $b_1 + b_2 = 1$ and $c_2 = 1$

$$
\begin{array}{c|cc}
c_1 & \dfrac{\cosh(c_2-c_1)\,\nu-\cosh c_2\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu} & \dfrac{\cosh c_1\,\nu-1}{\nu\,\sinh(c_1-c_2)\,\nu} \\
c_2 & \dfrac{1-\cosh c_2\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu} & \dfrac{\cosh c_1\,\nu-\cosh(c_2-c_1)\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu} \\
\hline
 & \dfrac{\cosh(1-c_2)\,\nu-\cosh c_2\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu} & \dfrac{\cosh c_1\,\nu-\cosh(1-c_1)\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu}
\end{array}
$$



$Z = \nu^2$

Case 2b : $b_1 + b_2 = 1$ and $c_2 = 1$

$$\begin{array}{c|cc} c_1 & \dfrac{\cosh(c_2-c_1)\,\nu - \cosh c_2\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu} & \dfrac{\cosh c_1\,\nu - 1}{\nu\,\sinh(c_1-c_2)\,\nu} \\ c_2 & \dfrac{1-\cosh c_2\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu} & \dfrac{\cosh c_1\,\nu - \cosh(c_2-c_1)\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu} \\ \hline & \dfrac{\cosh(1-c_2)\,\nu - \cosh c_2\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu} & \dfrac{\cosh c_1\,\nu - \cosh(1-c_1)\,\nu}{\nu\,\sinh(c_1-c_2)\,\nu} \end{array}$$

$$c_1 = \begin{cases} \dfrac{1}{\sqrt{Z}} \log\left(\dfrac{-G_1 + 1 + \sqrt{Z}G_1}{-1 + G_1 - \sqrt{Z}}\right) & Z > 0 \\ \dfrac{1}{3} & Z = 0 \\ \dfrac{-i}{\sqrt{-Z}} \log\left(\dfrac{-G_2 + 1 + i\sqrt{-Z}G_2}{-1 + G_2 - i\sqrt{-Z}}\right) & Z < 0 \end{cases}$$

$$G_1 = \exp(\sqrt{Z}) \qquad G_2 = \exp(i\sqrt{-Z})$$

$$c_1 = \frac{1}{3} - \frac{1}{405}Z + \frac{1}{34020}Z^2 + \mathcal{O}(Z^3)$$

The PLTE

$$plte(RadauIIA, exp, \text{case } 1) =$$
$$-\frac{h^4}{216}\left(-\omega^2 y^{(2)} + y^{(4)} - 4f_y(-\omega^2 y' + y^{(3)})\right)$$

The PLTE

$$plte(RadauIIA, exp, \text{case 1}) =$$

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$$plte(RadauIIA, exp, \text{case 2}) =$$

$$-\frac{h^4}{216}\left(-\omega^2 y^{(2)} + y^{(4)} - 4f_y(-\omega^2 y' + y^{(3)})\right)$$

The PLTE

$$plte(RadauIIA, exp, \text{case } 1) =$$
$$-\frac{h^4}{216}\left(-\omega^2 y^{(2)} + y^{(4)} - 4 f_y(-\omega^2 y' + y^{(3)})\right)$$

$$plte(RadauIIA, exp, \text{case } 2) =$$
$$-\frac{h^4}{216}\left(-\omega^2 y^{(2)} + y^{(4)} - 4 f_y(-\omega^2 y' + y^{(3)})\right)$$

- internal stages :
  $\{1, \exp(\omega\, x), \exp(-\omega\, x)\}$ are the independent solutions of $-\omega^2 y' + y^{(3)} = 0$
- final stage :
  $\{1, x, \exp(\omega\, x), \exp(-\omega\, x)\}$ are the independent solutions of $-\omega^2 y^{(2)} + y^{(4)} = 0$

EFRK methods

How to obtain higher order methods ?

