Exponentially fitted methods applied to fourth-order boundary value problems

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SCICADE, Beijing, 2009

Outline

Introduction

In the past 15 years, our research group has constructed modified versions of well-known

- linear multistep methods
- Runge-Kutta methods

Aim : build methods which perform very good when the solution has a known exponential of trigonometric behaviour.

A model problem

Consider the initial value problem

$$y'' + \omega^2 y = g(y) \qquad y(a) = y_a \qquad y(a) = y'_a \,.$$

If $|g(y)| \ll |\omega^2 y|$ then

$$y(t) \approx \alpha \cos(\omega t + \phi)$$

To mimic this oscillatory behaviour, we construct methods which yield exact results when the solution is of trigonometric (in the complex case : exponential) type.
These methods are called Exponentially-fitted methods.

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EF methods

To determine the coefficients of a method, we impose conditions on a linear functional. These conditions are related to the fitting space $\mathcal{S}$ which contains

- polynomials :

$$\{t^q | q = 0, \dots, K\}$$

- exponential or trigonometric functions, multiplied with powers of t :

$$\{t^q \exp(\pm\mu t) | q = 0, \dots, P\}$$

or, with $\omega = i\,\mu$,

$$\{t^q \cos(\omega t), t^q \sin(\omega t) | q = 0, \dots, P\}$$

EF method can be characterized by the couple (K, P)
Classical method : $P = -1$
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(K, P)				
$M = 2$	$M = 4$	$M = 6$	$M = 8$	$M = 10$
$(1, -1)$	$(3, -1)$	$(5, -1)$	$(7, -1)$	$(9, -1)$
$(-1, 1)$	$(1, 0)$	$(3, 0)$	$(5, 0)$	$(7, 0)$
	$(-1, 1)$	$(1, 1)$	$(3, 1)$	$(5, 1)$
		$(-1, 2)$	$(1, 2)$	$(3, 2)$
			$(-1, 3)$	$(1, 3)$
				$(-1, 4)$

$$(1, 2) \Longrightarrow \mathcal{S} = \left\{1,\ t,\ \exp(\pm\mu t),\ t\ \exp(\pm\mu t),\ t^2\ \exp(\pm\mu t)\right\}$$

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L. Ixaru and G. Vanden Berghe
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Kluwer Academic Publishers, Dordrecht, 2004

$$\eta_{-1}(Z) = \begin{cases} \cos(|Z|^{1/2}) & \text{if } Z < 0 \\ \cosh(Z^{1/2}) & \text{if } Z \geq 0 \end{cases}$$

$$\eta_0(Z) = \begin{cases} \sin(|Z|^{1/2})/|Z|^{1/2} & \text{if } Z < 0 \\ 1 & \text{if } Z = 0 \\ \sinh(Z^{1/2})/Z^{1/2} & \text{if } Z > 0 \end{cases} \qquad Z := (\mu h)^2 = -(\omega h)^2$$

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- local optimization
  based on local truncation error (lte)
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Fourth-order boundary value problem

$$y^{(4)} = F(t, y) \qquad a \leq t \leq b$$

$$\begin{array}{ll} y(a) = A_1 & y''(a) = A_2 \\ y(b) = B_1 & y''(b) = B_2 \end{array}$$

- special case : $y^{(4)} + f(t)y = g(t)$
- mathematical modeling of viscoelastic and inelastic flows, deformation of beams, plate deflection theory, ...
- work by Doedel, Usmani, Agarwal, Cherruault et al., Van Daele et al., ...
- finite differences, B-splines, ...

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The formulae

$$t_j = a + j\,h,\, j = 0,\,1,\,\ldots N+1 \qquad N \geq 5 \qquad h := \frac{b-a}{N+1}$$

- central formula for $j = 2, \ldots, N-1$

$$y_{j-2} + a_1\,y_{j-1} + a_0\,y_j + a_1\,y_{j+1} + y_{j+2} = h^4\,\left(b_2\,F_{j-2} + b_1\,F_{j-1} + b_0\,F_j + b_1\,F_{j+1} + b_2\,F_{j+2}\right)$$

  whereby y_j is approximate value of $y(t_j)$ and $F_j := F(t_j,\, y_j)$.

- begin formula

$$c_1\,y_0 + c_2\,y_1 + c_3\,y_2 + y_3 = d_1\,h^2\,y_0'' + h^4\,\left(d_2\,F_0 + d_3\,F_1 + d_4\,F_2 + d_5\,F_3 + d_6\,F_4 + d_7\,F_5\right)$$

