Exponentially fitted methods applied to fourth-order boundary value problems

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Outline

Introduction

In the past 15 years, our research group has constructed modified versions of well-known

- linear multistep methods
- Runge-Kutta methods

Aim : build methods which perform very good when the solution has a known exponential of trigonometric behaviour.

A model problem

Consider the initial value problem

$$y'' + \omega^2 y = g(y) \qquad y(a) = y_a \qquad y(a) = y'_a \,.$$

If $|g(y)| \ll |\omega^2 y|$ then

$$y(t) \approx \alpha \cos(\omega t + \phi)$$

To mimic this oscillatory behaviour, we construct methods which yield exact results when the solution is of trigonometric (in the complex case : exponential) type.
These methods are called Exponentially-fitted methods.

EF methods

To determine the coefficients of a method, we impose conditions on a linear functional. These conditions are related to the fitting space $\mathcal{S}$ which contains

- polynomials :

$$\{t^q | q = 0, \ \ldots, K\}$$

- exponential or trigonometric functions, multiplied with powers of t :

$$\{t^q \ \exp(\pm\mu t) | q = 0, \ \ldots, P\}$$

or, with $\omega = i\,\mu$,

$$\{t^q \ \cos(\omega t), \ t^q \ \sin(\omega t) | q = 0, \ \ldots, P\}$$

EF method can be characterized by the couple (K, P)
Classical method : $P = -1$
number of basis functions : $M = 2P + K + 3$

$$M = 2P + K + 3$$

(K, P)				
$M = 2$	$M = 4$	$M = 6$	$M = 8$	$M = 10$
$(1, -1)$	$(3, -1)$	$(5, -1)$	$(7, -1)$	$(9, -1)$
$(-1, 1)$	$(1, 0)$	$(3, 0)$	$(5, 0)$	$(7, 0)$
	$(-1, 1)$	$(1, 1)$	$(3, 1)$	$(5, 1)$
		$(-1, 2)$	$(1, 2)$	$(3, 2)$
			$(-1, 3)$	$(1, 3)$
				$(-1, 4)$

$$(1, 2) \Longrightarrow \mathcal{S} = \left\{1,\ t,\ \exp(\pm\mu t),\ t\ \exp(\pm\mu t),\ t^2\ \exp(\pm\mu t)\right\}$$

Exponential Fitting

L. Ixaru and G. Vanden Berghe
Exponential fitting
Kluwer Academic Publishers, Dordrecht, 2004

$$\eta_{-1}(Z) = \begin{cases} \cos(|Z|^{1/2}) & \text{if } Z < 0 \\ \cosh(Z^{1/2}) & \text{if } Z \geq 0 \end{cases}$$

$$\eta_0(Z) = \begin{cases} \sin(|Z|^{1/2})/|Z|^{1/2} & \text{if } Z < 0 \\ 1 & \text{if } Z = 0 \\ \sinh(Z^{1/2})/Z^{1/2} & \text{if } Z > 0 \end{cases} \qquad Z := (\mu h)^2 = -(\omega h)^2$$

$$\eta_n(Z) := \frac{1}{Z}[\eta_{n-2}(Z) - (2n-1)\eta_{n-1}(Z)], \quad n = 1, 2, 3, \ldots$$

$$\eta_n'(Z) = \frac{1}{2}\eta_{n+1}(Z), \quad n = 1, 2, 3, \ldots$$

Choice of ω

- local optimization
  based on local truncation error (lte)
  ω is step-dependent

- global optimization
  Preservation of geometric properties (periodicity, energy, ...)
  ω is constant over the interval of integration

Fourth-order boundary value problem

$$y^{(4)} = F(t, y) \qquad a \leq t \leq b$$

$$\begin{array}{ll} y(a) = A_1 & y''(a) = A_2 \\ y(b) = B_1 & y''(b) = B_2 \end{array}$$

- special case : $y^{(4)} + f(t)\,y = g(t)$
- mathematical modeling of viscoelastic and inelastic flows, deformation of beams, plate deflection theory, ...
- work by Doedel, Usmani, Agarwal, Cherruault et al., Van Daele et al., ...
- finite differences, B-splines, ...

The formulae

$$t_j = a + j\,h,\, j = 0,\, 1,\, \ldots\, N+1 \qquad N \geq 5 \qquad h := \frac{b-a}{N+1}$$

- central formula for $j = 2, \ldots, N-1$

$$y_{j-2} + a_1\, y_{j-1} + a_0\, y_j + a_1\, y_{j+1} + y_{j+2} = \\ h^4\, \left(b_2\, F_{j-2} + b_1\, F_{j-1} + b_0\, F_j + b_1\, F_{j+1} + b_2\, F_{j+2}\right)$$

  whereby y_j is approximate value of $y(t_j)$ and $F_j := F(t_j,\, y_j)$.

- begin formula

$$c_1\, y_0 + c_2\, y_1 + c_3\, y_2 + y_3 = \\ d_1\, h^2\, y_0'' + h^4\, (d_2\, F_0 + d_3\, F_1 + d_4\, F_2 + d_5\, F_3 + d_6\, F_4 + d_7\, F_5)$$

- end formula

Central formula

$$\mathcal{L}[y] := y(t-2h) + a_1\, y(t-h) + a_0\, y(t) + a_1\, y(t+h) + y(t+2h)$$
$$-h^4 \left(b_2\, y^{(4)}(t-2h) + b_1\, y^{(4)}(t-h) + b_0\, y^{(4)}(t) + b_1\, y^{(4)}(t+h) + b_2\, y^{(4)}(t+2h) \right)$$

$$P = -1 : \ \mathcal{S} = \left\{ 1,\, t,\, t^2,\, \ldots,\, t^{M-1} \right\}$$

case $M = 10$: order 6

$$y_{p-2} - 4\, y_{p-1} + 6\, y_p - 4\, y_{p+1} + y_{p+2} =$$
$$\frac{h^4}{720} \left(-y_{p-2}^{(4)} + 124\, y_{p-1}^{(4)} + 474\, y_p^{(4)} + 124\, y_{p+1}^{(4)} - y_{p+2}^{(4)} \right)$$

$$\mathcal{L}[y](t) = \frac{1}{3024}\, h^{10}\, y^{(10)}(t) + \mathcal{O}(h^{12})$$

case $M = 8$ and $b_2 = 0$: order 4

EF Central formula

$$\mathcal{L}[y] := y(t-2h) + a_1\, y(t-h) + a_0\, y(t) + a_1\, y(t+h) + y(t+2h)$$
$$-h^4 \left(b_2\, y^{(4)}(t-2h) + b_1\, y^{(4)}(t-h) + b_0\, y^{(4)}(t) + b_1\, y^{(4)}(t+h) + b_2\, y^{(4)}(t+2h) \right)$$

$$P = 0: \ \mathcal{S} = \left\{ \cos(\omega\, t),\ \sin(\omega\, t), 1,\ t,\ t^2,\ \ldots, t^{M-3} \right\}$$

case $M = 10$:

$$y_{p-2} - 4\, y_{p-1} + 6\, y_p - 4\, y_{p+1} + y_{p+2} =$$
$$h^4 \left(b_2\, y^{(4)}_{p-2} + b_1\, y^{(4)}_{p-1} + b_0\, y^{(4)}_p + b_1\, y^{(4)}_{p+1} + b_2\, y^{(4)}_{p+2} \right)$$

$$b_0 = \frac{4\,\cos^2\theta - 2 - 11\,\cos\theta}{6\,(\cos\theta - 1)^2} + \frac{6}{\theta^4} \quad b_1 = \frac{\cos^2\theta + 5}{6\,(\cos\theta - 1)^2} - \frac{4}{\theta^4} \quad b_2 = -\frac{\cos\theta + 2}{12\,(\cos\theta - 1)^2} + \frac{1}{\theta^4}$$

$$\mathcal{L}[y](t) = \frac{1}{3024} h^{10} \left(y^{(10)}(t) + \omega^2\, y^{(8)}(t) \right) + \mathcal{O}(h^{12})$$