Gauss methods

- case 1 : fixed c-values
- case 2 : ω-dependent c-values

Numerical examples

- fixed stepsize
- global errors in the endpoint $x = 1$
- ω obtained by annihilating the plte
- RadauIIA methods

Example 1

$$y' = y \qquad y(0) = 1$$

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$$\omega = 1$$

h	classical	case 1	case 2a	case 2b
1	$5.16\ 10^{-2}$	$1.33\ 10^{-15}$	0.00	$1.78\ 10^{-15}$
$\frac{1}{2}$	$5.55\ 10^{-2}$	$4.44\ 10^{-16}$	$8.88\ 10^{-16}$	$3.11\ 10^{-15}$
$\frac{1}{4}$	$6.33\ 10^{-4}$	$8.88\ 10^{-16}$	$1.78\ 10^{-15}$	$1.69\ 10^{-14}$
$\frac{1}{8}$	$7.63\ 10^{-5}$	0.00	$4.44\ 10^{-16}$	0.00
$\frac{1}{16}$	$9.37\ 10^{-6}$	$1.33\ 10^{-15}$	$1.33\ 10^{-15}$	$4.44\ 10^{-16}$

Example 2

$$\begin{cases} y_1' = -y_2 + \cos x + \sin 2\,x \\ y_2' = y_1 + 2\,\cos 2\,x - \sin x \end{cases} \qquad \begin{cases} y_1(0) = 0 \\ y_2(0) = 0 \end{cases}$$

Example 2

$$\begin{cases} y_1' = -y_2 + \cos x + \sin 2\,x \\ y_2' = y_1 + 2\ \cos 2\,x - \sin x \end{cases} \qquad \begin{cases} y_1(0) = 0 \\ y_2(0) = 0 \end{cases} \qquad \begin{cases} y_1(x) = \sin x \\ y_2(x) = \sin 2\,x \end{cases}$$

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$$\omega_1^2 = -1 \qquad \omega_2^2 = -4$$

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$$\begin{cases} y_1' = -y_2 + \cos x + \sin 2\,x \\ y_2' = y_1 + 2\,\cos 2\,x - \sin x \end{cases} \qquad \begin{cases} y_1(0) = 0 \\ y_2(0) = 0 \end{cases} \qquad \begin{cases} y_1(x) = \sin x \\ y_2(x) = \sin 2\,x \end{cases}$$

$$\omega_1^2 = -1 \qquad \omega_2^2 = -4$$

h	classical	case 1	case 2a	case 2b
1	$8.25\ 10^{-2}$	0.00	$3.98\ 10^{-3}$	$4.59\ 10^{-3}$
	$2.60\ 10^{-2}$	$1.11\ 10^{-16}$	$9.97\ 10^{-3}$	$3.07\ 10^{-3}$
$\frac{1}{2}$	$8.91\ 10^{-3}$	$1.11\ 10^{-16}$	$2.57\ 10^{-4}$	$1.12\ 10^{-4}$
	$1.83\ 10^{-3}$	$1.11\ 10^{-16}$	$1.00\ 10^{-3}$	$6.61\ 10^{-4}$
$\frac{1}{4}$	$1.11\ 10^{-3}$	$2.22\ 10^{-16}$	$2.48\ 10^{-5}$	$5.76\ 10^{-6}$
	$2.08\ 10^{-1}$	0.00	$1.21\ 10^{-4}$	$9.82\ 10^{-5}$
$\frac{1}{8}$	$1.40\ 10^{-4}$	0.00	$2.77\ 10^{-6}$	$1.69\ 10^{-6}$
	$2.57\ 10^{-5}$	$1.11\ 10^{-16}$	$1.50\ 10^{-5}$	$1.33\ 10^{-5}$
$\frac{1}{16}$	$1.77\ 10^{-5}$	$1.11\ 10^{-16}$	$3.29\ 10^{-7}$	$2.65\ 10^{-7}$
	$3.24\ 10^{-6}$	$1.11\ 10^{-16}$	$1.83\ 10^{-6}$	$1.73\ 10^{-6}$

What went wrong ?

