Exponentially fitted Obrechkoff methods

Marnix Van Daele, G. Vanden Berghe

Marnix.VanDaele@UGent.be

Vakgroep Toegepaste Wiskunde en Informatica

Universiteit Gent

Outline

- Introduction on exponentially fitted methods
- Obrechkoff methods
  - k-step methods for $y' = f(x, y)$
  - 2-step methods for $y'' = f(x, y)$
- Conclusions

Exponentially fitted methods

In the past 15 years, our research group has constructed modified versions of well-known

- linear multistep methods
- Runge-Kutta methods

Aim : build methods which perform very good when the solution has a known exponential of trigonometric behaviour.

Linear multistep methods

Well known methods to solve

$$y'' = f(y) \qquad y(a) = y_a \qquad y'(a) = y'_a$$

are

- Störmer-Verlet method (order 2)

$$y_{n+1} - 2\, y_n + y_{n-1} = h^2\, f(y_n)$$

- Numerov method (order 4)

$$y_{n+1} - 2\, y_n + y_{n-1} = \frac{h^2}{12}\, \left(f(y_{n-1}) + 10\, f(y_n) + f(y_{n+1})\right)$$

Numerov method : construction

several ways to obtain

$$y_{n+1} - 2\,y_n + y_{n-1} = \frac{h^2}{12}\,\left(f(y_{n-1}) + 10\,f(y_n) + f(y_{n+1})\right)$$

- starting from the identity

$$y(x_{n+1}) - 2\,y(x_n) + y(x_{n-1}) = \int_{x_n}^{x_{n+1}} (x_{n+1} - \tau)\,[y''(\tau) + y''(2\,x_n - \tau)]\,\mathrm{d}\tau$$

  Replace $y''(x) = f(y(x))$ by the interpolating polynomial $p(t) = a_0 + a_1\,x + a_2\,x^2$ at x_{n-1}, x_n, x_{n+1}.

- impose $\mathcal{L}[z(x);h] = 0$ for $z(x) = 1,\ x,\ x^2,\ x^3,\ x^4$ where

$$\begin{aligned}\mathcal{L}[z(x);h] :=\ & z(x+h) + \alpha_0\,z(x) + \alpha_{-1}\,z(x-h)\\ & -h^2\left(\beta_1\,z''(x+h) + \beta_0\,z''(x) + \beta_{-1}\,z''(x-h)\right)\end{aligned}$$

Exponential fitting

Consider the initial value problem

$$y'' + \omega^2\, y = g(y) \qquad y(a) = y_a \qquad y(a) = y'_a\,.$$

If $|g(y)| \ll |\omega^2\, y|$ then

$$y(x) \approx \alpha\, \cos(\omega\, x + \phi)$$

To mimic this oscillatory behaviour, one could replace polynomials by trigonometric (in the complex case : exponential) functions.

EF Numerov method

- starting from the identity

$$y(x_{n+1}) - 2\,y(x_n) + y(x_{n-1}) = \int_{x_n}^{x_{n+1}} (x_{n+1} - \tau)\,[y''(\tau) + y''(2\,x_n - \tau)]\,\mathrm{d}\tau$$

  Replace $y''(x) = f(y(x))$ by the interpolating function $p(x) = a\ \cos\omega\,x + b\ \sin\omega\,x + c_0$ at x_{n-1}, x_n, x_{n+1} :

- impose $\mathcal{L}[z(x);h] = 0$ for $z(x) = 1,\ x,\ x^2$, $\sin(\omega\,x)$, $\cos(\omega\,x)$

$$\begin{aligned}\mathcal{L}[z(x);h] &:= z(x+h) + \alpha_0\,z(x) + \alpha_{-1}\,z(x-h)\\ &\quad -h^2\left(\beta_1\,z''(x+h) + \beta_0\,z''(x) + \beta_{-1}\,z''(x-h)\right)\end{aligned}$$

$$y_{n+1} - 2\,y_n + y_{n-1} = h^2\ (\lambda f(y_{n-1}) + (1 - 2\,\lambda)\,f(y_n) + \lambda\,f(y_{n+1}))$$

$$\lambda = \left(\frac{1}{4\ \sin^2\frac{\theta}{2}} - \frac{1}{\theta^2}\right) = \frac{1}{12} + \frac{1}{240}\theta^2 + \frac{1}{6048}\theta^4 + \ldots \qquad \theta = \omega\,h$$

Choice of ω

1. local optimization

based on local truncation error (lte)

$$y(x_{n+1}) - y_{n+1} = -\frac{h^6}{240}\left(y^{(6)}(x_n) + \omega^2\, y^{(4)}(x_n)\right) + \ldots$$

$$\Longrightarrow \omega_n^2 = -\frac{y^{(6)}(x_n)}{y^{(4)}(x_n)}$$

ω is step-dependent

Choice of ω

2. global optimization

Preservation of geometric properties (periodicity, energy, …)

- backward error analysis
- linearisation : rewrite $y'' = f(y)$ as $y'' + \omega^2\, y = g(y)$ with $|g(y)|$ small
- Hamiltonian : minimize the leading term in $H_{n+1} - H_n$

ω is constant over the interval of integration

Obrechkoff methods

Nikola Obrechkoff (1896-1963)

Obrechkoff methods (OM) : $^\circ 1940$ for quadrature

Milne : OM for solving diff. eq. : 1949

Obrechkoff methods for $y' = f(x,\, y)$

$$\sum_{j=0}^{k} \alpha_j\, y_{n+j} = \sum_{i=1}^{l} h^i \sum_{j=0}^{k} \beta_{ij}\, y_{n+j}^{(i)}$$

$$\mathcal{L}[z(x); h] := \sum_{j=0}^{k} \alpha_j\, z(x + j\, h) - \sum_{i=1}^{l} h^i \sum_{j=0}^{k} \beta_{ij}\, z^{(i)}(x + j\, h)$$

order $p \Longleftrightarrow \mathcal{L}[x^q; h] = 0,\ q = 0,\ 1,\ \ldots,\ p$

$$p \leq (k+1)\,(l+1) - 2$$

lte $= C_{p+1}\, h^{p+1} y^{(p+1)}(x_n) + \mathcal{O}\left(h^{p+2}\right)$ $\qquad C_{p+1} = \dfrac{\mathcal{L}[x^{p+1}; h]}{(p+1)!\, h^{p+1}}$

The error constant C_{p+1} decreases more rapidly with increasing l than with increasing k.

Obrechkoff methods for $y' = f(x, y)$

- $k = 1, l = 2 \qquad p = 4, C_5 = \frac{1}{720}$

$$y_{n+1} - y_n = \frac{h}{2}\left(y_{n+1}^{(1)} + y_n^{(1)}\right) - \frac{h^2}{12}\left(y_{n+1}^{(2)} - y_n^{(2)}\right)$$

- $k = 1, l = 3 \qquad p = 6, C_7 = -\frac{1}{100800}$

$$y_{n+1} - y_n = \frac{h}{2}\left(y_{n+1}^{(1)} + y_n^{(1)}\right) - \frac{h^2}{10}\left(y_{n+1}^{(2)} - y_n^{(2)}\right) + \frac{h^3}{120}\left(y_{n+1}^{(3)} - y_n^{(3)}\right)$$

- $k = 2, l = 2 \qquad p = 6, C_7 = \frac{1}{9450}$

$$\begin{aligned} y_{n+2} - y_{n+1} \quad &= \quad \frac{h}{240}\left(101\, y_{n+2}^{(1)} + 128\, y_{n+1}^{(1)} + 11\, y_n^{(1)}\right) \\ &\quad + \frac{h^2}{240}\left(-13\, y_{n+2}^{(2)} + 40\, y_{n+1}^{(2)} + 3\, y_{n+2}^{(2)}\right) \end{aligned}$$