EF Central formula

$$\mathcal{L}[y] := y(t-2h) + a_1\, y(t-h) + a_0\, y(t) + a_1\, y(t+h) + y(t+2h)$$
$$-h^4 \left(b_2\, y^{(4)}(t-2h) + b_1\, y^{(4)}(t-h) + b_0\, y^{(4)}(t) + b_1\, y^{(4)}(t+h) + b_2\, y^{(4)}(t+2h) \right)$$

$$P = 1: \ \mathcal{S} = \left\{ \cos(\omega\, t), \sin(\omega\, t), t\, \cos(\omega\, t), t\, \sin(\omega\, t), 1, t, t^2, \ldots, t^{M-5} \right\}$$

case $M = 6$ and $b_1 = b_2 = 0$:

$$y_{p-2} + a_1\, y_{p-1} + a_0\, y_p + a_1\, y_{p+1} + y_{p+2} = b_0\, h^4\, y_p^{(4)}$$

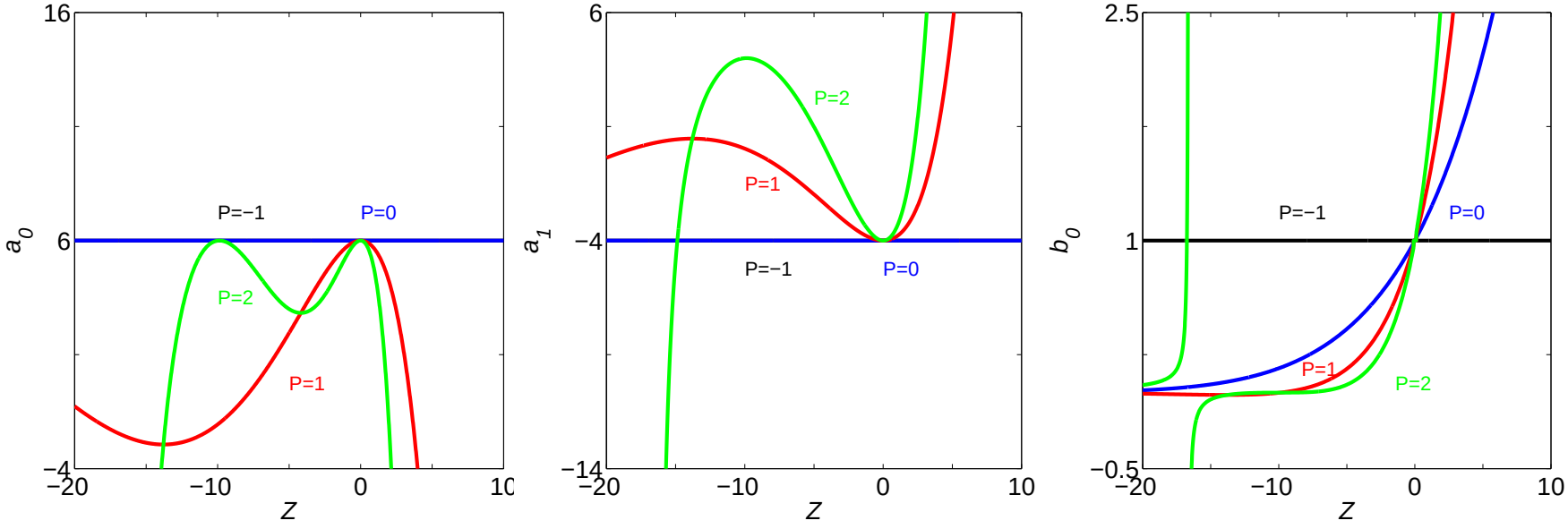
$$a_0 = 2\, \frac{-8\, \sin^2\theta + \theta\, (4\, \cos\theta - 1) \sin\theta - 4\, \cos\theta + 4}{\theta\, \sin\theta + 4\, \cos\theta - 4} \qquad a_1 = -4\, \frac{\sin\theta\, (\theta\, \cos\theta - 2\, \sin\theta)}{\theta\, \sin\theta + 4\, \cos\theta - 4}$$

$$b_0 = 4\, \frac{\sin\theta \left(\sin^2\theta - 2 + 2\, \cos\theta \right)}{\theta^3\, (\theta\, \sin\theta + 4\, \cos\theta - 4)}$$

Coefficients of central formula

e.g. case $M = 6$:

$$y_{p-2} + a_1\, y_{p-1} + a_0\, y_p + a_1\, y_{p+1} + y_{p+2} = b_0\, h^4\, y_p^{(4)}$$



$$Z = (\mu\, h)^2 = -(\omega\, h)^2$$

Coefficients of central formula

e.g. case $M = 6$

- $P = -1$:

$$b_0 = 1$$

- $P = 0$:

$$b_0 = 4\,\frac{(\cos\theta - 1)^2}{\theta^4}$$

- $P = 1$:

$$b_0 = -4\,\frac{\sin\theta\,(\cos\theta - 1)^2}{\theta^3\,(4\,\cos\theta - 4 + \theta\,\sin\theta)}$$

- $P = 2$:

$$b_0 = -2\,\frac{\sin^3\theta}{\theta^2\,(\theta\,\cos\theta - 3\,\sin\theta)}$$

Central formula : coefficients

e.g. case $M = 6$

- $P = -1$:

$$b_0 = 1$$

- $P = 0$:

$$b_0 = 1 - \frac{1}{6}\theta^2 + \frac{1}{80}\theta^4 + \mathcal{O}(\theta^6)$$

- $P = 1$:

$$b_0 = 1 - \frac{1}{3}\theta^2 + \frac{37}{720}\theta^4 + \mathcal{O}(\theta^6)$$

- $P = 2$:

$$b_0 = 1 - \frac{1}{2}\theta^2 + \frac{7}{60}\theta^4 + \mathcal{O}(\theta^6)$$

Central formula : local truncation error

$$\text{lte} = \mathcal{L}[y](t)$$

As an inifinite series :

$$\text{lte} = h^M \, C_M \, D^{K+1} \, (D^2 + \omega^2)^{P+1} y(t) + \mathcal{O}(h^{M+2})$$

In closed form : (Coleman and Ixaru)

$$\text{lte} = h^M \, \Phi_{K,P}(Z) \, D^{K+1} \, (D^2 + \omega^2)^{P+1} y(\xi)$$

$Z \in$ some interval $\qquad \Phi_{K,P}(0) \neq 0 \qquad \xi \in (t - 2\,h,\, t + 2\,h)$

Local truncation error

$$\text{lte} = h^M\, C_M\, D^{K+1}\,(D^2+\omega^2)^{P+1} y(t) + \mathcal{O}(h^{M+2})\,,$$

At t_j : $D^{(K+1)}\,(D^2+\omega_j^2)^{(P+1)} y(t)\Big|_{t=t_j} = 0 \qquad j = 2,\ldots,N-1$

- $P = 0$:

$$y^{(K+3)}(t_j) + y^{(K+1)}(t_j)\,\omega_j^2 = 0$$

- $P = 1$:

$$y^{(K+5)}(t_j) + 2\,y^{(K+3)}(t_j)\,\omega_j^2 + y^{(K+1)}(t_j)\,\omega_j^4 = 0$$

- $P = 2$:

$$y^{(K+7)}(t_j) + 3\,y^{(K+5)}(t_j)\,\omega_j^4 + 3\,y^{(K+3)}(t_j)\,\omega_j^4 + y^{(K+1)}(t_j)\,\omega_j^6 = 0$$

Local truncation error

$$\text{lte} = h^M \, C_M \, D^{K+1} \, (D^2 + \omega^2)^{P+1} y(t) + \mathcal{O}(h^{M+2}) \, ,$$

$$\text{At } t_j : D^{(K+1)} \, (D^2 + \omega_j^2)^{(P+1)} y(t) \Big|_{t=t_j} = 0 \qquad j = 2, \ldots, N-1$$

ω_j^2 is solution of equation of degree $P+1$.

- Which value of P should be chosen ?
- Which root ω_j should be chosen ?

Parameter selection

$$\text{lte} = h^M \, C_M \, D^{K+1} \, (D^2 - \mu^2)^{P+1} y(t) + \mathcal{O}(h^{M+2})$$

Suppose $y(t)$ takes the form $t^{P_0} \, e^{\mu_0 \, t}$

Then lte$= 0$ for any EF rule with $P \geq P_0$ and $\mu_j = \mu_0$

Theorem

If $y(t) = t^{P_0} \, e^{\mu_0 \, t}$ then $\nu = \mu_0^2$ is a root of multiplicity $P - P_0 + 1$ of $D^{K+1} \, (D^2 - \nu)^{P+1} y(t) = 0$.