How to choose ω when an EFRK is applied to a system ?

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- apply the same ω for each component
- choose a separate ω for each component

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$$\begin{cases} y' = f(x, y, z) \\ z' = g(x, y, z) \end{cases}$$

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$$\begin{cases} y' = f(x, y, z) \\ z' = g(x, y, z) \end{cases}$$

The order of a PRK is determined by

- order conditions for each of the RK methods in the PRK method
- coupling conditions

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The order of our PRK methods is always the same as the order of the RK methods.

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The order of a PRK is determined by

- order conditions for each of the RK methods in the PRK method
- coupling conditions

The order of our PRK methods is always the same as the order of the RK methods.

However : there are coupling terms

Case 1

$$y' = f(x,\, y) \qquad plte(RadauIIA, exp, \text{case } 1) =$$

$$-\frac{h^4}{216}\left(-\omega^2 y^{(2)} + y^{(4)} - 4 f_y(-\omega^2 y' + y^{(3)})\right)$$

Case 1

$$y' = f(x,\, y) \qquad plte(RadauIIA, exp, \text{case } 1) =$$

$$-\frac{h^4}{216}\left(-\omega^2 y^{(2)} + y^{(4)} - 4f_y(-\omega^2 y' + y^{(3)})\right)$$

$$\begin{cases} y' = f(x, y, z) \\ z' = g(x, y, z) \end{cases} \qquad plte(RadauIIA, exp, \text{case } 1) =$$

$$-\frac{h^4}{216}\left\{-\begin{pmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{pmatrix}\begin{pmatrix} y^{(2)} \\ z^{(2)} \end{pmatrix} + \begin{pmatrix} y^{(4)} \\ z^{(4)} \end{pmatrix}\right.$$

$$\left. -4\begin{pmatrix} f_y & f_z \\ g_y & g_z \end{pmatrix}\left[-\begin{pmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{pmatrix}\begin{pmatrix} y' \\ z' \end{pmatrix} + \begin{pmatrix} y^{(3)} \\ z^{(3)} \end{pmatrix}\right]\right\}$$

Case 2a

$$y' = f(x,\, y) \qquad plte(RadauIIA, exp, \text{case 2a}) =$$
$$-\frac{h^4}{216}\left(-\omega^2 y^{(2)} + y^{(4)} - 4 f_y(-\omega^2 y' + y^{(3)})\right)$$

Case 2a

$$y' = f(x,\, y) \qquad plte(RadauIIA, exp, \text{case 2a}) =$$
$$-\frac{h^4}{216}\left(-\omega^2 y^{(2)} + y^{(4)} - 4 f_y(-\omega^2 y' + y^{(3)})\right)$$

$$\begin{cases} y' = f(x, y, z) \\ z' = g(x, y, z) \end{cases} \qquad plte(RadauIIA, exp, \text{case } 2a) =$$

$$-\frac{h^4}{216}\left\{ - \begin{pmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{pmatrix} \begin{pmatrix} y^{(2)} \\ z^{(2)} \end{pmatrix} + \begin{pmatrix} y^{(4)} \\ z^{(4)} \end{pmatrix} \right.$$
$$-4 \begin{pmatrix} f_y & f_z \\ g_y & g_z \end{pmatrix} \left[- \begin{pmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{pmatrix} \begin{pmatrix} y' \\ z' \end{pmatrix} + \begin{pmatrix} y^{(3)} \\ z^{(3)} \end{pmatrix} \right]$$
$$\left. -\frac{2}{5} \begin{pmatrix} (\omega_1^2 - \omega_2^2)\, f_z\, g \\ (\omega_2^2 - \omega_1^2)\, g_y\, f \end{pmatrix} \right\}$$

Case 2b

$$y' = f(x,\, y) \qquad plte(RadauIIA, exp, \text{case 2b}) =$$

$$-\frac{h^4}{216}\left(-\omega^2 y^{(2)} + y^{(4)} - 4 f_y(-\omega^2 y' + y^{(3)})\right)$$