Lambert and Mitchell (1962)

Expon. fitted OM for $y' = f(x, y)$

The coefficients are determined by $\mathcal{L}[z(x); h] = 0$

$$\mathcal{L}[z(x); h] := \sum_{j=0}^{k} \alpha_j \, z(x + j\,h) - \sum_{i=1}^{l} h^i \sum_{j=0}^{k} \beta_{ij} \, z^{(i)}(x + j\,h)$$

Polynomial methods : $M = p + 1$ functions

$$z(x) = x^q, \; q = 0, \, 1, \, 2, \, \ldots, \, p$$

Exponentially fitted methods : $M = K + 2\,(P + 1) + 1$ functions

$$z(x) = x^q \, \exp(\pm \mu \, x) = x^q \begin{cases} \cos \omega \, x \\ \sin \omega \, x \end{cases} , \; q = 0, \, 1, \, \ldots, \, P$$

$$\omega = i \, \mu$$

and $z(x) = x^q$, $q = 0, \, 1, \, 2, \, \ldots, \, K$

example : if $M = 5$ then $(K, P) = (4, \, -1)$, $(2, \, 0)$, or $(0, \, 1)$

Expon. fitted OM for $y' = f(x,y)$

$\mathcal{L}[z(x);h] = 0$ for $M = K + 2\,(P+1) + 1$ functions

$$z(x) = x^q \left\{ \begin{array}{l} \cos \omega\, x \\ \sin \omega\, x \end{array} \right. ,\ q = 0,\ 1,\ \ldots,\ P$$

and $z(x) = x^q$, $q = 0,\ 1,\ 2,\ \ldots,\ K$

$$lte = \frac{\mathcal{L}^*[x^{K+1};h]}{(K+1)!\theta^{2\,(P+1)}} h^M\ D^{K+1}(D^2+\omega^2)^{P+1} y(x_n) + \mathcal{O}\left(h^{M+1}\right)$$

where $\mathcal{L}^*[x^q;h] = \dfrac{1}{h^q}\mathcal{L}[x^q;h]$

L. Ixaru and G. Vanden Berghe

EFOM for $y' = f(x,y)$ with $k=1, l=2$

- $\{1,\ x,\ x^2,\ \ x^3,\ x^4\}$ $\qquad (M=5, P=-1,\ K=4)$

$$y_{n+1} - y_n = \frac{h}{2}\left(y^{(1)}_{n+1} + y^{(1)}_n\right) - \frac{h^2}{12}\left(y^{(2)}_{n+1} - y^{(2)}_n\right)$$

- $\{1,\ x,\ x^2,\ \cos\omega\, x,\ \sin\omega\, x\}$ $\qquad (M=5, P=0,\ K=2)$

$$\begin{aligned} y_{n+1} - y_n \quad &= \quad \frac{h}{2}\left(y^{(1)}_{n+1} + y^{(1)}_n\right) \\ &\quad -h^2\,\frac{\theta + \theta\ \cos\theta - 2\ \sin\theta}{2\,\theta^2\ \sin\theta}\left(y^{(2)}_{n+1} - y^{(2)}_n\right) \end{aligned}$$

- $\{1,\ \cos\omega\, x,\ \sin\omega\, x,\ x\ \cos\omega\, x,\ x\ \sin\omega\, x\}$ $\qquad (M=5, P=1,\ K=0)$

$$\begin{aligned} y_{n+1} - y_n \quad &= \quad \frac{2\,(1-\cos\theta)}{\theta\,(\sin\theta + \theta)}\,h\ \left(y^{(1)}_{n+1} + y^{(1)}_n\right) \\ &\quad -\frac{\theta - \sin\theta}{\theta^2\,(\sin\theta+\theta)}\,h^2\ \left(y^{(2)}_{n+1} + y^{(2)}_n\right) \end{aligned}$$

EFOM for $y' = f(x,y)$ with $k=1, l=2$

- $\{1,\, x,\, x^2,\, x^3,\, x^4\}$ $\quad (P=-1,\, K=4)$

$$lte = \frac{1}{720} h^5\, y^{(5)}(x_n) + \mathcal{O}\left(h^6\right)$$

- $\{1,\, x,\, x^2,\, \cos\omega\, x,\, \sin\omega\, x\}$ $\quad (P=0,\, K=2)$

$$lte = \frac{6\,\theta\,\cos\theta - 6\,\theta - \sin\theta\,\theta^2 + 12\,\sin\theta}{12\,\theta^4\,\sin\theta}\, h^5 \times$$
$$\left(y^{(5)}(x_n) + \omega^2\, y^{(3)}(x_n)\right) + \mathcal{O}\left(h^6\right)$$

- $\{1,\, \cos\omega\, x,\, \sin\omega\, x,\, x\,\cos\omega\, x,\, x\,\sin\omega\, x\}$ $\quad (P=1,\, K=0)$

$$lte = \left(\frac{1}{\theta^4} + \frac{4\,(\cos\theta - 1)}{\theta^5\,(\sin\theta + \theta)}\right) h^5 \times$$
$$\left(y^{(5)}(x_n) + 2\,\omega^2\, y^{(3)}(x_n) + \omega^4\, y^{(1)}(x_n)\right) + \mathcal{O}\left(h^6\right)$$

EFOM for $y' = f(x, y)$ with $k = 1, l = 2$

- $\{1,\ x,\ x^2,\ \ x^3,\ x^4\}$ $\qquad \beta_{10} = \beta_{11} = \dfrac{1}{2} \qquad \beta_{21} = -\beta_{20} = -\dfrac{1}{12}$
- $\{1,\ x,\ x^2,\ \cos\omega\, x,\ \sin\omega\, x\}$

  $$\beta_{10} = \beta_{11} = \frac{1}{2}$$

  $$\beta_{21} = -\beta_{20} = \frac{2\ \sin\theta - \theta - \theta\cos\theta}{2\,\theta^2\ \sin\theta} = -\frac{1}{12} - \frac{\theta^2}{720} - \frac{\theta^4}{30240} + \mathcal{O}(\theta^6)$$
- $\{1,\ \cos\omega\, x,\ \sin\omega\, x,\ x\ \cos\omega\, x,\ x\ \sin\omega\, x\}$

  $$\beta_{10} = \beta_{11} = \frac{2\,(1 - \cos\theta)}{\theta\,(\sin\theta + \theta)} = \frac{1}{2} - \frac{\theta^4}{1440} + \mathcal{O}\left(\theta^6\right)$$

  $$\beta_{21} = -\beta_{20} = \frac{\theta - \sin\theta}{\theta^2\,(\sin\theta + \theta)} = -\frac{1}{12} - \frac{\theta^2}{360} + \frac{\theta^4}{60480} + \mathcal{O}\left(\theta^6\right)$$

EFOM for $y' = f(x, y)$ with $k = 1$, $l = 2$

- $\{1,\, x,\, x^2,\; x^3,\, x^4\}$ $\qquad (P = -1,\, K = 4)$

$$lte = \frac{1}{720} h^5\, y^{(5)}(x_n) + \mathcal{O}\left(h^6\right)$$

- $\{1,\, x,\, x^2,\, \cos\omega\, x,\, \sin\omega\, x\}$ $\qquad (P = 0,\, K = 2)$