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- end formula

Central formula

$$\mathcal{L}[y] := y(t-2\,h) + a_1\,y(t-h) + a_0\,y(t) + a_1\,y(t+h) + y(t+2\,h)$$
$$-h^4\,\left(b_2\,y^{(4)}(t-2\,h) + b_1\,y^{(4)}(t-h) + b_0\,y^{(4)}(t) + b_1\,y^{(4)}(t+h) + b_2\,y^{(4)}(t+2\,h)\right)$$

$$P = -1: \;\mathcal{S} = \left\{1,\, t,\, t^2,\, \ldots,\, t^{M-1}\right\}$$

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case $M = 10$: order 6

$$y_{p-2} - 4\, y_{p-1} + 6\, y_p - 4\, y_{p+1} + y_{p+2} =$$
$$\frac{h^4}{720} \left(-y^{(4)}_{p-2} + 124\, y^{(4)}_{p-1} + 474\, y^{(4)}_{p} + 124\, y^{(4)}_{p+1} - y^{(4)}_{p+2} \right)$$

$$\mathcal{L}[y](t) = \frac{1}{3024}\, h^{10}\, y^{(10)}(t) + \mathcal{O}(h^{12})$$

Central formula

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case $M = 8$ and $b_2 = 0$: order 4

$$y_{p-2} - 4\, y_{p-1} + 6\, y_p - 4\, y_{p+1} + y_{p+2} = \frac{h^4}{6}\left(y^{(4)}_{p-1} + 4\, y^{(4)}_p + y^{(4)}_{p+1}\right)$$

$$\mathcal{L}[y](t) = -\frac{1}{720}\, h^8\, y^{(8)}(t) + \mathcal{O}(h^{10})$$

Central formula

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case $M = 6$ and $b_1 = b_2 = 0$: order 2

$$y_{p-2} - 4\, y_{p-1} + 6\, y_p - 4\, y_{p+1} + y_{p+2} = h^4\, y_p^{(4)}$$

$$\mathcal{L}[y](t) = \frac{1}{6}\, h^6\, y^{(6)}(t) + \mathcal{O}(h^8)$$

EF Central formula

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$$b_0 = \frac{4\,\cos^2\theta - 2 - 11\,\cos\theta}{6\,(\cos\theta - 1)^2} + \frac{6}{\theta^4} \quad b_1 = \frac{\cos^2\theta + 5}{6\,(\cos\theta - 1)^2} - \frac{4}{\theta^4} \quad b_2 = -\frac{\cos\theta + 2}{12\,(\cos\theta - 1)^2} + \frac{1}{\theta^4}$$

$$\mathcal{L}[y](t) = \frac{1}{3024} h^{10}\, \left(y^{(10)}(t) + \omega^2\, y^{(8)}(t)\right) + \mathcal{O}(h^{12})$$

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$$b_0 = \frac{\cos\theta}{\cos\theta - 1} + \frac{4\,(1-\cos\theta)}{\theta^4} \qquad b_1 = \frac{1}{2\,(1-\cos\theta)} + \frac{2\,(\cos\theta - 1)}{\theta^4}$$

$$\mathcal{L}[y](t) = -\frac{1}{720}\, h^8\left(y^{(8)}(t) + \omega^2\, y^{(6)}(t)\right) + \mathcal{O}(h^{10})$$

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$$P = 0: \ \mathcal{S} = \left\{\cos(\omega\, t),\, \sin(\omega\, t), 1,\, t,\, t^2,\, \ldots,\, t^{M-3}\right\}$$

case $M = 6$ and $b_1 = b_2 = 0$:

$$y_{p-2} - 4\, y_{p-1} + 6\, y_p - 4\, y_{p+1} + y_{p+2} = \frac{\sin^4(\theta/2)}{(\theta/2)^4}\, h^4\, y_p^{(4)}$$

$$\mathcal{L}[y](t) = \frac{1}{6}\, h^6\, (y^{(6)}(t) + \omega^2\, y^{(4)}(t)) + \mathcal{O}(h^8)$$

EF Central formula

$$\mathcal{L}[y] := y(t-2h) + a_1 y(t-h) + a_0 y(t) + a_1 y(t+h) + y(t+2h)$$
$$-h^4 \left(b_2 y^{(4)}(t-2h) + b_1 y^{(4)}(t-h) + b_0 y^{(4)}(t) + b_1 y^{(4)}(t+h) + b_2 y^{(4)}(t+2h)\right)$$

$$P = 1: \ \mathcal{S} = \left\{\cos(\omega t), \sin(\omega t), t\ \cos(\omega t), t\ \sin(\omega t), 1, t, t^2, \ldots, t^{M-5}\right\}$$

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$$\mathcal{L}[y] := y(t-2h) + a_1\, y(t-h) + a_0\, y(t) + a_1\, y(t+h) + y(t+2h)$$
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case $M = 6$ and $b_1 = b_2 = 0$:

$$y_{p-2} + a_1\, y_{p-1} + a_0\, y_p + a_1\, y_{p+1} + y_{p+2} = b_0\, h^4\, y_p^{(4)}$$