- if $P = P_0$, then $\mu = \mu_0$ will be a single root
- if $P = P_0 + 1$, then $\mu = \mu_0$ will be a double root
- if $P = P_0 + 2$, then $\mu = \mu_0$ will be a triple root
- ...

Parameter selection

$$\text{lte} = h^M \, C_M \, D^{K+1} \, (D^2 - \mu^2)^{P+1} y(t) + \mathcal{O}(h^{M+2})$$

Suppose $y(t)$ does not take the form $t^{P_0} \, e^{\mu_0 \, t}$.

Then $y(t) \notin \mathcal{S}$ for any P.

For a given value of P :

$$D^{(K+1)} \, (D^2 - \mu_j^2)^{(P+1)} y(t) \Big|_{t=t_j} = 0$$

At each point t_j, this gives $P + 1$ values for μ_j^2.

Idea : keep $|\mu_j \, h|$ as small as possible.
If possible, choose $P \geq 1$ to avoid too large values for $|\mu_j|$.

Numerical Illustrations

Problem 1

$$y^{(4)} - \frac{384\,t^4}{(2+t^2)^4} y = 24\,\frac{2-11\,t^2}{(2+t^2)^4}$$

$$y(-1) = \frac{1}{3} \qquad y(1) = \frac{1}{3}$$

$$y''(-1) = \frac{2}{27} \qquad y''(1) = \frac{2}{27}$$

$$\text{Solution}: y(t) = \frac{1}{2+t^2}$$

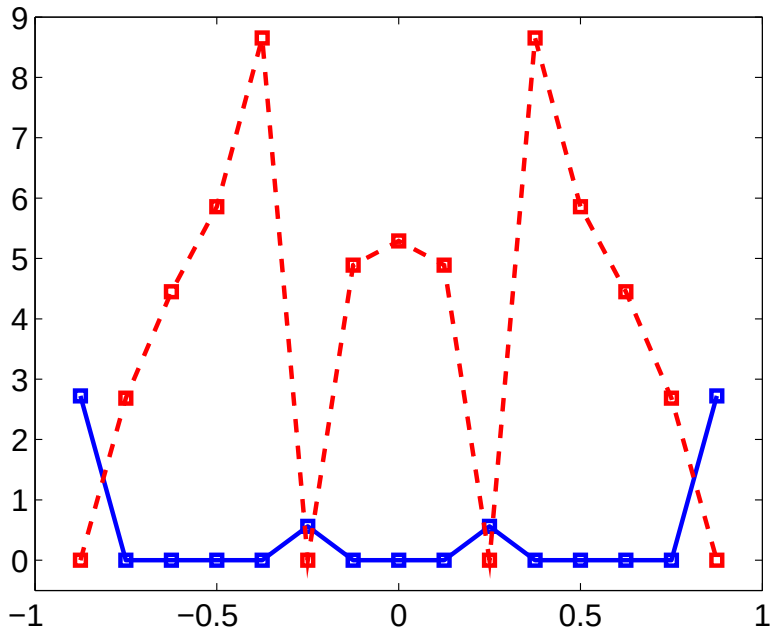
Since $y(t)$ does not to belong to the fitting space of a EF-rule, the parameter μ will not be constant over the interval of integration.

Solution obtained by a fourth-order EF method

Computation of μ_j with $M = 8$:

$$P = 0 : y^{(8)}(t_j) - y^{(6)}(t_j)\,\mu_j^2 = 0$$

- re-express higher order derivatives in terms of y, y', y'' and y'''
- approximate y', y'' and y''' in terms of y
- an initial approximation for y can be computed with the classical, polynomial rule

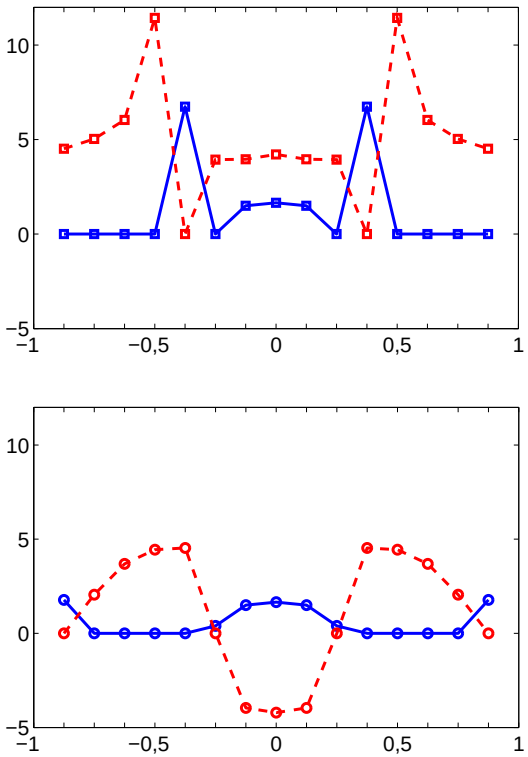


Real and imaginary part of μ_j

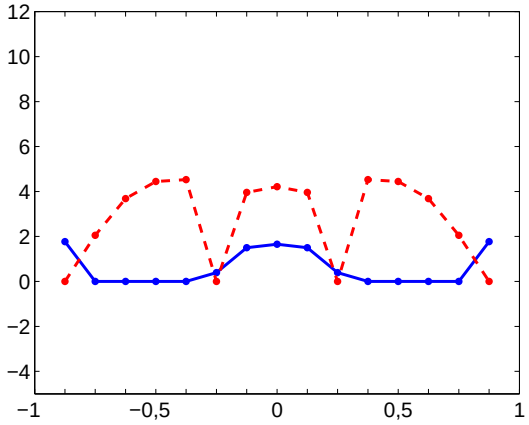
Solution obtained by a fourth-order EF method

Computation of μ_j with $M = 8$:

$$P = 1 : y^{(8)}(t_j) - 2\,y^{(6)}(t_j)\,\mu_j^2 + y^{(4)}(t_j)\,\mu_j^4 = 0$$



Real and imag. part of $\mu_{1,j}$ and $\mu_{2,j}$

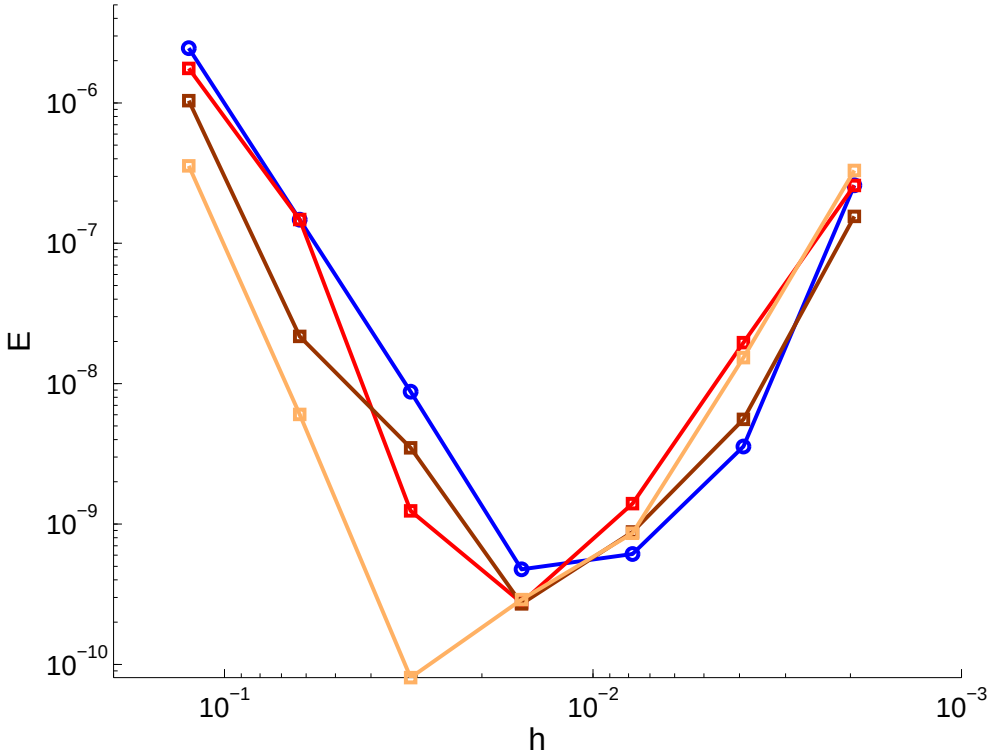


Real and imaginary part of μ_j with smallest norm

Solution obtained by a fourth-order EF method

Computation of μ_j with $M = 8$:

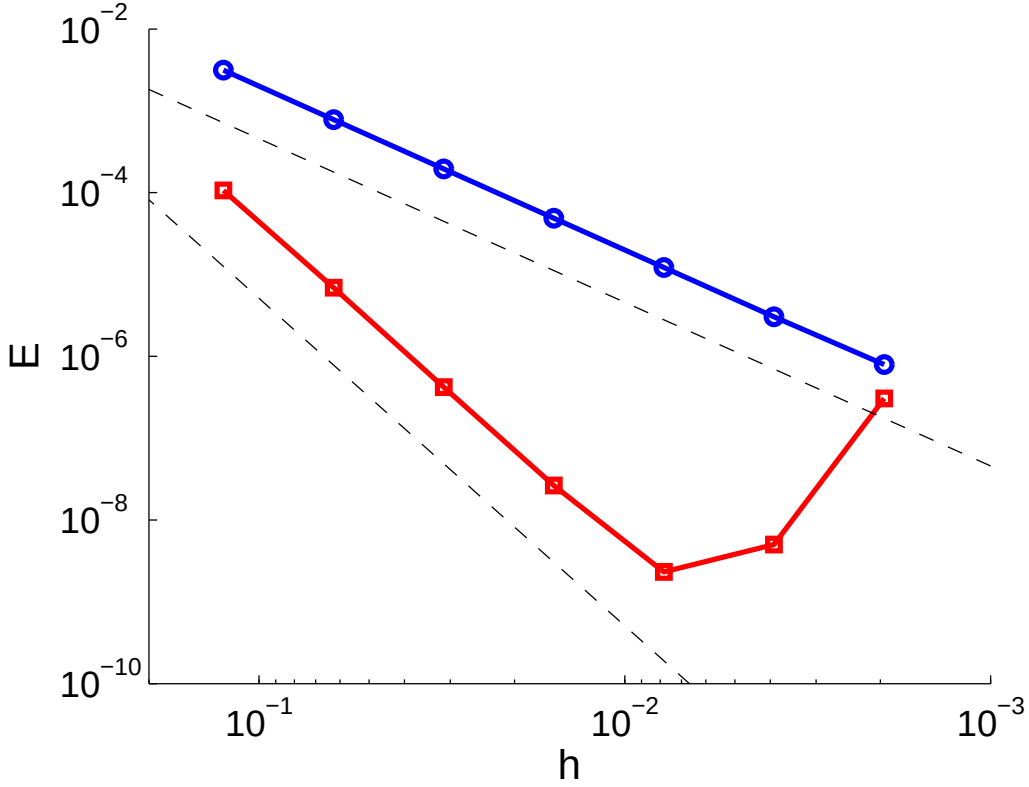
$$P = 1 : y^{(8)}(t_j) - 2\,y^{(6)}(t_j)\,\mu_j^2 + y^{(4)}(t_j)\,\mu_j^4 = 0$$



error obtained with $\mu_{1,j}$, $\mu_{2,j}$ and μ with smallest norm

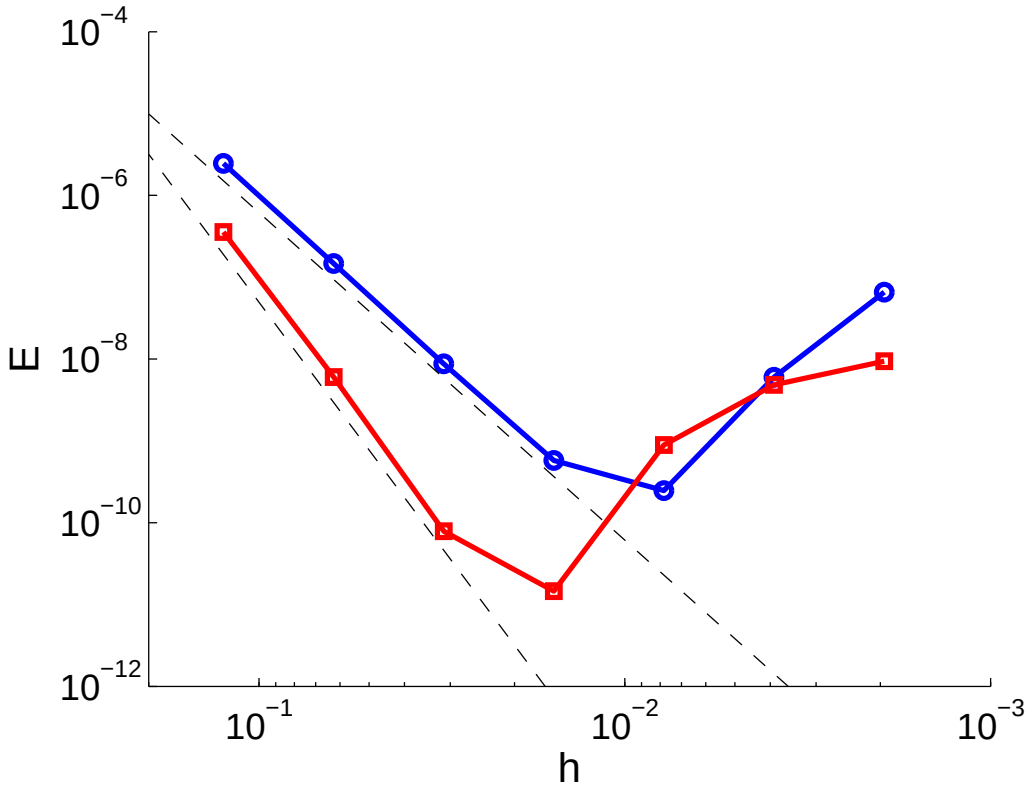
Global error

$M = 6$: $(K, P) = (5, -1)$: second-order method
$(K, P) = (1, 1)$: fourth-order method



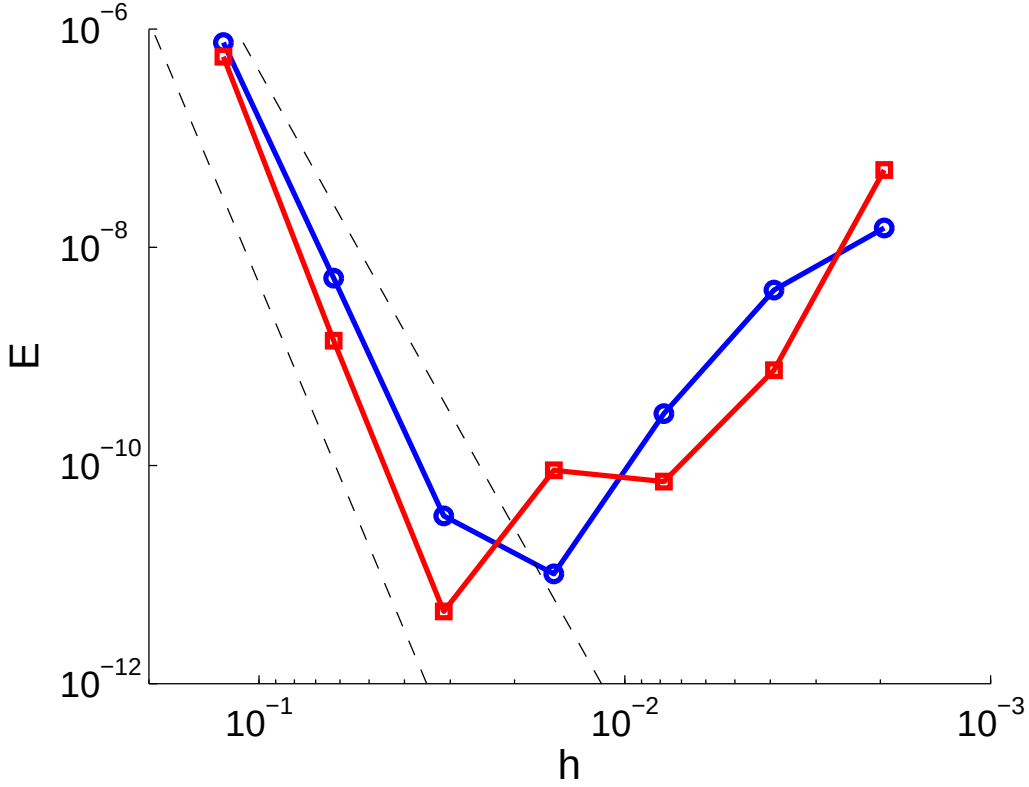
Global error

$M = 8$: $(K, P) = (7, -1)$: fourth-order method
$(K, P) = (3, 1)$: sixth-order method