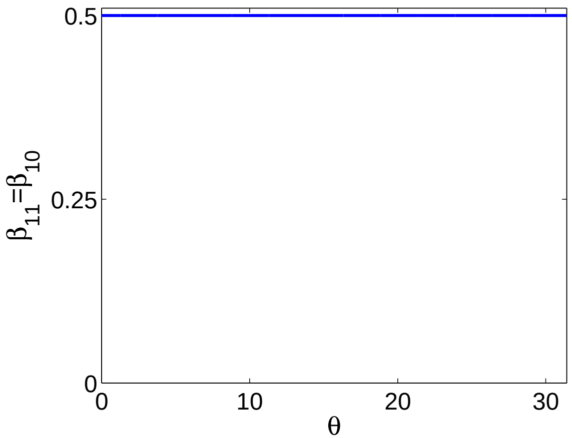
$$lte = \left(\frac{1}{720} + \frac{1}{30240}\theta^2 + \frac{1}{1209600}\theta^4 + \mathcal{O}\left(\theta^6\right)\right) h^5 \times \left(y^{(5)}(x_n) + \omega^2\, y^{(3)}(x_n)\right) + \mathcal{O}\left(h^6\right)$$

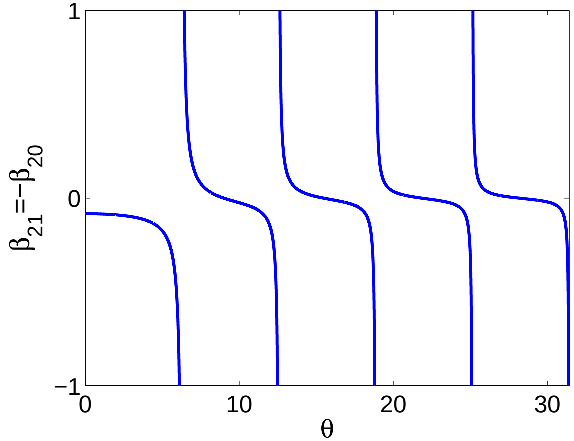
- $\{1,\, \cos\omega\, x,\, \sin\omega\, x,\, x\, \cos\omega\, x,\, x\, \sin\omega\, x\}$ $\qquad (P = 1,\, K = 0)$

$$lte = \left(\frac{1}{720} + \frac{1}{15120}\theta^2 + \frac{1}{1814400}\theta^4 + \mathcal{O}\left(\theta^6\right)\right) h^5 \times \left(y^{(5)}(x_n) + 2\,\omega^2\, y^{(3)}(x_n) + \omega^4\, y^{(1)}(x_n)\right) + \mathcal{O}\left(h^6\right)$$

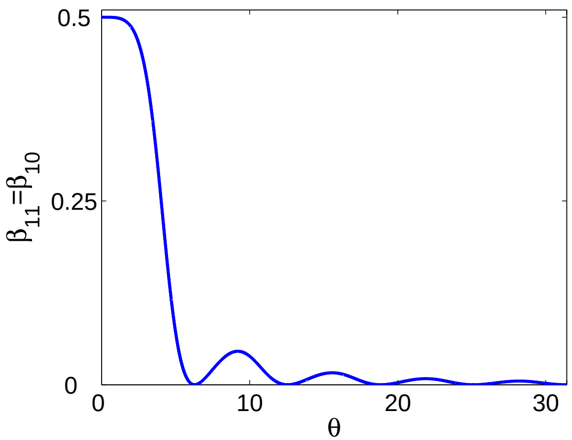
EFOM for $y' = f(x,y)$ with $k = 1$, $l = 2$

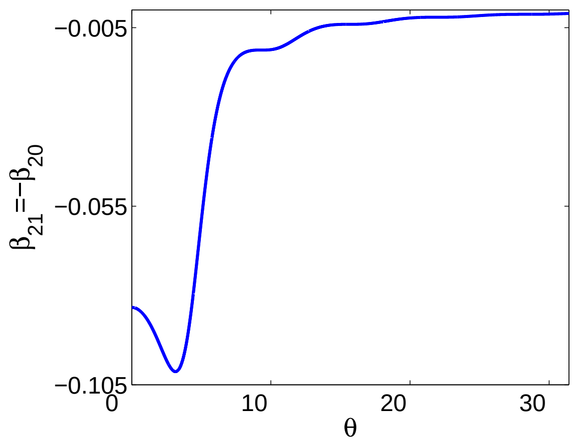
- $\{1,\ x,\ x^2,\ \ x^3,\ x^4\}$ $\qquad \beta_{10} = \beta_{11} = \dfrac{1}{2} \qquad \beta_{21} = -\beta_{20} = -\dfrac{1}{12}$
- $\{1,\ x,\ x^2,\ \cos\omega\, x,\ \sin\omega\, x\}$





- $\{1,\ \cos\omega\, x,\ \sin\omega\, x,\ x\ \cos\omega\, x,\ x\ \sin\omega\, x\}$





Obrechkoff methods for $y'' = f(x,y)$

Two-step methods

$$y_{n+1} - 2y_n + y_{n-1} = \sum_{i=1}^{m} h^{2\,i} \left(\beta_{i0}\, y_{n+1}^{(2i)} + 2\,\beta_{i1}\, y_n^{(2i)} + \beta_{i0}\, y_{n-1}^{(2i)} \right)$$

$$\begin{aligned} \mathcal{L}[z(x);h] &:= z(x+h) - 2\,z(x) + z(x-h) \\ &\quad - \sum_{i=1}^{m} h^{2\,i} \left(\beta_{i0}\, z^{(2i)}(x+h) + 2\,\beta_{i1}\, z^{(2i)}(x) + \beta_{i0}\, z^{(2i)}(x-h) \right) \end{aligned}$$

symmetric method : $\mathcal{L}[z(x);h] \equiv 0$ if $z(x)$ is odd

$$\mathcal{L}[1;h] \equiv 0$$

order $p \iff \mathcal{L}[x^q;h] = 0,\ q = 0,\ 1,\ \ldots,\ p+1$

$$\text{lte} = C_{p+2}\, h^{p+2} y^{(p+2)}(x_n) + \mathcal{O}\left(h^{p+3}\right) \qquad C_{p+2} = \frac{\mathcal{L}[x^{p+2};h]}{(p+2)!\, h^{p+2}}$$

Two-step OM for $y'' = f(x,y)$

- $m = 1 : p = 4,\ C_6 = -\dfrac{1}{240}$ (Numerov method)

$$y_{n+1} - 2y_n + y_{n-1} = \frac{h^2}{12}\left(y_{n+1}^{(2)} + 10\, y_n^{(2)} + y_{n-1}^{(2)}\right)$$

- $m = 2 : p = 8,\ C_{10} = \dfrac{59}{76204800}$

$$y_{n+1} - 2y_n + y_{n-1} = \frac{h^2}{252}\left(11\, y_{n+1}^{(2)} + 230\, y_n^{(2)} + 11\, y_{n-1}^{(2)}\right) - \frac{h^4}{15120}\left(13\, y_{n+1}^{(4)} - 626\, y_n^{(4)} + 13\, y_{n-1}^{(4)}\right)$$

Two-step OM for $y'' = f(x, y)$

- $m = 3 : p = 12,\, C_{14} = -\dfrac{45469}{1697361329664000}$

$$
\begin{aligned}
y_{n+1} - 2y_n + y_{n-1} = {} \\
& \frac{h^2}{7788}\left(229\, y_{n+1}^{(2)} + 7330\, y_n^{(2)} + 229\, y_{n-1}^{(2)}\right) \\
& -\frac{h^4}{25960}\left(11\, y_{n+1}^{(4)} - 1422\, y_n^{(4)} + 11\, y_{n-1}^{(4)}\right) \\
& +\frac{h^6}{39251520}\left(127\, y_{n+1}^{(6)} + 4846\, y_n^{(6)} + 127\, y_{n-1}^{(6)}\right)
\end{aligned}
$$