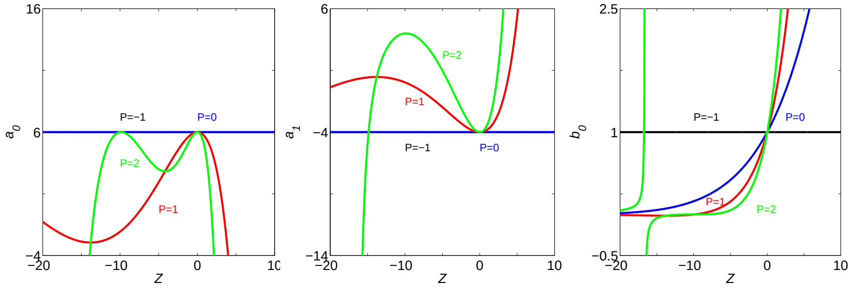
$$a_0 = 2\,\frac{-8\,\sin^2\theta + \theta\,(4\,\cos\theta - 1)\sin\theta - 4\,\cos\theta + 4}{\theta\,\sin\theta + 4\,\cos\theta - 4} \qquad a_1 = -4\,\frac{\sin\theta\,(\theta\,\cos\theta - 2\,\sin\theta)}{\theta\,\sin\theta + 4\,\cos\theta - 4}$$

$$b_0 = 4\,\frac{\sin\theta\left(\sin^2\theta - 2 + 2\,\cos\theta\right)}{\theta^3\,(\theta\,\sin\theta + 4\,\cos\theta - 4)}$$

Coefficients of central formula

e.g. case $M = 6$:

$$y_{p-2} + a_1\, y_{p-1} + a_0\, y_p + a_1\, y_{p+1} + y_{p+2} = b_0\, h^4\, y_p^{(4)}$$



$$Z = (\mu\, h)^2 = -(\omega\, h)^2$$

Coefficients of central formula

e.g. case $M = 6$

- $P = -1$:

$$b_0 = 1$$

- $P = 0$:

$$b_0 = 4\,\frac{(\cos\theta - 1)^2}{\theta^4}$$

- $P = 1$:

$$b_0 = -4\,\frac{\sin\theta\,(\cos\theta - 1)^2}{\theta^3\,(4\,\cos\theta - 4 + \theta\,\sin\theta)}$$

- $P = 2$:

$$b_0 = -2\,\frac{\sin^3\theta}{\theta^2\,(\theta\,\cos\theta - 3\,\sin\theta)}$$

Central formula : coefficients

e.g. case $M = 6$

- $P = -1$:

$$b_0 = 1$$

- $P = 0$:

$$b_0 = 1 - \frac{1}{6}\theta^2 + \frac{1}{80}\theta^4 + \mathcal{O}(\theta^6)$$

- $P = 1$:

$$b_0 = 1 - \frac{1}{3}\theta^2 + \frac{37}{720}\theta^4 + \mathcal{O}(\theta^6)$$

- $P = 2$:

$$b_0 = 1 - \frac{1}{2}\theta^2 + \frac{7}{60}\theta^4 + \mathcal{O}(\theta^6)$$

Central formula : local truncation error

$$\text{lte} = \mathcal{L}[y](t)$$

As an inifinite series :

$$\text{lte} = h^M\, C_M\, D^{K+1}\,(D^2+\omega^2)^{P+1} y(t) + \mathcal{O}(h^{M+2})$$

In closed form : (Coleman and Ixaru)

$$\text{lte} = h^M\, \Phi_{K,P}(Z)\, D^{K+1}\,(D^2+\omega^2)^{P+1} y(\xi)$$

$Z \in$ some interval $\quad \Phi_{K,P}(0) \neq 0 \quad \xi \in (t-2\,h, t+2\,h)$

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Local truncation error

$$\text{lte} = h^M\, C_M\, D^{K+1}\,(D^2+\omega^2)^{P+1} y(t) + \mathcal{O}(h^{M+2})\,,$$

$$\text{At } t_j : D^{(K+1)}\,(D^2+\omega_j^2)^{(P+1)} y(t)\Big|_{t=t_j} = 0 \qquad j = 2,\ldots,N-1$$

- $P = 0$:

$$y^{(K+3)}(t_j) + y^{(K+1)}(t_j)\,\omega_j^2 = 0$$

- $P = 1$:

$$y^{(K+5)}(t_j) + 2\,y^{(K+3)}(t_j)\,\omega_j^2 + y^{(K+1)}(t_j)\,\omega_j^4 = 0$$

- $P = 2$:

$$y^{(K+7)}(t_j)+3\,y^{(K+5)}(t_j)\,\omega_j^4+3\,y^{(K+3)}(t_j)\,\omega_j^4+y^{(K+1)}(t_j)\,\omega_j^6 = 0$$

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ω_j^2 is solution of equation of degree $P + 1$.

- Which value of P should be chosen ?
- Which root ω_j should be chosen ?

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Parameter selection

$$\text{lte} = h^M \, C_M \, D^{K+1} \, (D^2 - \mu^2)^{P+1} y(t) + \mathcal{O}(h^{M+2})$$

Suppose $y(t)$ takes the form $t^{P_0} \, e^{\mu_0 t}$

Then lte$= 0$ for any EF rule with $P \geq P_0$ and $\mu_j = \mu_0$

Theorem

If $y(t) = t^{P_0} \, e^{\mu_0 t}$ then $\nu = \mu_0^2$ is a root of multiplicity $P - P_0 + 1$ of $D^{K+1} \, (D^2 - \nu)^{P+1} y(t) = 0$.