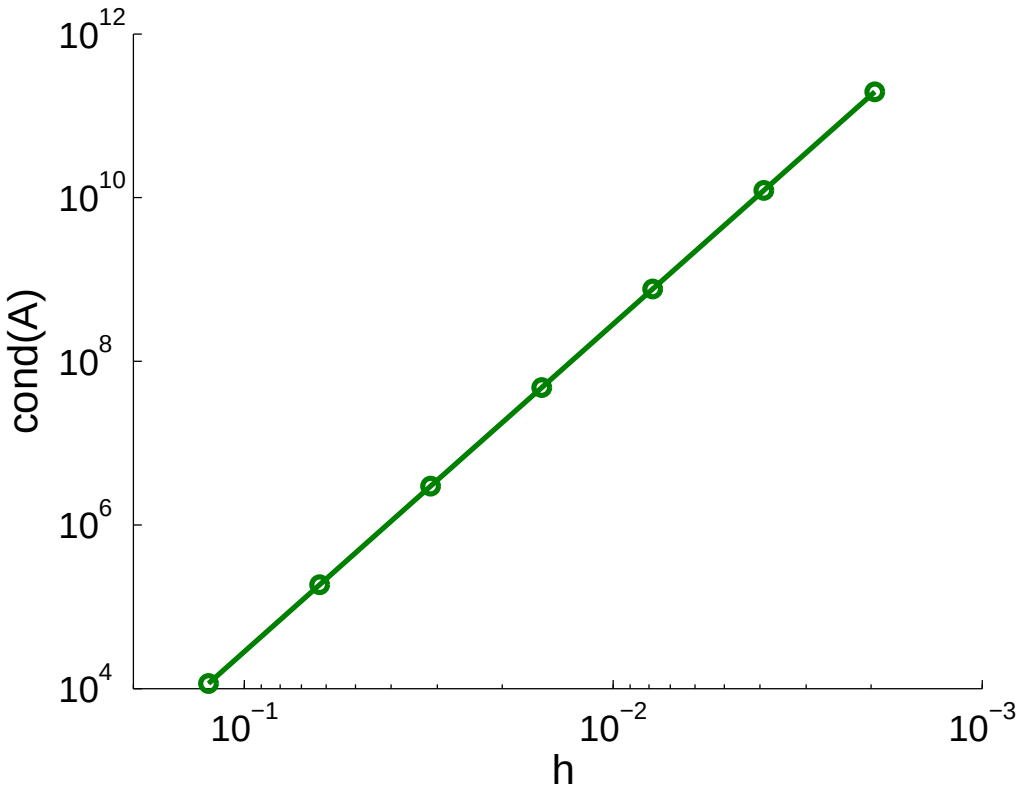


Global error

$M = 10$: $(K, P) = (9, -1)$: sixth-order method
$(K, P) = (5, 1)$: eighth-order method



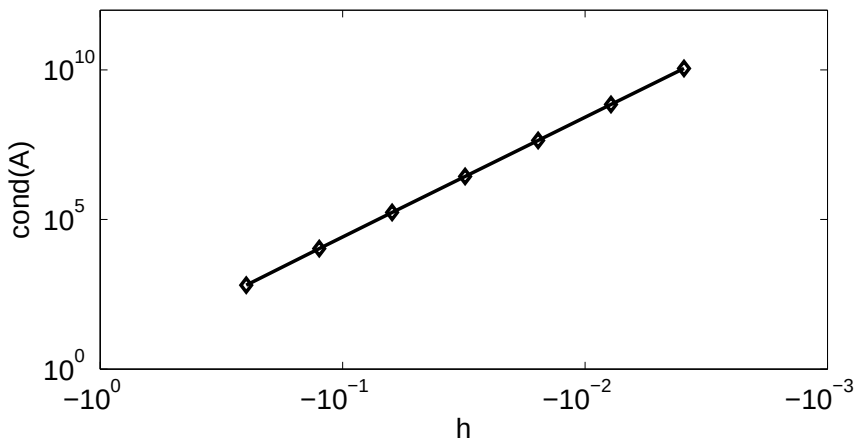
Condition number of the coefficient matrix

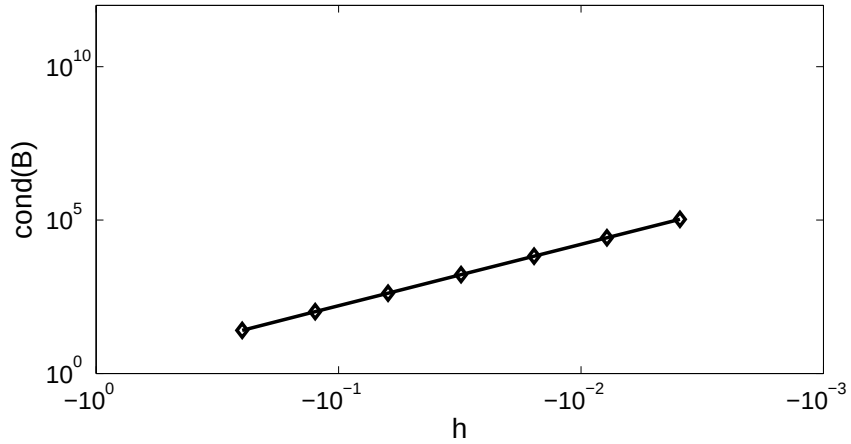


Condition number of the coefficient matrix

The (classical) discretisation of $y_p^{(4)}$ gives rise to the coefficient matrix

$$\begin{pmatrix} 5 & -4 & 1 & & & & \\ -4 & 6 & -4 & 1 & & & \\ 1 & -4 & 6 & -4 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & -4 & 6 & -4 & 1 \\ & & & 1 & -4 & 6 & -4 \\ & & & & 1 & -4 & 5 \end{pmatrix} = \begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{pmatrix}^2$$





Factorisation of the coefficient matrix

$$\begin{pmatrix} c_2 & c_3 & 1 & & & & \\ a_1 & a_2 & a_1 & 1 & & & \\ 1 & a_1 & a_2 & a_1 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & a_1 & a_2 & a_1 & 1 \\ & & & 1 & a_1 & a_2 & a_1 \\ & & & & 1 & c_3 & c_2 \end{pmatrix} = \begin{pmatrix} a_2-1 & a_1 & 1 & & & & \\ a_1 & a_2 & a_1 & 1 & & & \\ 1 & a_1 & a_2 & a_1 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & a_1 & a_2 & a_1 & 1 \\ & & & 1 & a_1 & a_2 & a_1 \\ & & & & 1 & a_1 & a_2-1 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha & 1 & & & \\ 1 & \alpha & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & \alpha & 1 \\ & & & 1 & \alpha \end{pmatrix} \begin{pmatrix} \beta & 1 & & & \\ 1 & \beta & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & \beta & 1 \\ & & & 1 & \beta \end{pmatrix}$$

$M = 6$:

P	α	β
-1	-2	-2
0	-2	-2
1	$-2 - 4\,\frac{\sin\theta(2\,\cos\theta - 2 + \theta\,\sin\theta)}{4\,\sin\theta - \theta\,(1+\cos\theta))}$	-2

Problem 2

$$y^{(4)} - t = 4\,e^t$$

$$y(-1) = -1/e \qquad y(1) = e$$
$$y''(-1) = 1/e \qquad y''(1) = 3\,e$$

$$\text{Solution}: y(t) = e^t\,t$$

In theory, this problem is solved up to machine accuracy by any EF-rule with $P \geq 1$ and $\mu_j = 1$.