2-step EFOM for $y'' = f(x,y)$, $m = 2$

$$
\begin{aligned}
y_{n+1} - 2y_n + y_{n-1} = \\
h^2 \left(\beta_{10}\, y_{n+1}^{(2)} + 2\,\beta_{11}\, y_n^{(2)} + \beta_{10}\, y_{n-1}^{(2)} \right) \\
+ h^4 \left(\beta_{20}\, y_{n+1}^{(4)} + 2\,\beta_{21}\, y_n^{(4)} + \beta_{20}\, y_{n-1}^{(4)} \right)
\end{aligned}
$$

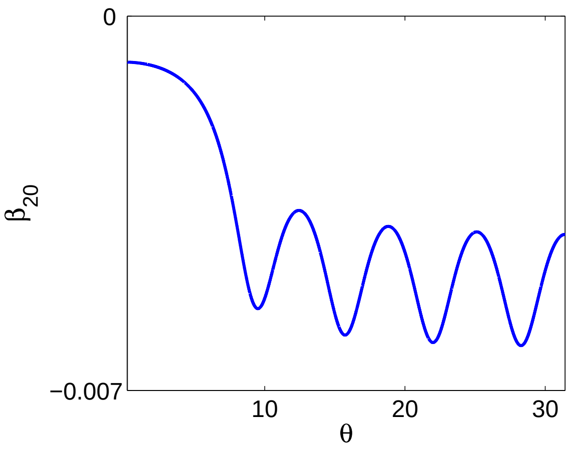
2-step EFOM for $y'' = f(x,y)$, $m = 2$

$$\{x^2,\, x^4\} \Longrightarrow \beta_{10} = \frac{1}{12} - 2\,\beta_{20} - 2\,\beta_{21},\, \beta_{11} = \frac{5}{12} + 2\,\beta_{20} + 2\,\beta_{21}$$

- $\{x^6,\, x^8\} \Longrightarrow \beta_{21} = \dfrac{313}{15120}, \beta_{20} = -\dfrac{13}{15120}$
- $\{x^6,\, \cos\omega\, x\} \Longrightarrow$

$$\begin{aligned}
\beta_{21} &= \frac{1}{40} + 5\,\beta_{20} \\
\beta_{20} &= -\frac{120 - 4\,\cos(\theta)\,\theta^2 - 56\,\theta^2 - 120\,\cos(\theta) + 3\,\theta^4}{120\,\theta^2\,(12\,\cos(\theta) - 12 + \cos(\theta)\,\theta^2 + 5\,\theta^2)} \\
&= -\frac{13}{15120} - \frac{59}{3810240}\theta^2 + \mathcal{O}\left(\theta^4\right)
\end{aligned}$$

continuous coefficients



2-step EFOM for $y'' = f(x,y)$, $m=2$

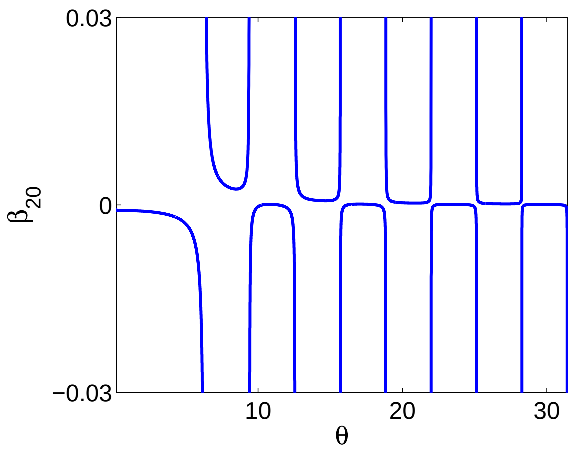
$$\{x^2,\ x^4\} \Longrightarrow \beta_{10} = \frac{1}{12} - 2\,\beta_{20} - 2\,\beta_{21},\, \beta_{11} = \frac{5}{12} + 2\,\beta_{20} + 2\,\beta_{21}$$

- $\{\cos\omega\, x,\ x\ \sin\omega\, x\}$

$$\beta_{20} = \frac{\theta^5 \sin\theta + 2\,(\cos\theta + 5)\,\theta^4 + 48\,(\cos\theta - 1)\,\theta^2 + 48\,(\cos\theta - 1)^2}{12\,\theta^4\,(\theta^3\ \sin\theta - 4\,(1-\cos\theta)^2)}$$

$$\beta_{21} = \frac{5\,\theta^5 \sin\theta - 2\ \cos\theta\,(\cos\theta + 5)\,\theta^4 - 48\ \cos\theta\,(\cos\theta - 1)\,\theta^2 - 48\,(\cos\theta - 1)^2}{12\,\theta^4\,(\theta^3\ \sin\theta - 4\,(1-\cos\theta)^2)}$$

discontinuous coefficients



- $\{x^2,\ \cos\omega\, x,\ x\ \sin\omega\, x,\ x^2\ \cos\omega\, x\}$

discontinuous coefficients

Stability of OM for $y'' = f(x, y)$

$$y_{n+1} - 2\, y_n + y_{n-1} = \sum_{i=1}^{m} h^{2\,i} \left(\beta_{i0}\, y_{n+1}^{(2i)} + 2\, \beta_{i1}\, y_n^{(2i)} + \beta_{i0}\, y_{n-1}^{(2i)} \right)$$

applied to $y'' = -\lambda^2\, y$ gives

$$y_{n+1} - 2 R_{mm}(\nu^2)\, y_n + y_{n-1} = 0$$

$$R_{mm}(\nu^2) = \frac{1 + \displaystyle\sum_{i=1}^{m} (-1)^i \beta_{i1}\, \nu^{2i}}{1 + \displaystyle\sum_{i=1}^{m} (-1)^{i+1} \beta_{i0}\, \nu^{2i}}$$

$$R_{22} = \frac{1 - \beta_{11}\, \nu^2 + \beta_{21}\, \nu^4}{1 + \beta_{10}\, \nu^2 - \beta_{20}\, \nu^4} \qquad R_{33} = \frac{1 - \beta_{11}\, \nu^2 + \beta_{21}\, \nu^4 - \beta_{31}\, \nu^6}{1 + \beta_{10}\, \nu^2 - \beta_{20}\, \nu^4 + \beta_{30}\, \nu^6}$$

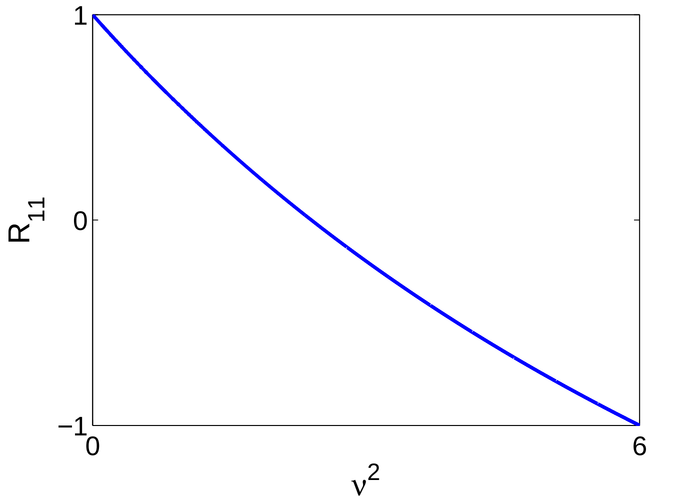
A method has the interval of periodicity $(0, \nu_0^2)$ if

$|R_{mm}(\nu^2)| < 1$ for $0 < \nu^2 < \nu_0^2$.

The method is P-stable if $|R_{mm}(\nu^2)| < 1$ for all real $\nu \neq 0$.