- if $P = P_0$, then $\mu = \mu_0$ will be a single root
- if $P = P_0 + 1$, then $\mu = \mu_0$ will be a double root
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$$\text{lte} = h^M \, C_M \, D^{K+1} \, (D^2 - \mu^2)^{P+1} y(t) + \mathcal{O}(h^{M+2})$$

Suppose $y(t)$ does not take the form $t^{P_0} \, e^{\mu_0 \, t}$.

Then $y(t) \notin \mathcal{S}$ for any P.

For a given value of P :

$$D^{(K+1)} \, (D^2 - \mu_j^2)^{(P+1)} y(t) \Big|_{t=t_j} = 0$$

At each point t_j, this gives $P + 1$ values for μ_j^2.

Idea : keep $|\mu_j \, h|$ as small as possible.
If possible, choose $P \geq 1$ to avoid too large values for $|\mu_j|$.

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Numerical Illustrations

Problem 1

$$y^{(4)} - \frac{384\,t^4}{(2+t^2)^4}y = 24\,\frac{2-11\,t^2}{(2+t^2)^4}$$

$$y(-1) = \frac{1}{3} \qquad y(1) = \frac{1}{3}$$

$$y''(-1) = \frac{2}{27} \qquad y''(1) = \frac{2}{27}$$

$$\text{Solution : } y(t) = \frac{1}{2+t^2}$$

Since $y(t)$ does not to belong to the fitting space of a EF-rule, the parameter μ will not be constant over the interval of integration.

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Solution obtained by a fourth-order EF method

Computation of μ_j with $M = 8$:

$$P = 0 : y^{(8)}(t_j) - y^{(6)}(t_j)\,\mu_j^2 = 0$$

- re-express higher order derivatives in terms of y, y', y'' and y'''
- approximate y', y'' and y''' in terms of y
- an initial approximation for y can be computed with the classical, polynomial rule

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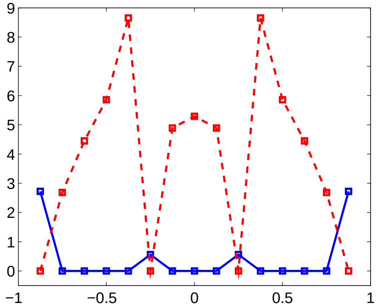
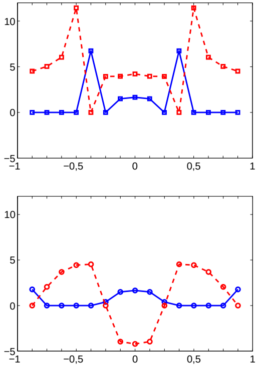


Real and imaginary part of μ_j

Solution obtained by a fourth-order EF method

Computation of μ_j with $M = 8$:

$$P = 1 : y^{(8)}(t_j) - 2\,y^{(6)}(t_j)\,\mu_j^2 + y^{(4)}(t_j)\,\mu_j^4 = 0$$

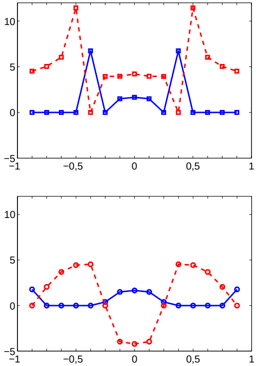


Real and imag. part of $\mu_{1,j}$ and $\mu_{2,j}$

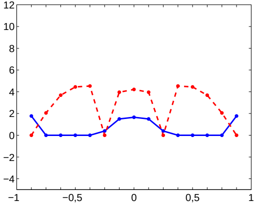
Solution obtained by a fourth-order EF method

Computation of μ_j with $M = 8$:

$$\color{red}{P = 1} : y^{(8)}(t_j) - 2\,y^{(6)}(t_j)\,\mu_j^2 + y^{(4)}(t_j)\,\mu_j^4 = 0$$



Real and imag. part of $\mu_{1,j}$ and $\mu_{2,j}$

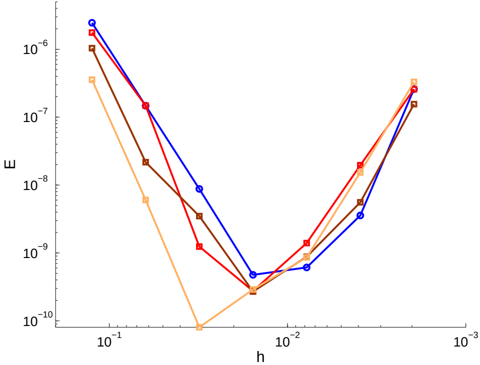


Real and imaginary part of μ_j with smallest norm

Solution obtained by a fourth-order EF method

Computation of μ_j with $M = 8$:

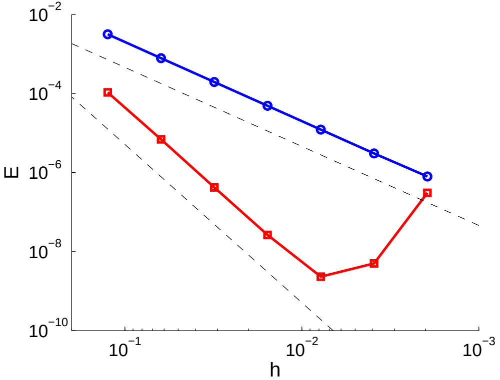
$$P = 1 : y^{(8)}(t_j) - 2\,y^{(6)}(t_j)\,\mu_j^2 + y^{(4)}(t_j)\,\mu_j^4 = 0$$



error obtained with $\mu_{1,j}$, $\mu_{2,j}$ and μ with smallest norm

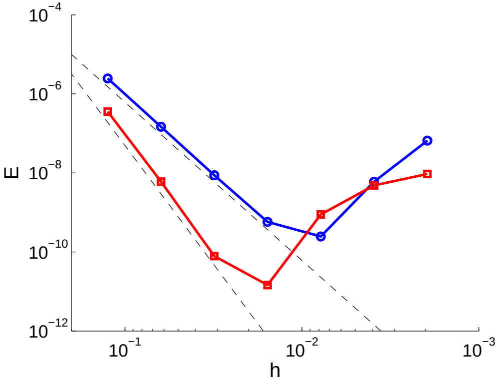
Global error

$$M = 6: \quad \begin{array}{ll} (K,P) = (5,-1) & : \text{second-order method} \\ (K,P) = (1,1) & : \text{fourth-order method} \end{array}$$



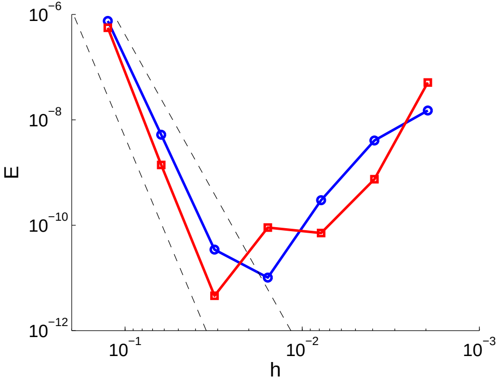
Global error

$$M = 8: \quad \begin{array}{ll} (K,P) = (7,-1) & : \text{fourth-order method} \\ (K,P) = (3,1) & : \text{sixth-order method} \end{array}$$

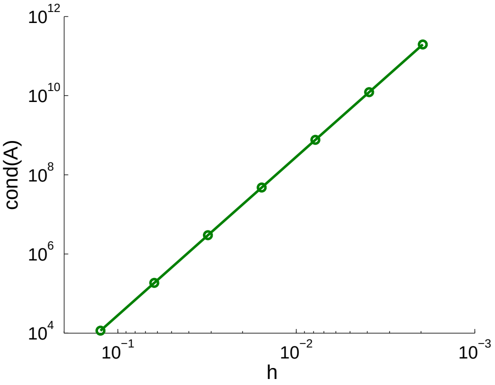


Global error

$$M = 10 : \begin{array}{ll} (K, P) = (9, -1) & : \text{sixth-order method} \\ (K, P) = (5, 1) & : \text{eighth-order method} \end{array}$$



Condition number of the coefficient matrix



Condition number of the coefficient matrix

The (classical) discretisation of $y_p^{(4)}$ gives rise to the coefficient matrix

$$\begin{pmatrix} 5 & -4 & 1 & & & & & \\ -4 & 6 & -4 & 1 & & & & \\ 1 & -4 & 6 & -4 & 1 & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & 1 & -4 & 6 & -4 & 1 \\ & & & 1 & -4 & 6 & -4 \\ & & & & 1 & -4 & 5 \end{pmatrix}$$

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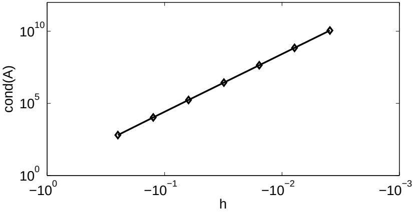

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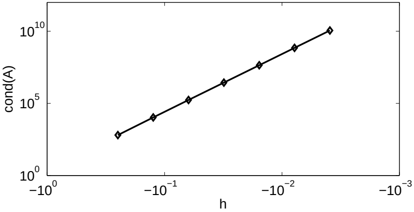
$$\begin{pmatrix} 5 & -4 & 1 & & & & \\ -4 & 6 & -4 & 1 & & & \\ 1 & -4 & 6 & -4 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & -4 & 6 & -4 & 1 \\ & & & 1 & -4 & 6 & -4 \\ & & & & 1 & -4 & 5 \end{pmatrix} = \begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{pmatrix}^2$$



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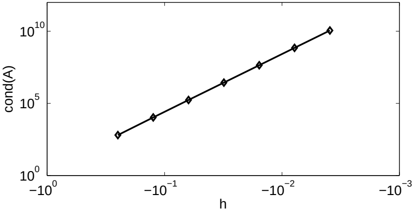

