On Orthonormal Müntz–Laguerre Filters

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Abstract—When the Müntz–Szász condition holds, the Müntz–Laguerre filters form a uniformly bounded orthonormal basis in Hardy space. This has consequences in terms of optimal pole-cancellation schemes, and it also allows for a generalization of Lerch’s theorem.

Index Terms—Completeness, filters, orthogonal functions.

I. INTRODUCTION

RECENTLY, an upsurge in research relating to Laguerre filters for use in system optimization [1], system identification [2], and reduced-order modeling [3] has been noticed. The main reason for the good performance of Laguerre filters is that they form a uniformly bounded orthonormal basis in Hardy space [1]. Another reason, and this represents the novelty of the work carried out in this paper, is that there is a fundamental link with the well-known Müntz–Szász theorem [4]–[8]. Here, we show that the ordinary Laguerre filters belong to the more general class of Müntz–Laguerre filters, which are closely related to the Müntz–Legendre quasipolynomials [7]. We prove that when the Müntz–Szász condition [6] holds, the Müntz–Laguerre filters form a uniformly bounded orthonormal basis in Hardy space. This has consequences in terms of optimal pole-cancellation schemes, and the results also imply a generalization of Lerch’s theorem [9].

II. MAIN RESULT

We work in the Hilbert space $L_2(R_+)$ of square integrable functions over $[0, \infty)$ with scalar product

$$\langle f|g \rangle = \int_0^\infty f(t)g^*(t) \, dt.$$  

A closely associated Hilbert space is the Hardy space $H_2$ [1] consisting of all analytic and square-integrable functions in the open right half plane $\Re s > 0$ with scalar product

$$\langle F|G \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(i\omega)G^*(i\omega) \, d\omega.$$  

Simply speaking, we can consider the Hardy space $H_2$ to be the Laplace transform of $L_2(R_+)$. It is known [6] that the completeness of a set of exponentials $\{e^{\beta_n t}\}$, $\Re (\beta_n) > 0$ in $L_2(R_+)$ is equivalent with the Müntz–Szász condition

$$\sum_{n=0}^{\infty} \frac{\Re (\beta_n)}{1 + |\beta_n|^2} = \infty$$  

where $\Re (b)$ stands for the real part of $b$.

In Hardy space $H_2$, the Müntz–Szász condition (3) naturally governs the completeness of the pole system $\{1/(s + \beta_n)\}$. This pole system can be orthonormalized to yield a complete orthonormal basis (COB). We thus have the following theorem.

Theorem 1: Let $\{\beta_0, \beta_1, \beta_2, \ldots\}$ be a complex sequence with $\Re (\beta_n) \geq \epsilon > 0$ such that the condition (3) holds. Then, the Müntz–Laguerre filters

$$\Phi_n(s) = \frac{\sqrt{2\Re (\beta_n)}}{s + \beta_n} \prod_{k=0}^{n-1} \left( \frac{s - b_k^*}{s + b_k^*} \right) \quad n = 0, 1, 2, \ldots$$  

form a uniformly bounded orthonormal basis for the Hardy space $H_2$. Note that

$$\Phi_0(s) = \frac{\sqrt{2\Re (\lambda_0)}}{s + \lambda_0}. $$  

Proof: Let $\{\lambda_0, \lambda_1, \lambda_2, \ldots\}$ be a complex sequence with $\Re (\lambda_n) > -1/2$. We define the $n$th Müntz–Legendre quasipolynomial [7] on $(0, 1]$ as

$$M_n(x) = \frac{\sqrt{1 + 2\Re (\lambda_n)}}{2\pi i} \int_{\Gamma} \prod_{k=0}^{n-1} \left( \frac{z + \lambda_k^* + 1}{z - \lambda_k} \right) \frac{z^2 \, dz}{z - \lambda_n}$$  

where the simple contour $\Gamma$ surrounds all the poles of the integrand. In [8], it is proved that the $M_n(x)$ are orthonormal in $L_2(0, 1]$, i.e.,

$$\int_0^1 M_n(x)M_{m}^*(x) \, dx = \delta_{n,m} \quad n, m = 0, 1, 2, \ldots.$$  

With the change of variable $x = e^{-u}$, (7) can be written as

$$\int_0^\infty M_n(e^{-u})M_{m}^*(e^{-u})e^{-u} \, du = \delta_{n,m} \quad n, m = 0, 1, 2, \ldots.$$  