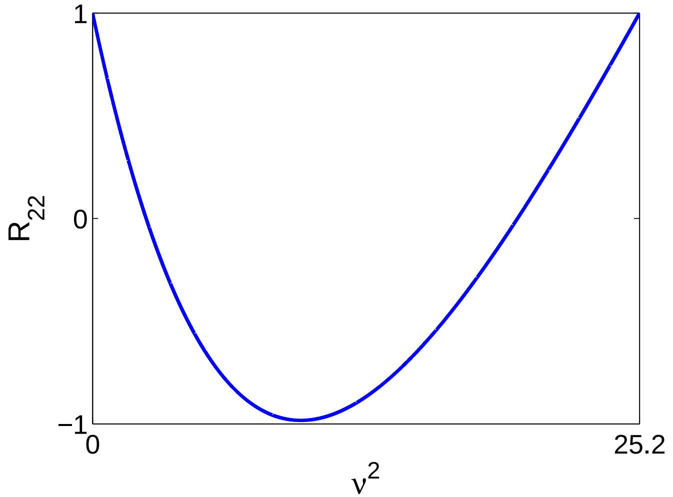
Two-step OM for $y'' = f(x,y)$

- $m = 1 : p = 4$, $C_6 = -\dfrac{1}{240}$ (Numerov method) $\nu_0^2 = 6$



$$y_{n+1} - 2y_n + y_{n-1} = \frac{h^2}{12}\left(y_{n+1}^{(2)} + 10\, y_n^{(2)} + y_{n-1}^{(2)}\right)$$

- $m = 2 : p = 8$, $C_{10} = \dfrac{59}{76204800}$ $\nu_0^2 = 25.2$

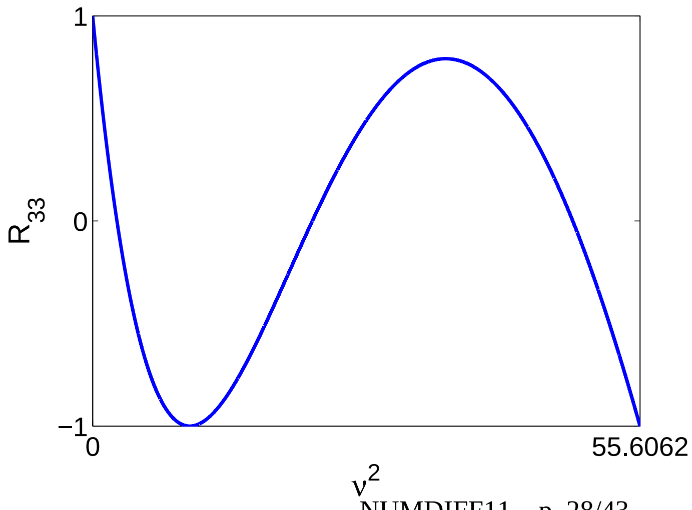


$$y_{n+1} - 2y_n + y_{n-1} = \frac{h^2}{252}\left(11\, y_{n+1}^{(2)} + 115\, y_n^{(2)} + 11\, y_{n-1}^{(2)}\right) - \frac{h^4}{15120}\left(13\, y_{n+1}^{(4)} - 626\, y_n^{(4)} + 13\, y_{n+1}^{(4)}\right)$$

Two-step OM for $y'' = f(x,y)$

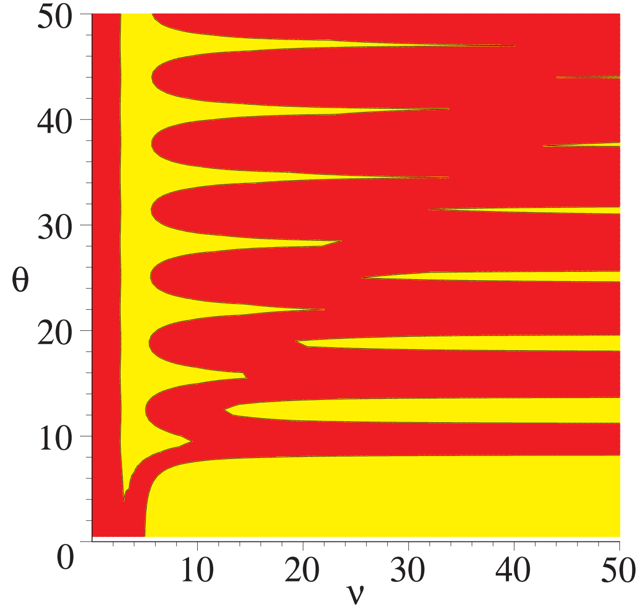
- $m = 3 : p = 12,\ C_{14} = -\dfrac{45469}{1697361329664000}$ $\qquad \nu_0^2 = 55.60\ldots$

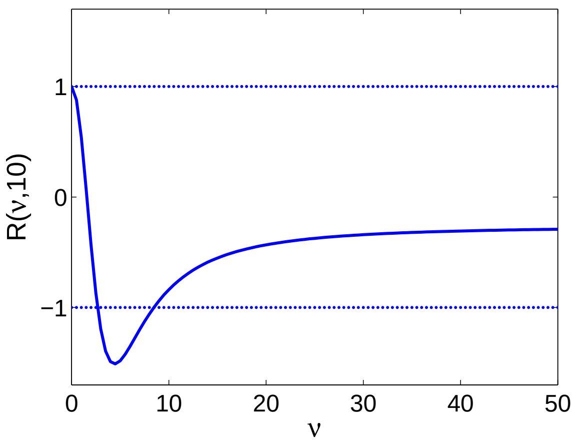
$$
\begin{aligned}
y_{n+1} - 2y_n + y_{n-1} = \\
&\frac{h^2}{7788}\left(229\, y_{n+1}^{(2)} + 7330\, y_n^{(2)} + 229\, y_{n-1}^{(2)}\right) \\
&+\frac{h^4}{25960}\left(-11\, y_{n+1}^{(4)} + 1422\, y_n^{(4)} - 11\, y_{n-1}^{(4)}\right) \\
&+\frac{h^6}{39251520}\left(127\, y_{n+1}^{(6)} + 4846\, y_n^{(6)} + 127\, y_{n-1}^{(6)}\right)
\end{aligned}
$$