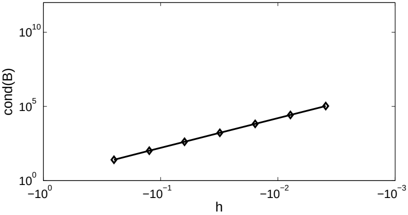

Factorisation of the coefficient matrix

$$\begin{pmatrix} c_2 & c_3 & 1 & & & & \\ a_1 & a_2 & a_1 & 1 & & & \\ 1 & a_1 & a_2 & a_1 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & a_1 & a_2 & a_1 & 1 \\ & & & 1 & a_1 & a_2 & a_1 \\ & & & & 1 & c_3 & c_2 \end{pmatrix} = \begin{pmatrix} a_2 - 1 & a_1 & 1 & & & & \\ a_1 & a_2 & a_1 & 1 & & & \\ 1 & a_1 & a_2 & a_1 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & a_1 & a_2 & a_1 & 1 \\ & & & 1 & a_1 & a_2 & a_1 \\ & & & & 1 & a_1 & a_2 - 1 \end{pmatrix}$$

Factorisation of the coefficient matrix

$$
\begin{pmatrix}
c_2 & c_3 & 1 & & & & \\
a_1 & a_2 & a_1 & 1 & & & \\
1 & a_1 & a_2 & a_1 & 1 & & \\
 & \ddots & \ddots & \ddots & \ddots & \ddots & \\
 & & 1 & a_1 & a_2 & a_1 & 1 \\
 & & & 1 & a_1 & a_2 & a_1 \\
 & & & & 1 & c_3 & c_2
\end{pmatrix}
=
\begin{pmatrix}
a_2-1 & a_1 & 1 & & & & \\
a_1 & a_2 & a_1 & 1 & & & \\
1 & a_1 & a_2 & a_1 & 1 & & \\
 & \ddots & \ddots & \ddots & \ddots & \ddots & \\
 & & 1 & a_1 & a_2 & a_1 & 1 \\
 & & & 1 & a_1 & a_2 & a_1 \\
 & & & & 1 & a_1 & a_2-1
\end{pmatrix}
$$

$$
=
\begin{pmatrix}
\alpha & 1 & & & \\
1 & \alpha & 1 & & \\
 & \ddots & \ddots & \ddots & \\
 & & 1 & \alpha & 1 \\
 & & & 1 & \alpha
\end{pmatrix}
\begin{pmatrix}
\beta & 1 & & & \\
1 & \beta & 1 & & \\
 & \ddots & \ddots & \ddots & \\
 & & 1 & \beta & 1 \\
 & & & 1 & \beta
\end{pmatrix}
$$

Factorisation of the coefficient matrix

$$\begin{pmatrix} c_2 & c_3 & 1 & & & & \\ a_1 & a_2 & a_1 & 1 & & & \\ 1 & a_1 & a_2 & a_1 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & a_1 & a_2 & a_1 & 1 \\ & & & 1 & a_1 & a_2 & a_1 \\ & & & & 1 & c_3 & c_2 \end{pmatrix} = \begin{pmatrix} a_2-1 & a_1 & 1 & & & & \\ a_1 & a_2 & a_1 & 1 & & & \\ 1 & a_1 & a_2 & a_1 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & a_1 & a_2 & a_1 & 1 \\ & & & 1 & a_1 & a_2 & a_1 \\ & & & & 1 & a_1 & a_2-1 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha & 1 & & & \\ 1 & \alpha & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & \alpha & 1 \\ & & & 1 & \alpha \end{pmatrix} \begin{pmatrix} \beta & 1 & & & \\ 1 & \beta & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & \beta & 1 \\ & & & 1 & \beta \end{pmatrix}$$

$M = 6:$

P	α	β
-1	-2	-2

Factorisation of the coefficient matrix

$$\begin{pmatrix} c_2 & c_3 & 1 & & & & \\ a_1 & a_2 & a_1 & 1 & & & \\ 1 & a_1 & a_2 & a_1 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & a_1 & a_2 & a_1 & 1 \\ & & & 1 & a_1 & a_2 & a_1 \\ & & & & 1 & c_3 & c_2 \end{pmatrix} = \begin{pmatrix} a_2-1 & a_1 & 1 & & & & \\ a_1 & a_2 & a_1 & 1 & & & \\ 1 & a_1 & a_2 & a_1 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & a_1 & a_2 & a_1 & 1 \\ & & & 1 & a_1 & a_2 & a_1 \\ & & & & 1 & a_1 & a_2-1 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha & 1 & & & \\ 1 & \alpha & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & \alpha & 1 \\ & & & 1 & \alpha \end{pmatrix} \begin{pmatrix} \beta & 1 & & & \\ 1 & \beta & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & \beta & 1 \\ & & & 1 & \beta \end{pmatrix}$$