Hence, the set of functions $\{\phi_n(t) = e^{-t/2}M_n(e^{-t})\}$ is orthonormal over $L_2(R_+)$. Taking $\beta_n = \lambda_n + 1/2$ and $u = z + 1/2$, we can write

$$\phi_n(t) = M_n(e^{-t})e^{-t/2}$$  

$$= \frac{\sqrt{2\Re (\beta_n)}}{2\pi i} \int_{\Gamma'} \prod_{k=0}^{n-1} \frac{s + b_k^*}{s - b_k^*} \frac{e^{-s t} \, ds}{s - \beta_n}$$  

where $\Gamma'$ surrounds all the poles of the integrand. This contour can be deformed to a semicircle at infinity in the right halfplane, with no contribution to the integral, plus the imaginary axis in the sense $+i\infty \rightarrow -i\infty$. Taking the Laplace transform of (9) yields

$$\Phi_n(s) = \frac{\sqrt{2\Re (\beta_n)}}{2\pi i} \int_{-i\infty}^{i\infty} \prod_{k=0}^{n-1} \frac{u + b_k^*}{u - b_k^*} \left( u - \beta_n \right) (u + s) \, du.$$  

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Transforming the integral over the imaginary axis to a closed contour in the left half-plane by adding a semicircle at infinity, again with no contribution to the integral, invoking Cauchy’s theorem and the Müntz–Szász condition establishes the fact that \( \{\Phi_n(s)\} \) is an orthonormal basis for \( \mathcal{H}_2 \). The uniform boundedness follows from the fact that
\[
|\Phi_n(s)| \leq \sqrt{2/\Re(b_n)} \leq \sqrt{2/\epsilon} \quad \Re(s) \geq 0. \tag{11}
\]
Note that if the \( b_n \) are all distinct, then \( \phi_n(t) \) can be written as
\[
\phi_n(t) = \sum_{k=0}^{n} C_{n,k} e^{-b_k t} \tag{12}
\]
This follows from the residue theorem applied to (9).

**Corollary 1:** Given a set of distinct complex exponents \( \{b_0, b_1, \cdots, b_N\} \) and residues \( \{d_0, d_1, \cdots, d_N\} \), consider the minimization problem
\[
J = \min_{\{d_n\}} \int_0^\infty \left| \sum_{n=0}^N d_n e^{-b_n t} - \sum_{n=0}^M \hat{d}_n e^{-b_n t} \right|^2 dt \tag{13}
\]
with respect to \( \{\hat{d}_0, \hat{d}_1, \cdots, \hat{d}_M\} \), with \( M \leq N \). The solution is
\[
\sum_{n=0}^M \hat{d}_n e^{-b_n t} = \sum_{n=0}^M D_n \phi_n(t) \tag{14}
\]
where
\[
D_n = \sum_{k=0}^N d_k (\Phi_n(b_k))^* \quad n = 0, 1, \cdots, N. \tag{15}
\]
**Proof:** Clearly, if
\[
\sum_{n=0}^N d_n e^{-b_n t} = \sum_{n=0}^M D_n \phi_n(t) \tag{16}
\]
then (15) is valid. This follows from the orthonormality relation (8) and the fact that \( \Phi_n(s) \) is the Laplace transform of \( \phi_n(t) \). In the same vein, we can write
\[
\sum_{n=0}^M \hat{d}_n e^{-b_n t} = \sum_{n=0}^M \hat{D}_n \phi_n(t). \tag{17}
\]
By orthonormality, the minimization problem then becomes
\[
J = \min_{D_n} \left\{ \sum_{n=0}^M |\hat{D}_n - D_n|^2 + \sum_{n=M+1}^N |D_n|^2 \right\} \tag{18}
\]
and the result follows. Note that
\[
J_{\min} = \sum_{n=M+1}^N |D_n|^2. \tag{19}
\]
This also has consequences at the filtering level. Suppose we have a filter \( H(s) \) in \( \mathcal{H}_2 \) with unwanted (for some reason) poles \( \{b_{M+1}, b_{M+2}, \cdots, b_N\} \), which we want to remove, while approximately maintaining the overall performance of \( H(s) \). The optimal way to do this is to expand \( H(s) \) in the Müntz–Laguerre filter basis associated with its own poles as
\[
H(s) = \sum_{n=0}^N D_n \Phi_n(s) \tag{20}
\]
and to retain only the Fourier segment
\[
\hat{H}_{\text{opt}}(s) = \sum_{n=0}^M D_n \Phi_n(s). \tag{21}
\]
This optimal pole-cancellation scheme can be considered to be a form of reduced-order modeling [3].