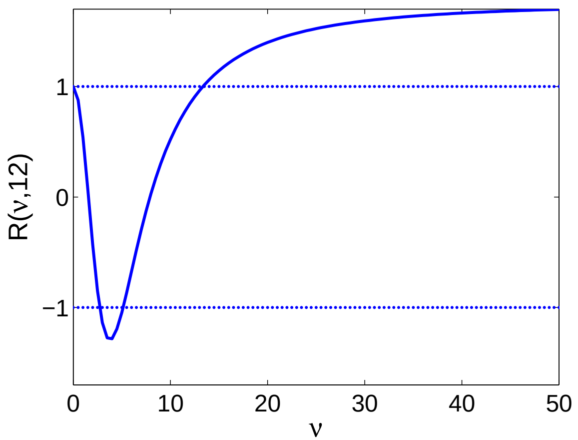


Two-step EFOM for $y'' = f(x,y)$

$$m = 2, \qquad \{x^2,\, x^4,\, x^6,\, \cos\omega\, x\}$$

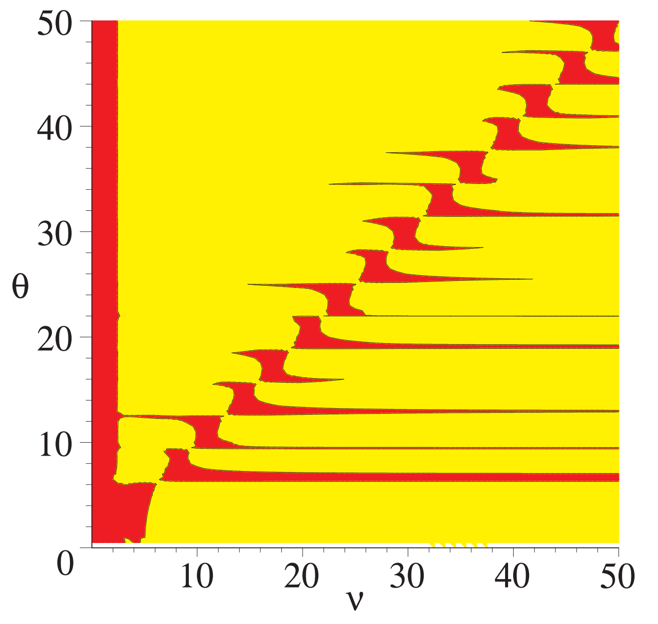






2-step EFOM for $y'' = f(x,y)$

$m = 2, \qquad \{x^2,\, x^4,\, \cos\omega\, x,\, x\, \sin\omega\, x\}$



P-stable 2-step OM for $y'' = f(x, y)$

Simos : use one parameter to integrate the test equation exactly

$$y_{n+1} - 2y_n + y_{n-1} = \sum_{i=1}^{3} h^{2i} \left[\beta_{i0}\, y_{n+1}^{(2i)} + 2\beta_{i1} y_n^{(2i)} + \beta_{i0}\, y_{n-1}^{(2i)} \right]$$

$$\begin{array}{ll} \beta_{10} = \frac{89}{1878} - \frac{15120}{313}\beta_{31} & \beta_{11} = \frac{425}{939} + \frac{15120}{313}\beta_{31} \\ \beta_{20} = -\frac{1907}{1577520} + \frac{660}{313}\beta_{31} & \beta_{21} = \frac{30257}{1577520} + \frac{6900}{313}\beta_{31} \\ \beta_{30} = \frac{59}{3155040} - \frac{13}{313}\beta_{31} & \end{array}$$

$$\begin{aligned} \beta_{31} \quad = \quad & \frac{1}{3568320\,\nu^{12}} \;\; 190816819200[1 - \cos\,\nu)] - 95408409600\,\nu^2 \\ & +7950700800\,\nu^4 - 265023360\,\nu^6 + 4732560\,\nu^8 - 52584\,\nu^{10} + 1727\,\nu^{12} + \ldots \end{aligned}$$

$$lte = \left(-\frac{2923}{209898501120} + \frac{59}{1577520}\beta_{31} \right) h^{12}\, y_n^{(12)} + \mathcal{O}\left(h^{13}\right)$$

$$\{x^2,\, x^4,\, x^6,\, x^8,\, x^{10},\, \cos\lambda\, x\} \qquad \nu = \lambda\, h$$

P-stable 2-step OM for $y'' = f(x,y)$

Ananthakrishnaiah (1987)

idea : introduce extra parameters to obtain P-stability.

$m = 3$ and $m = 4$

e.g. $m = 3$

$$
\begin{aligned}
y_{n+1} - 2y_n + y_{n-1} = \\
& h^2\left(\beta_{10}\, y_{n+1}^{(2)} + 2\,\beta_{11}\, y_n^{(2)} + \beta_{10}\, y_{n-1}^{(2)}\right) \\
& + h^4\left(\beta_{20}\, y_{n+1}^{(4)} + 2\,\beta_{21}\, y_n^{(4)} + \beta_{20}\, y_{n-1}^{(4)}\right) \\
& + h^6\left(\beta_{30}\, y_{n+1}^{(6)} + 2\,\beta_{31}\, y_n^{(6)} + \beta_{30}\, y_{n-1}^{(6)}\right)
\end{aligned}
$$

impose $p = 6 : \{x^2,\, x^4,\, x^6\}$,

then 3 parameters (β_{20}, β_{30} and β_{31}) remain

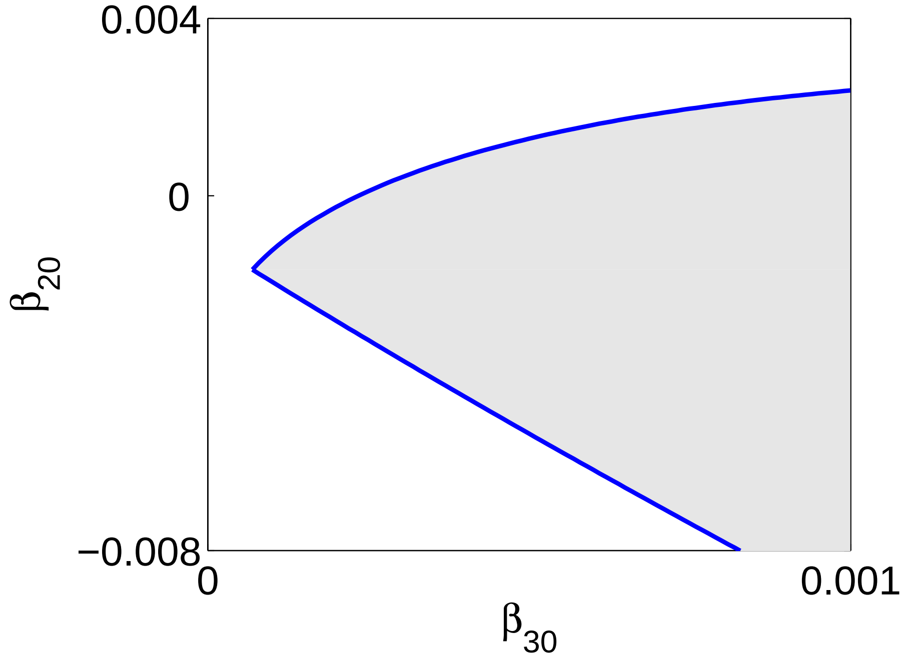
$$R = \frac{1 - \beta_{11}\,\nu^2 + \beta_{21}\,\nu^4 - \beta_{31}\,\nu^6}{1 + \beta_{10}\,\nu^2 - \beta_{20}\,\nu^4 + \beta_{30}\,\nu^6}$$

let $\beta_{31} = \beta_{30}$

P-stable 2-step OM for $y'' = f(x, y)$

Choose these 2 remaining parameters β_{20}, β_{30} such that the method becomes P-stable

$$|R| = \left|\frac{N}{D}\right| < 1 \iff (D - N)\,(D + N) > 0$$

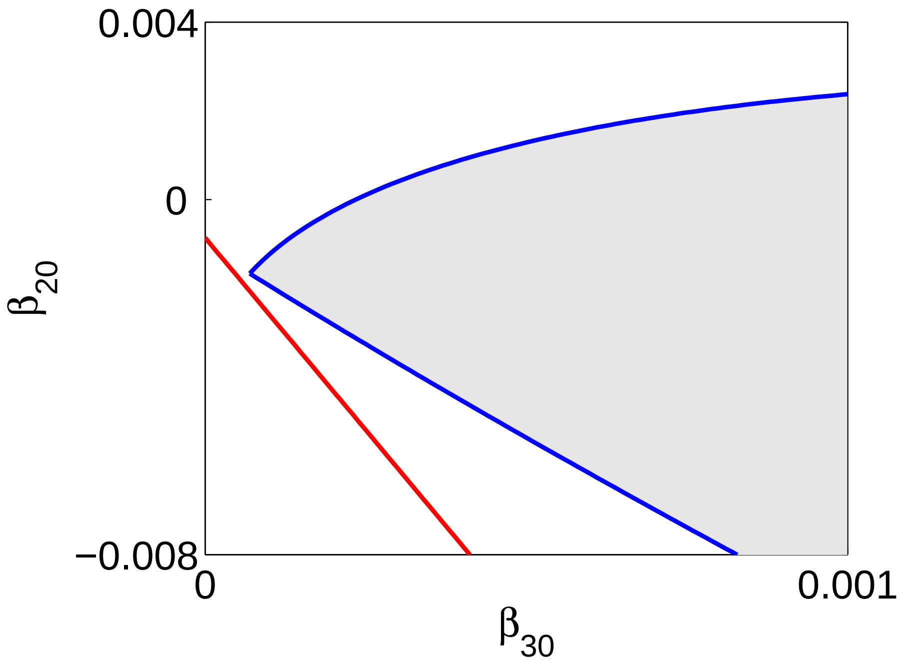


P-stable 2-step OM for $y'' = f(x,y)$

Find the solution for which the phase-lag,

$$\nu - \arccos R = \left(\frac{13}{604800} + \frac{13}{30}\,\beta_{30} + \frac{1}{40}\,\beta_{20} \right) \nu^7 + \mathcal{O}\left(\nu^9\right)$$

becomes minimal.