$M = 6:$

P	α	β
-1	-2	-2
0	-2	-2

Factorisation of the coefficient matrix

$$
\begin{pmatrix}
c_2 & c_3 & 1 & & & & \\
a_1 & a_2 & a_1 & 1 & & & \\
1 & a_1 & a_2 & a_1 & 1 & & \\
 & \ddots & \ddots & \ddots & \ddots & \ddots & \\
 & & 1 & a_1 & a_2 & a_1 & 1 \\
 & & & 1 & a_1 & a_2 & a_1 \\
 & & & & 1 & c_3 & c_2
\end{pmatrix}
=
\begin{pmatrix}
a_2-1 & a_1 & 1 & & & & \\
a_1 & a_2 & a_1 & 1 & & & \\
1 & a_1 & a_2 & a_1 & 1 & & \\
 & \ddots & \ddots & \ddots & \ddots & \ddots & \\
 & & 1 & a_1 & a_2 & a_1 & 1 \\
 & & & 1 & a_1 & a_2 & a_1 \\
 & & & & 1 & a_1 & a_2-1
\end{pmatrix}
$$

$$
= \begin{pmatrix}
\alpha & 1 & & & \\
1 & \alpha & 1 & & \\
 & \ddots & \ddots & \ddots & \\
 & & 1 & \alpha & 1 \\
 & & & 1 & \alpha
\end{pmatrix}
\begin{pmatrix}
\beta & 1 & & & \\
1 & \beta & 1 & & \\
 & \ddots & \ddots & \ddots & \\
 & & 1 & \beta & 1 \\
 & & & 1 & \beta
\end{pmatrix}
$$

$M = 6:$

P	α	β
-1	-2	-2
0	-2	-2
1	$-2-4\,\frac{\sin\theta(2\cos\theta-2+\theta\sin\theta)}{4\sin\theta-\theta(1+\cos\theta)}$	-2

Problem 2

$$y^{(4)} - t = 4\,e^t$$

$$y(-1) = -1/e \qquad y(1) = e$$

$$y''(-1) = 1/e \qquad y''(1) = 3\,e$$

Solution : $y(t) = e^t\,t$

In theory, this problem is solved up to machine accuracy by any EF-rule with $P \geq 1$ and $\mu_j = 1$.

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$$P = 1 : y^{(6)}(t_j) - 2\,y^{(4)}(t_j)\,\mu_j^2 + y^{(2)}(t_j)\,\mu_j^4 = 0$$

differentiating the differential equation :

$$(y^{(2)}(t_j) + 4\,e^{t_j}) - 2\,(y_j + 4\,e^{t_j})\mu_j^2 + y^{(2)}(t_j)\,\mu_j^4 = 0$$

approximating $y^{(2)}(t_j)$ by $\mathcal{O}(h^2)$ finite difference scheme

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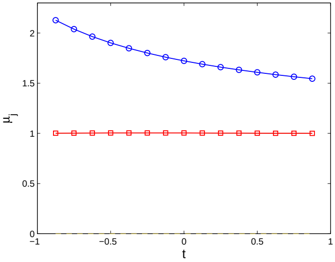
approximating $y^{(2)}(t_j)$ by $\mathcal{O}(h^2)$ finite difference scheme



two real roots $\mu^{(1)}$ and $\mu^{(2)}$

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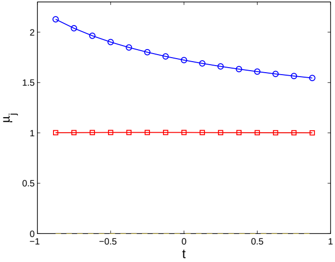

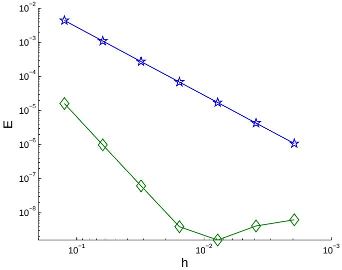

two real roots $\mu^{(1)}$ and $\mu^{(2)}$

How to improve the accuracy ?

$$y^{(0)} \xrightarrow{\mu^{(1)}} y^{(1)} \xrightarrow{\mu^{(2)}} y^{(2)} \ldots$$

- $y^{(0)}$ is obtained from classical, second order method
- compute $\mu^{(1)}$ from $y^{(0)}$ with classical second order schemes to approximate the derivatives that appear in the lte
- $y^{(1)}$ is obtained from EF method with $P = 1$ and $\mu = \mu^{(1)}$
- compute $\mu^{(2)}$ from $y^{(1)}$ with EF schemes ($P = 1$) to approximate the derivatives that appear in the lte
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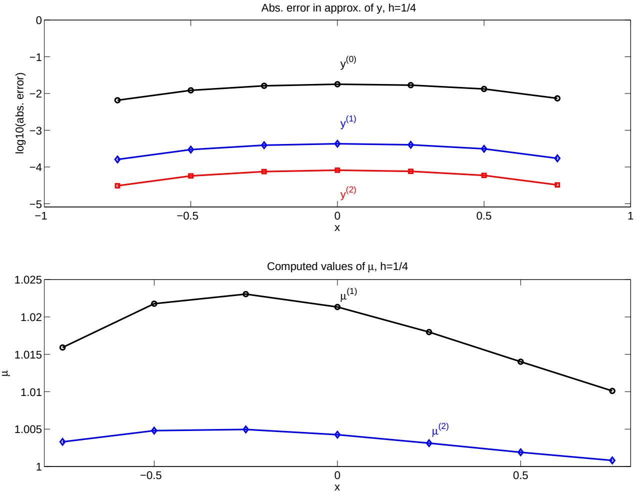
  $\ldots$