**Corollary 2:** Suppose that \( F(s) \) in \( \mathcal{H}_2 \) is such that
\[
F(b_n) = 0 \quad n = 0, 1, 2, \cdots, \infty \tag{22}
\]
on a set of distinct complex numbers \( \{b_n\} \) with \( \Re(b_n) \geq \epsilon > 0 \), such that the Müntz–Szász condition (3) holds. Then, \( F(s) = 0 \).

**Proof:** Since the Müntz–Szász condition is invariant under complex conjugation, we can consider \( F(b_n^*) = 0 \) and expand \( F(s) \) as
\[
F(s) = \sum_{n=0}^\infty c_n \Phi_n(s). \tag{23}
\]
Since \( \Phi_n(b_k^*) = 0 \) for \( n > k \), we have
\[
0 = F(b_k^*) = \sum_{n=0}^k c_n \Phi_n(b_k^*) \quad k = 0, 1, \cdots, \infty. \tag{24}
\]
Since all the points \( b_k \) are distinct and since the Müntz–Laguerre basis is uniformly bounded, we have that \( \Phi_n(b_k^*) \neq 0 \) and is bounded. It follows that the only solution to the infinite triangular linear system (24) is \( c_n = 0 \) for \( n = 0, 1, 2, \cdots \). Hence, \( F(s) = 0 \), and the proof is complete.

Corollary 2 is in fact a generalization of the theorem of Lerch [9, p. 208], which utilizes the set \( \{b_n = s_0 + n\lambda\} \) with \( \lambda \) real and positive.

It should be noted by inspection of (4) that all Müntz–Laguerre filters have the same form, i.e., a simple one-pole low-pass filter multiplied by a product of allpass filters.

An illustrative example of Müntz–Laguerre filters arises when we take \( b_n = \alpha \), with \( \alpha \) real and positive. The corresponding orthonormal basis in \( \mathcal{H}_2 \) is
\[
\Phi_n(s) = \frac{\sqrt{2\alpha}}{s+\alpha} \left( \frac{s-\alpha}{s+\alpha} \right)^n \quad n = 0, 1, \cdots. \tag{25}
\]
In \( L_2(R_+) \), this yields the orthonormal basis [10]
\[
\phi_n(t) = \sqrt{2\alpha} e^{-\alpha t} \ell_n(2\alpha t) \quad n = 0, 1, \cdots \tag{26}
\]
where \( \ell_n(t) \) is the Laguerre polynomial
\[
\ell_n(t) = \frac{e^t}{n!} \frac{d^n}{dt^n} (e^{-t} t^n). \tag{27}
\]
This is the rationale for calling a basis of the general type (4) a Müntz–Laguerre basis.

### III. DISCUSSION

Müntz–Laguerre bases are also known as Kautz [11], [12] bases since they can be obtained by first Gram–Schmidt orthonormalizing a set of exponentials and afterwards taking the
Laplace transform. The main novelty of our approach is that we start with Müntz–Legendre quasipolynomials [7], [8] that are orthonormal in \( L_2[0, 1] \)—a mere interval of the real line—and end up directly in Hardy space \( \mathcal{H}_2 \) with orthonormality over the imaginary axis. It is important to note that the results for the open right halfplane can be extended to the open unit disk and, in fact, to any open simply connected domain that is not the entire open complex plane. This has been proved in [13] by making use of the Riemann mapping theorem. Since the materials presented in the SPS2000 symposium can be hard to come by, it has been added here as Theorem 2 of the Appendix. We have worked in \( L_2 \) and \( \mathcal{H}_2 \) since the natural place for COBs is Hilbert space, but it can happen that a COB is also complete in other Banach spaces. For example, the Müntz–Szász condition also governs the completeness of a Kautz system of exponentials in \( L_p \)-spaces [6]. Recently, similar results with respect to completeness in \( L_p \) and \( \mathcal{H}_p \) spaces have been presented in [14]. It is puzzling, however, that the publications concerning COBs in the control community rarely mention the names of Müntz and Szász.