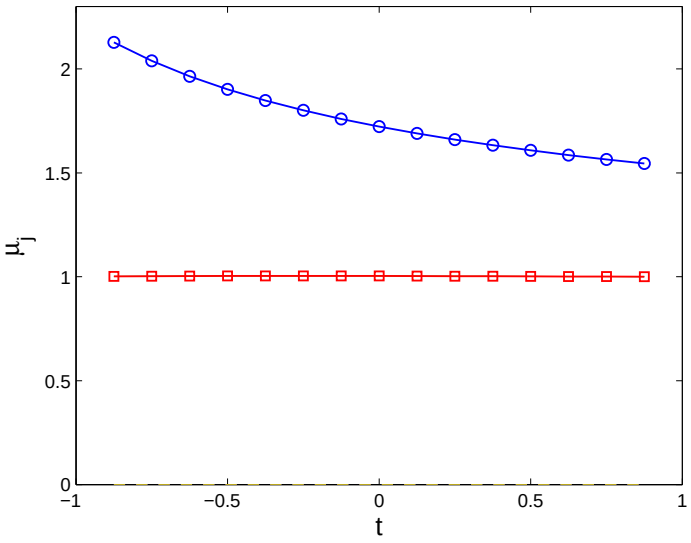
$M = 6$

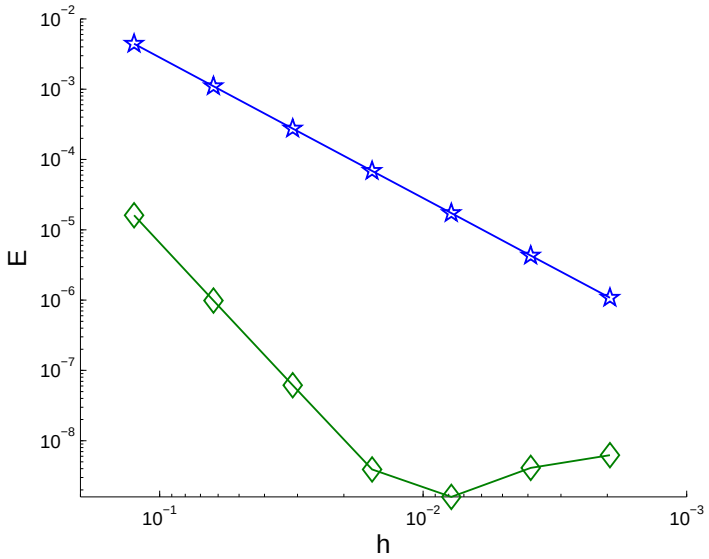
$$P = 1 : y^{(6)}(t_j) - 2\,y^{(4)}(t_j)\,\mu_j^2 + y^{(2)}(t_j)\,\mu_j^4 = 0$$

differentiating the differential equation :

$$(y^{(2)}(t_j) + 4\,e^{t_j}) - 2\,(y_j + 4\,e^{t_j})\mu_j^2 + y^{(2)}(t_j)\,\mu_j^4 = 0$$

approximating $y^{(2)}(t_j)$ by $\mathcal{O}(h^2)$ finite difference scheme





two real roots $\mu^{(1)}$ and $\mu^{(2)}$

How to improve the accuracy ?

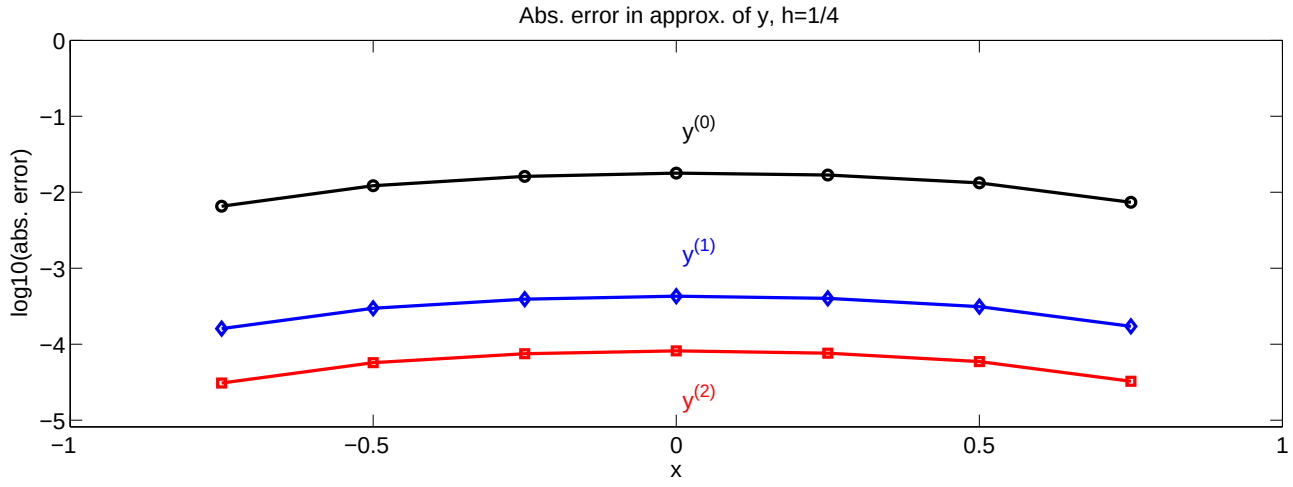
$$y^{(0)} \overset{\mu^{(1)}}{\longrightarrow} y^{(1)} \overset{\mu^{(2)}}{\longrightarrow} y^{(2)} \dots$$

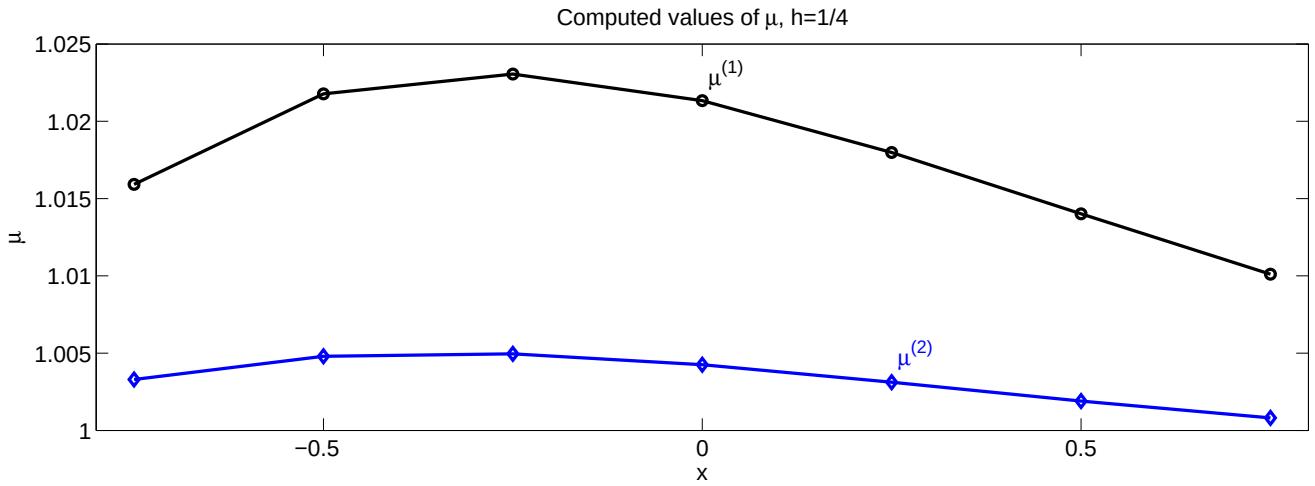
- $y^{(0)}$ is obtained from classical, second order method
- compute $\mu^{(1)}$ from $y^{(0)}$ with classical second order schemes to approximate the derivatives that appear in the lte
- $y^{(1)}$ is obtained from EF method with $P = 1$ and $\mu = \mu^{(1)}$
- compute $\mu^{(2)}$ from $y^{(1)}$ with EF schemes ($P = 1$) to approximate the derivatives that appear in the lte
- $y^{(2)}$ is obtained from EF method with $P = 1$ and $\mu = \mu^{(2)}$

  $\dots$

How to improve the accuracy ?