Case 2b

$$y' = f(x,\, y) \qquad plte(RadauIIA, exp, \text{case 2b}) =$$

$$-\frac{h^4}{216}\left(-\omega^2 y^{(2)} + y^{(4)} - 4 f_y(-\omega^2 y' + y^{(3)})\right)$$

$$\begin{cases} y' = f(x,y,z) \\ z' = g(x,y,z) \end{cases} \qquad plte(RadauIIA, exp, \text{case } 2b) =$$

$$-\frac{h^4}{216}\left\{-\begin{pmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{pmatrix}\begin{pmatrix} y^{(2)} \\ z^{(2)} \end{pmatrix} + \begin{pmatrix} y^{(4)} \\ z^{(4)} \end{pmatrix}\right.$$

$$-4\begin{pmatrix} f_y & f_z \\ g_y & g_z \end{pmatrix}\left[-\begin{pmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{pmatrix}\begin{pmatrix} y' \\ z' \end{pmatrix} + \begin{pmatrix} y^{(3)} \\ z^{(3)} \end{pmatrix}\right]$$

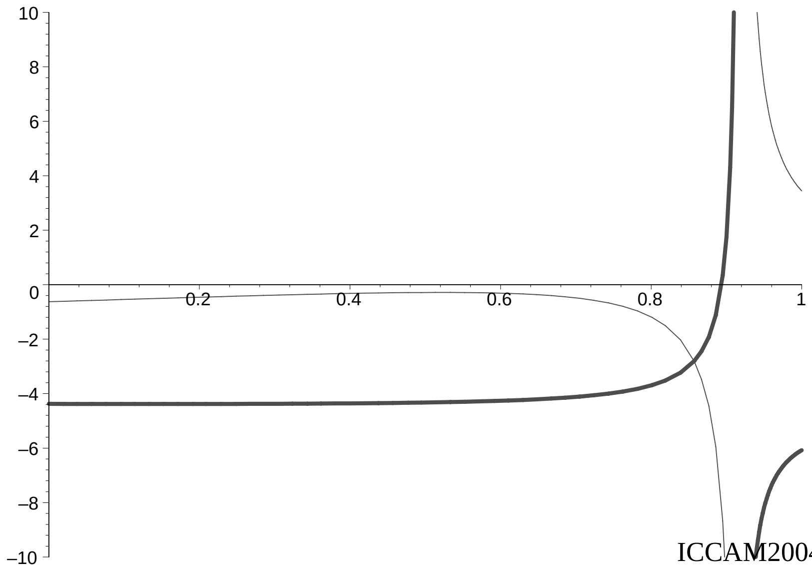
$$\left. + \frac{2}{5}\begin{pmatrix} (\omega_1^2 - \omega_2^2)\, f_z\, g \\ (\omega_2^2 - \omega_1^2)\, g_y\, f \end{pmatrix}\right\}$$

Example 2

$$\begin{cases} y_1' = -y_2 + \cos x + \sin 2\,x \\ y_2' = y_1 + 2\ \cos 2\,x - \sin x \end{cases} \qquad \begin{cases} y_1(0) = 0 \\ y_2(0) = 0 \end{cases} \qquad \begin{cases} y_1(x) = \sin x \\ y_2(x) = \sin 2\,x \end{cases}$$

For case 2b :