**APPENDIX**

The notation is as follows: If \( z \) is a complex number, then \( \overline{z} \), \( \Re z \), \( \Im z \), and \( \arg z \), respectively, stand for the complex conjugate, the real part, the imaginary part, and the argument of \( z \). \( U \) is the open unit disk \( |z| < 1 \), and \( \mathcal{Y} \) is the unit circle \( |z| = 1 \). In what follows, we suppose that \( \Omega \) is a simply connected open domain in the complex plane closed by a Jordan curve \( \Gamma \) [9]. The Hardy space [15] of functions analytic inside \( \Omega \) with \( \mathcal{H}_2 \)-extension on \( \Gamma \) with scalar product

\[
\langle u|v \rangle = \frac{1}{2\pi} \int_{\Gamma} u(z) \overline{v(z)} |dz|
\]

is denoted \( \mathcal{H}_2(\Omega) \). The well-known Riemann mapping theorem [9] states that there is a unique conformal mapping \( h_\Omega(z) \) of \( \Omega \) onto the unit disk \( U \) such that \( h_\Omega(a) = 0 \) and \( \Im h_\Omega'(a) = 0 \). \( \Re h_\Omega'(a) > 0 \)—in short, \( h_\Omega'(a) > 0 \)—for any point \( a \) of \( \Omega \), provided \( \Omega \) is not the whole complex plane. Moreover, by the Osgood–Carathéodory theorem [9], these mappings can be extended to the boundary in the sense that \( z \in \Gamma \) implies \( h_\Omega(z) \in \mathcal{Y} \) for all \( \Omega \). Note that this means that \( |h_\Omega(z)| = 1 \) for \( z \in \Gamma \). To obtain the Riemann mappings \( h_\Omega(z) \) for all \( \Omega \), it is sufficient to have one of them, say, the Riemann mapping associated with \( a = a_0 \). This follows from the following lemma.

**Lemma 1:** Let \( h(z) \) be the Riemann mapping associated with some point \( a_0 \) in \( \Omega \). Then, the Riemann mapping associated with any \( a \in \Omega \) is given by

\[
h_a(z) = e^{-i\arg h'(a)} \frac{h(z) - h(a)}{1 - h(a)h(z)}.
\]

**Proof:** It is known [9] that the Riemann mappings associated with the unit disk \( U \) are the Möbius mappings

\[
m_b(z) = \frac{z - b}{1 - b\overline{z}}, \quad |b| < 1.
\]

Hence, the Riemann mapping for \( \Omega \) associated with \( a \) must have the form

\[
h_a(z) = e^{i\arg h'(a)} \frac{h(z) - h(a)}{1 - h(a)h(z)}.
\]

The requirements \( h_a(a) = 0 \) and \( h_a'(a) > 0 \) then yield the result (A2).

An important function derived from the Riemann mapping \( h_a(z) \) is the square root of its derivative, i.e., \( r_a(z) = \sqrt{h'_a(z)} \), where the root with \( r_a'(a) > 0 \) is chosen. The importance of the function \( r_a(z) \) is a consequence of the reproducing property in the following lemma.

**Lemma 2:** Let \( v(z) \in \mathcal{H}_2(\Omega) \). Then

\[
\langle v|h'_a \rangle = v(a)/r_a(a),
\]

**Proof:** See [9].

Lemma 2 enables us to state the main result.