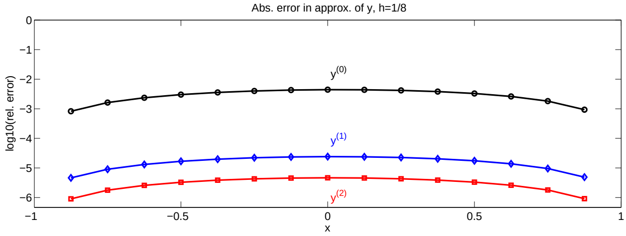
How to improve the accuracy ?

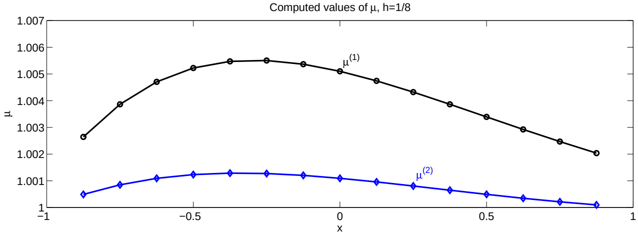
case $M = 6$, $h = 1/4$



How to improve the accuracy ?

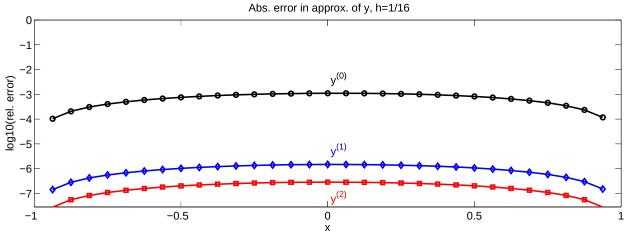
case $M = 6$, $h = 1/8$

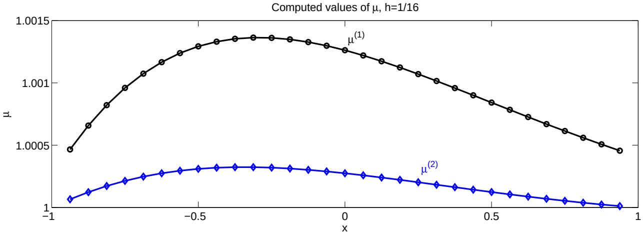




How to improve the accuracy ?

case $M = 6$, $h = 1/16$





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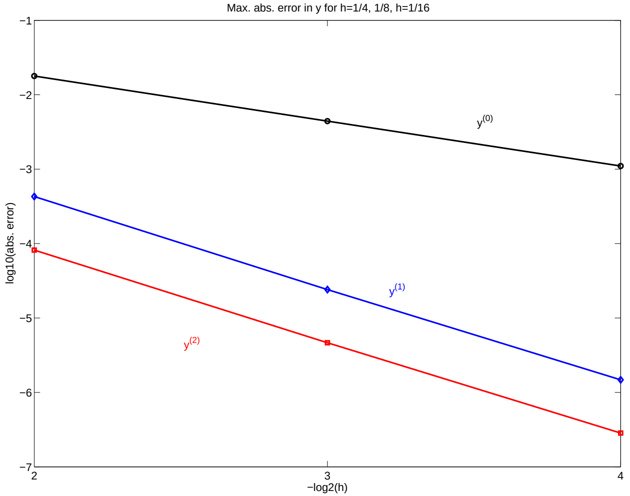
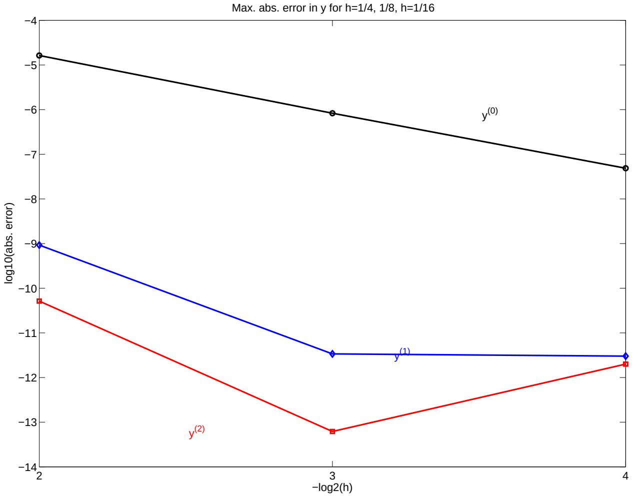
case $M = 6$



How to improve the accuracy ?

case $M = 8$



Conclusions

- Fourth-order boundary value problems are solved by means of parameterized exponentially-fitted methods.
- A suitable value for the parameter can be found from the roots of the leading term of the local truncation error.
- If a constant value is found, then a very accurate solution can be obtained.
- However, the methods strongly suffer from the fact that the system to be solved is ill-conditioned for small values of the mesh size.

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