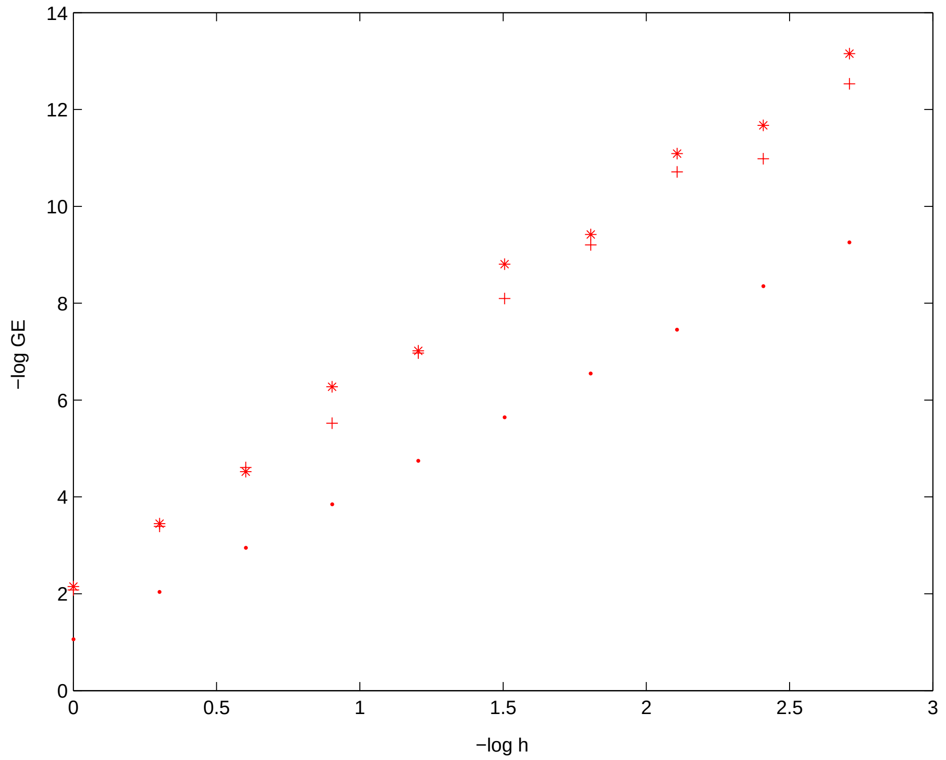
$$\begin{cases} \omega_1^2 = -\dfrac{128\ \sin^3 x - 60\ \sin^2 x - 63\ \sin x + 40}{32\ \sin^3 x - 108\ \sin^2 x - 15\ \sin x + 64} \\ \omega_2^2 = -\dfrac{128\ \sin^3 x - 480\ \sin^2 x - 63\ \sin x + 280}{32\ \sin^3 x - 108\ \sin^2 x - 15\ \sin x + 64} \end{cases}$$



Example 2

$$\begin{cases} y_1' = -y_2 + \cos x + \sin 2\,x \\ y_2' = y_1 + 2\,\cos 2\,x - \sin x \end{cases} \qquad \begin{cases} y_1(0) = 0 \\ y_2(0) = 0 \end{cases} \qquad \begin{cases} y_1(x) = \sin x \\ y_2(x) = \sin 2\,x \end{cases}$$

h	case 2a	case 2b
1	$7.11\ 10^{-3}$	$4.06\ 10^{-4}$
	$1.71\ 10^{-4}$	$8.30\ 10^{-3}$
$\frac{1}{2}$	$3.53\ 10^{-4}$	$1.02\ 10^{-4}$
	$5.07\ 10^{-5}$	$3.93\ 10^{-4}$
$\frac{1}{4}$	$2.91\ 10^{-5}$	$1.72\ 10^{-5}$
	$7.92\ 10^{-6}$	$1.78\ 10^{-5}$
$\frac{1}{8}$	$4.45\ 10^{-7}$	$3.00\ 10^{-6}$
	$2.88\ 10^{-7}$	$3.35\ 10^{-7}$
$\frac{1}{16}$	$8.73\ 10^{-8}$	$4.64\ 10^{-8}$
	$4.01\ 10^{-8}$	$9.64\ 10^{-8}$



Conclusion

- we constructed several EF versions of well-known 2-stage RK methods
- two classes : fixed knot points and ω dependent knot points
- methods with ω dependent knot points behave like partitioned RK methods

  An accurate computation of the frequencies is no longer possible due to the presence of coupling terms
- for systems of equations : methods with fixed knot points should be preferred