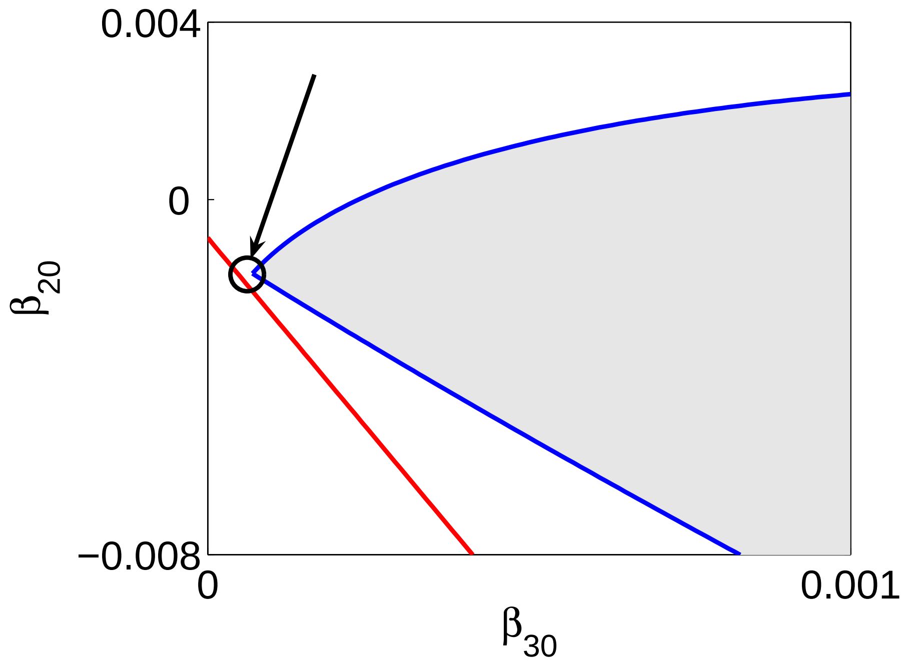


P-stable 2-step OM for $y'' = f(x, y)$

Find the solution for which the phase-lag,

$$\nu - \arccos R = \left(\frac{13}{604800} + \frac{13}{30}\beta_{30} + \frac{1}{40}\beta_{20} \right) \nu^7 + \mathcal{O}\left(\nu^9\right)$$

becomes minimal.



This gives $(\beta_{30}, \beta_{20}) = \left(\frac{1}{14400}, -\frac{1}{600} \right)$.

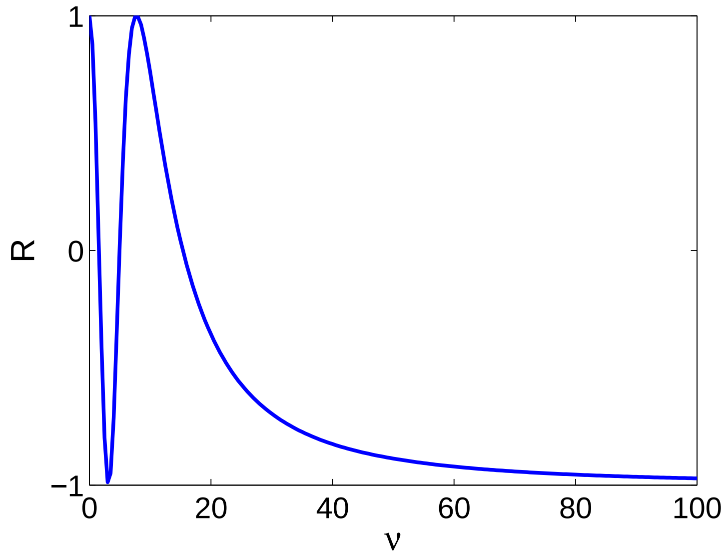
P-stable 2-step OM for $y'' = f(x,y)$

Ananthakrishnaiah $m = 3 : p = 6$, P-stable, $C_8 = -\dfrac{1}{50400}$

$$
\begin{aligned}
y_{n+1} - 2y_n + y_{n-1} = \\
&\frac{h^2}{20}\left(y_{n+1}^{(2)} + 18\,y_n^{(2)} + y_{n-1}^{(2)}\right) \\
&-\frac{h^4}{600}\left(y_{n+1}^{(4)} - 22\,y_n^{(4)} + y_{n-1}^{(4)}\right) \\
&+\frac{h^6}{14400}\left(y_{n+1}^{(6)} + 2\,y_n^{(6)} + y_{n-1}^{(6)}\right)
\end{aligned}
$$

P-stable 2-step OM for $y'' = f(x, y)$

$$R = \frac{1 - \frac{9}{20}\,\nu^2 + \frac{11}{600}\,\nu^4 - \frac{1}{14400}\,\nu^6}{1 + \frac{1}{20}\,\nu^2 + \frac{1}{600}\,\nu^4 + \frac{1}{14400}\,\nu^6}$$



$$\nu - \arccos R = \frac{1}{100800}\,\nu^7 + \mathcal{O}\left(\nu^9\right)$$

$$\text{and } |R| = \left|\frac{N}{D}\right| < 1 \text{ since}$$

$$(D - N)\,(D + N) = \frac{\nu^2\,(\nu^2 - 10)^2\,(\nu^2 - 60)^2}{360000}$$

P-stable 2-step EFOM for $y'' = f(x,y)$

Following the ideas of Ananthakrishnaiah

$$m = 3$$

$$\begin{aligned} y_{n+1} - 2y_n + y_{n-1} = \\ & h^2 \left(\beta_{10}\, y^{(2)}_{n+1} + 2\,\beta_{11}\, y^{(2)}_n + \beta_{10}\, y^{(2)}_{n-1} \right) \\ & + h^4 \left(\beta_{20}\, y^{(4)}_{n+1} + 2\,\beta_{21}\, y^{(4)}_n + \beta_{20}\, y^{(4)}_{n-1} \right) \\ & + h^6 \left(\beta_{30}\, y^{(6)}_{n+1} + 2\,\beta_{31}\, y^{(6)}_n + \beta_{30}\, y^{(6)}_{n-1} \right) \end{aligned}$$

impose $p = 6 : \{x^2,\, x^4,\; \cos\omega\, x\}$,

then 3 parameters (β_{20}, β_{30} and β_{31}) remain

$$R = \frac{1 - \beta_{11}\,\nu^2 + \beta_{21}\,\nu^4 - \beta_{31}\,\nu^6}{1 + \beta_{10}\,\nu^2 - \beta_{20}\,\nu^4 + \beta_{30}\,\nu^6}$$

let $\beta_{31} = \beta_{30}$

P-stable 2-step EFOM for $y'' = f(x, y)$

The two remaining parameters β_{20}, β_{30} are chosen in such a way that the method becomes P-stable