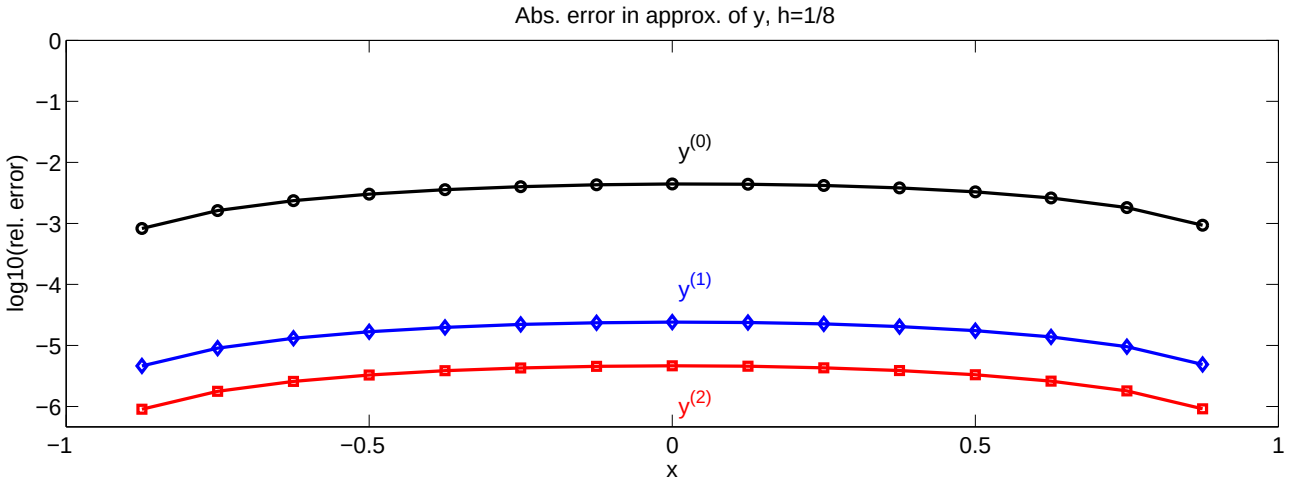
case $M = 6$, $h = 1/4$

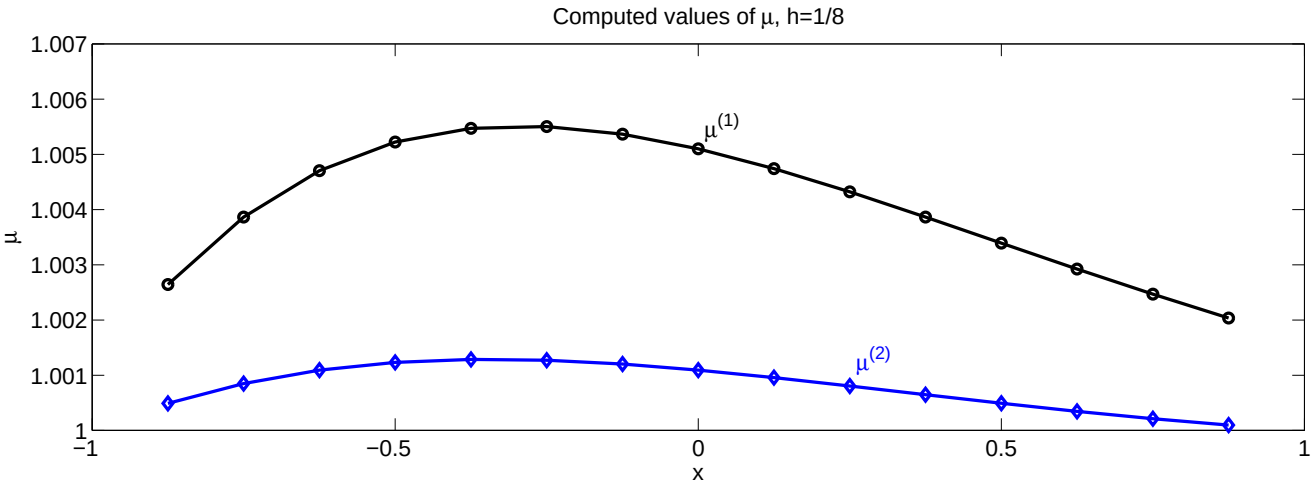




How to improve the accuracy ?

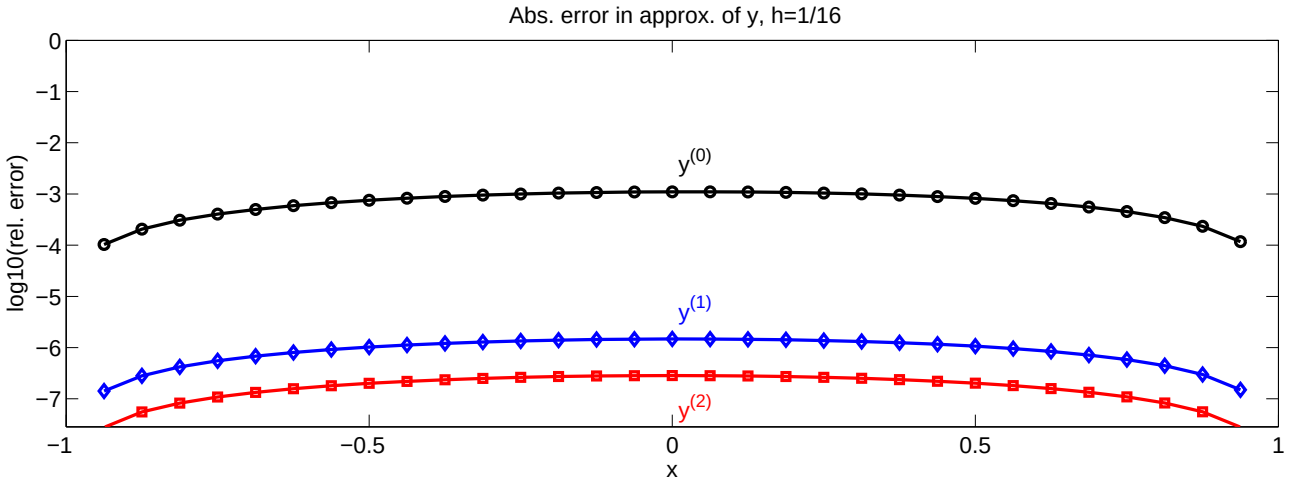
case $M = 6$, $h = 1/8$

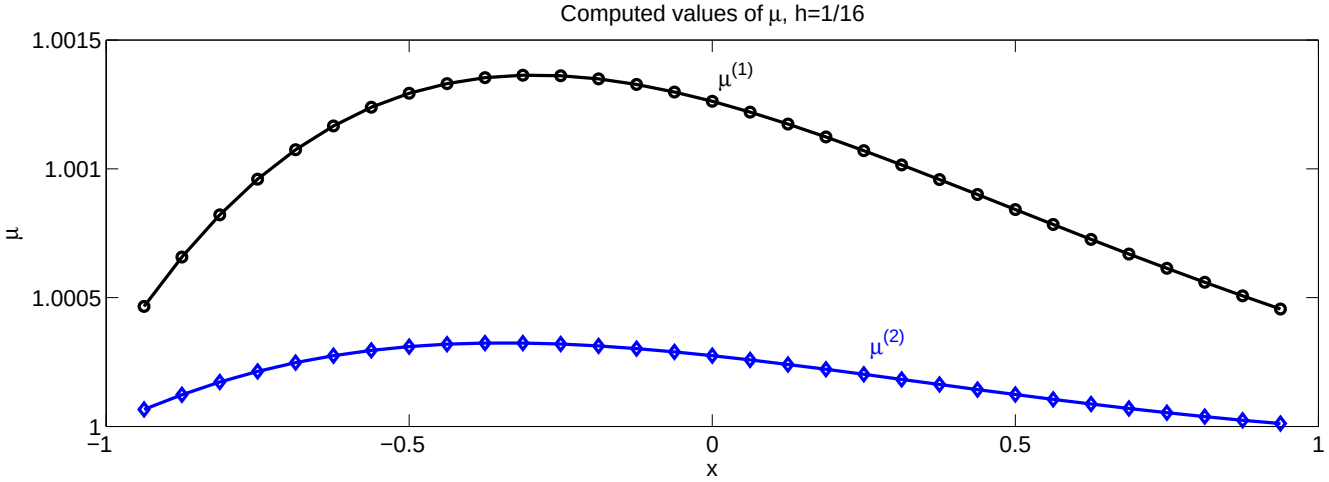




How to improve the accuracy ?