$$|R| = \left|\frac{N}{D}\right| < 1 \Longleftrightarrow (D - N)\,(D + N) > 0$$

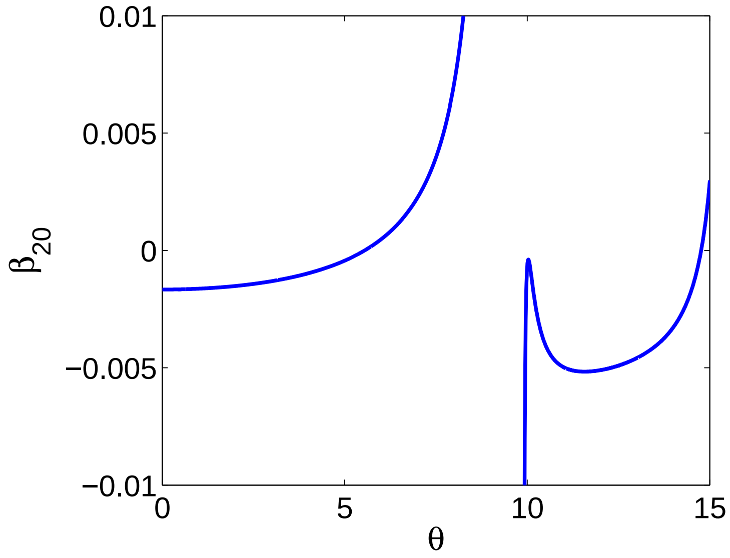
Within this set, we choose a couple $(\beta_{20},\ \beta_{30})$ such that $(D - N)\,(D + N)$ is a perfect square.

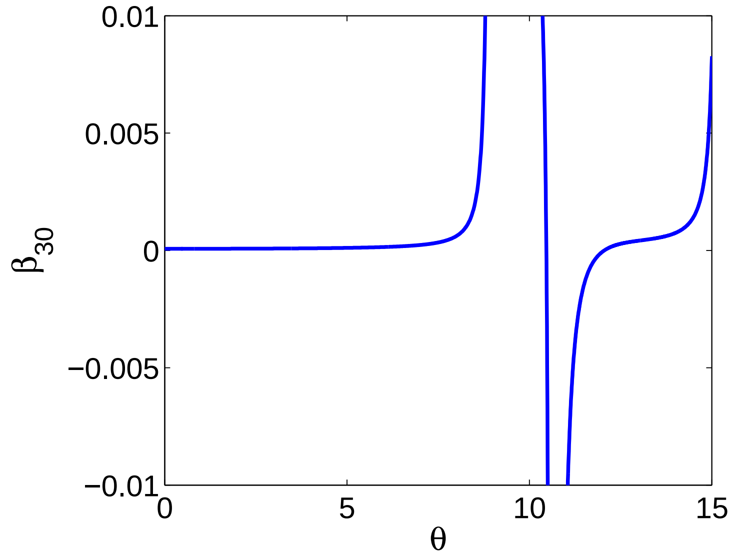
Then

$\beta_{30} =$ a linear function of β_{20}

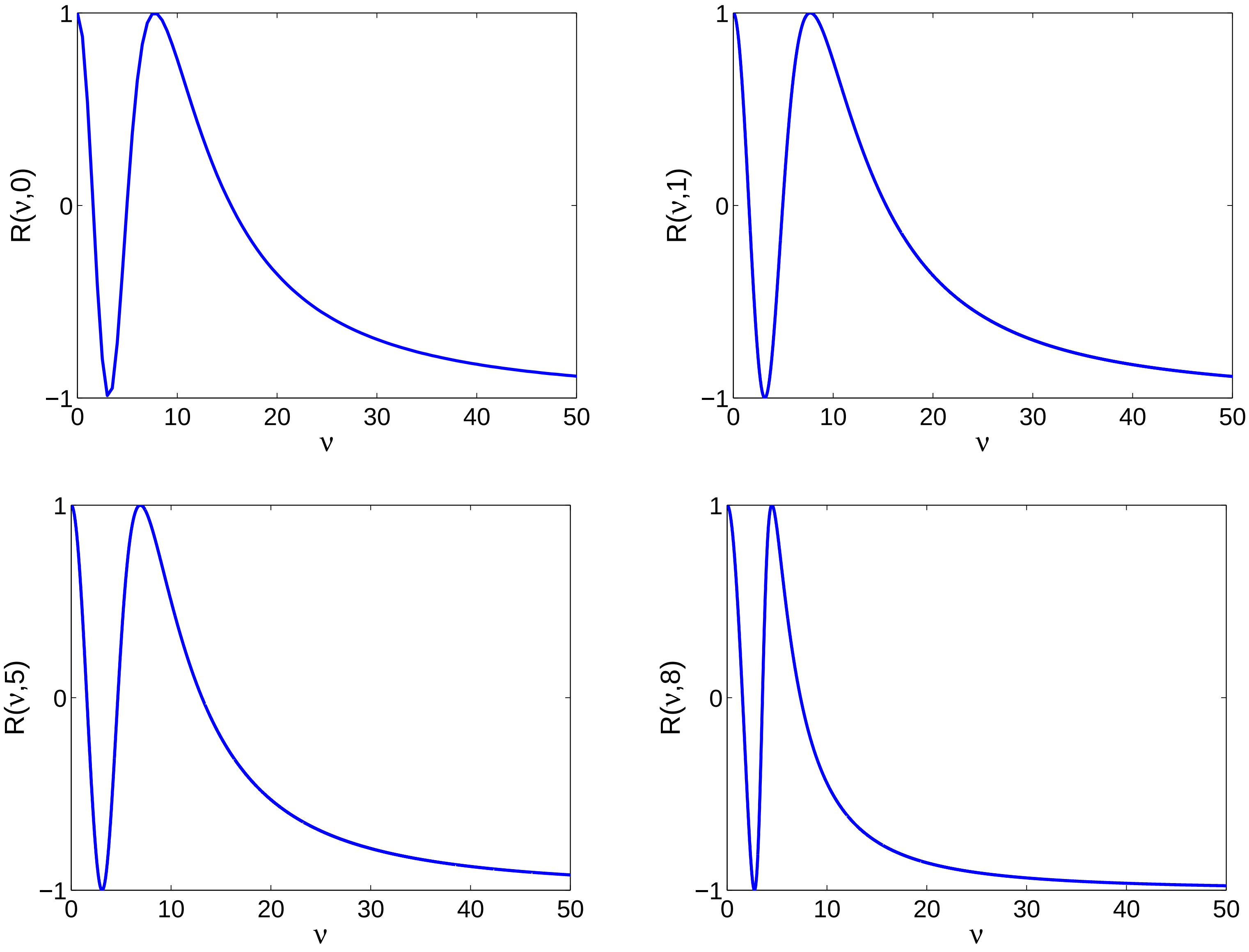
$\beta_{20} =$ the root of a quadratic equation

P-stable 2-step EFOM for $y'' = f(x, y)$





P-stable 2-step EFOM for $y'' = f(x, y)$



P-stable 2-step EFOM for $y'' = f(x, y)$

$m = 3$, approach of Ananthakrishnaiah

- $\{x^2,\, x^4,\, x^6\}$:
  P-stable method of Ananthakrishnaiah
- $\{x^2,\, x^4,\, \cos\omega\, x\}$:
  P-stable method with coefficients that can be computed analytically for a given θ
- $\{x^2,\, \cos\omega\, x,\, x\, \sin\omega\, x\}$:
  P-stable method with coefficients that have to be computed numerically for a given θ
  $\beta_{30} =$ a rational function of β_{20}
  $\beta_{20} =$ the root of a fourth-order equation

Conclusion

Exponentially fitted Obrechkoff methods :

- methods of any order can be constructed
- the coefficients depend on a parameter θ
- the coefficients may become discontinuous functions of θ
- stability can be increased by increasing the number of parameters, but it may become difficult to compute the coefficients : a numerical procedure may be needed.