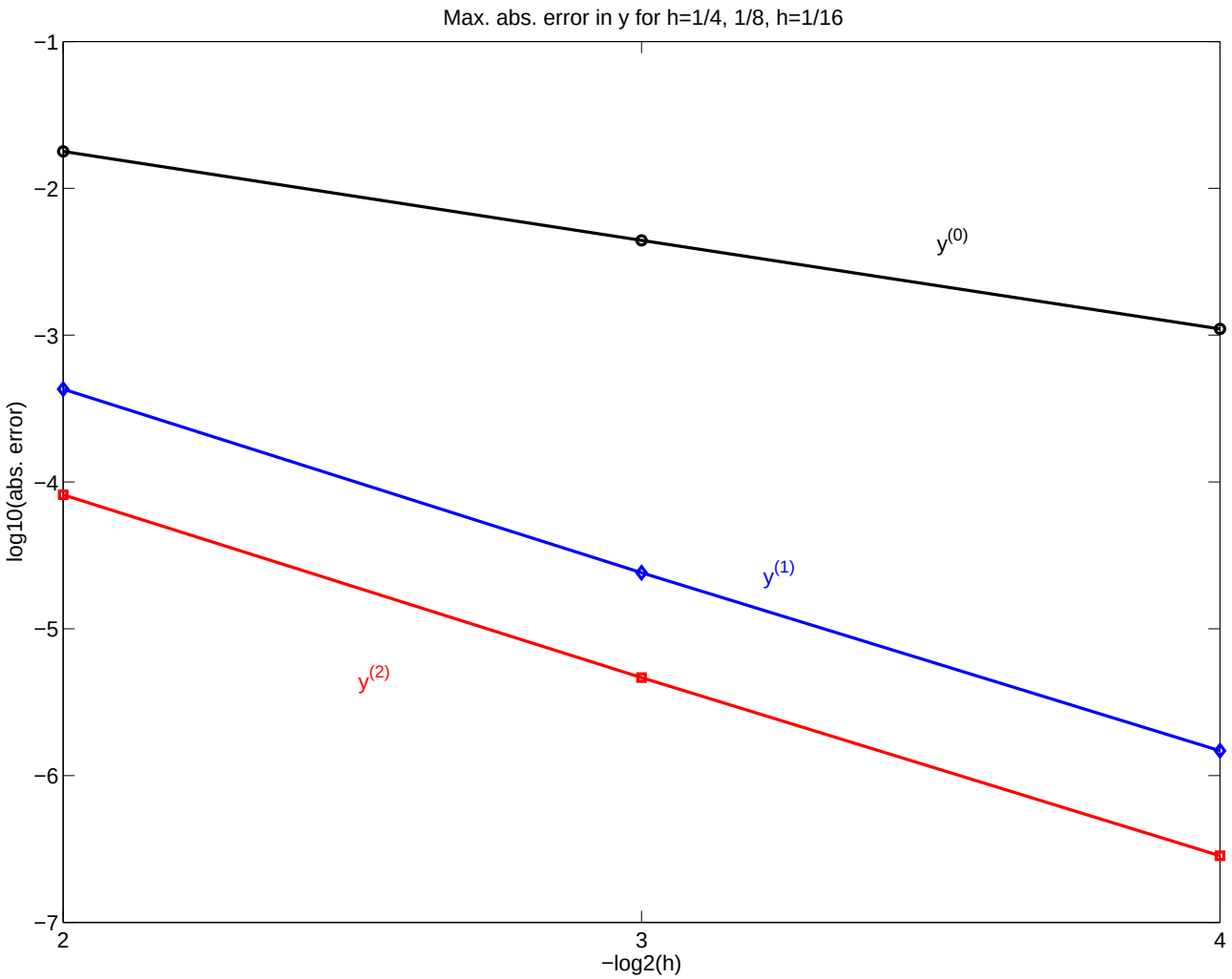
case $M = 6$, $h = 1/16$





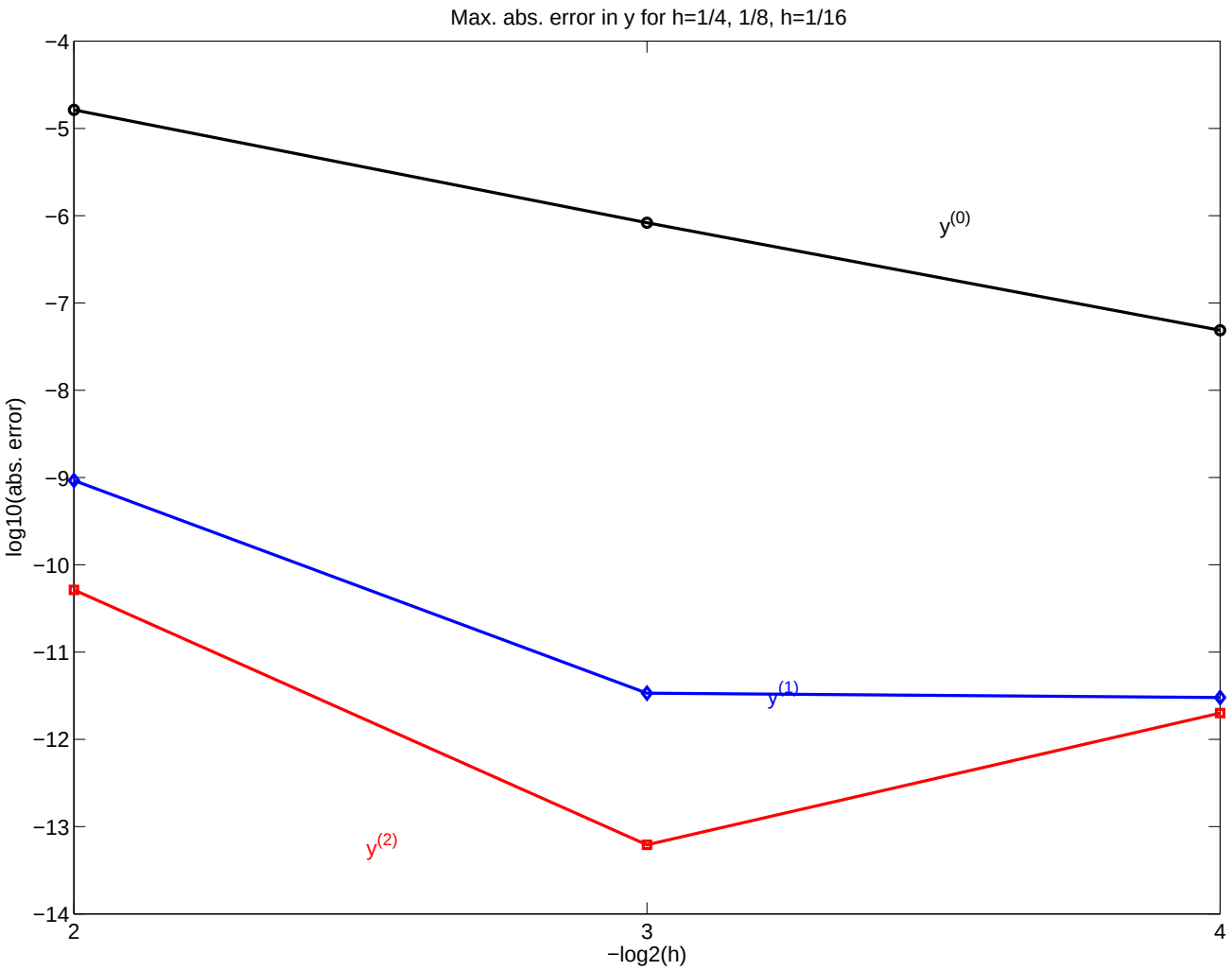
How to improve the accuracy ?

case $M = 6$



How to improve the accuracy ?

case $M = 8$



Conclusions

- Fourth-order boundary value problems are solved by means of parameterized exponentially-fitted methods.
- A suitable value for the parameter can be found from the roots of the leading term of the local truncation error.
- If a constant value is found, then a very accurate solution can be obtained.
- However, the methods strongly suffer from the fact that the system to be solved is ill-conditioned for small values of the mesh size.