**Theorem 2:** Let \( a_k, k = 0, 1, \cdots \) be a sequence of complex numbers in \( \Omega \). Then, the functions

\[
\Phi_n(z) = r_{a_0}(z)
\]

and

\[
\Phi_n(z) = r_{a_n}(z) \prod_{k=0}^{n-1} h_{a_k}(z) \quad n > 0
\]

are orthonormal in \( \mathcal{H}_2(\Omega) \).

**Proof:** Since \( \langle \Phi_n | \Phi_m \rangle = \langle \Phi_m | \Phi_n \rangle \), we can consider \( n \geq m \). When \( n = m \), we have

\[
\langle \Phi_n | \Phi_n \rangle = \langle r_{a_n} | r_{a_n} \rangle = 1
\]

by Lemma 2 and the fact that \( |h_{a_n}(z)| = 1 \) on the boundary \( \Gamma \). When \( n > m \), we have

\[
\langle \Phi_n | \Phi_n \rangle = \left\langle r_{a_n} \prod_{k=m}^{n-1} h_{a_k}(z) | r_{a_m} \right\rangle
\]

\[
= \left( r_{a_n}(a_m) \prod_{k=m}^{n-1} h_{a_k}(a_m) \right) / r_{a_m}(a_m) = 0
\]

by Lemma 2 and the fact that \( h_{a_m}(a_m) = 0 \).

**Corollary 1:** Let \( \Omega = U \) (the open unit disk). Then, the functions defined in (A6) and (A7) form a complete orthonormal basis in \( \mathcal{H}_2(U) \) if and only if

\[
\sum_{n=0}^{\infty} (1 - |a_n|) = \infty.
\]

**Proof:** For the unit circle, we have \( h_a(z) = m_a(z) \) and \( r_a(z) = \sqrt{1 - |a|^2/(1 - \overline{a}z)} \). Hence, the functions defined in (A6) and (A7) correspond with the well-known Takenaka–Walsh orthonormal basis on the unit circle [16], [17]. The condition (A10) is known as the Szász condition [18].

**Corollary 2:** Let \( \Omega = \mathcal{R} \) (the open right halfplane \( \Re z > 0 \)). Then, the functions defined in (A6) and (A7) form a complete orthonormal basis in \( \mathcal{H}_2(\mathcal{R}) \) if and only if

\[
\sum_{n=0}^{\infty} \left( 1 - \frac{a_n - 1}{a_n + 1} \right) = \infty.
\]
Proof: The bilinear mapping \((z - 1)/(z + 1)\) is the Riemann mapping for \(\mathcal{R}\) associated with \(z = 1\). By Lemma 1, we readily obtain

\[
h_n(z) = \frac{z - \alpha_n}{z + \alpha_n}, \quad r_n(z) = \frac{\sqrt{2\Re \alpha_n}}{z + \alpha_n} \quad \Re \alpha_n > 0. \quad (A12)
\]

Hence, the functions defined in (A6) and (A7) correspond with the Kautz orthonormal basis [11], which is complete if and only if the condition

\[
\sum_{n=0}^{\infty} \frac{\Re(\alpha_n)}{1 + |\alpha_n - 1/2|^2} = \infty. \quad (A13)
\]

is valid [19]. In [20], it is shown that the formulations (A13) and (A11) are equivalent.

Corollary 3: Let \(h(z)\) be the Riemann mapping associated with some point \(\alpha_0\) in \(\Omega\). Then, the functions defined in (A6) and (A7) form a complete orthonormal basis in \(\mathcal{H}_2(\Omega)\) if and only if

\[
\sum_{n=0}^{\infty} (1 - \|h(\alpha_n)\|) = \infty. \quad (A14)
\]

Proof: Let the sequence \(\{\Phi_n(z)\}\) be the orthonormal basis of Corollary 1. The orthonormality can be expressed as

\[
\frac{1}{2\pi} \int_{\Gamma} \Phi_n(z)\overline{\Phi_m(z)}|dz| = \delta_{n,m}. \quad (A15)
\]

With the change of variable \(z = h(w)\), this can be written as

\[
\frac{1}{2\pi} \int_{\Gamma} \Phi_n(h(w))\overline{\Phi_m(h(w))}|h'(w)|\,|dw| = \delta_{n,m} \quad (A16)
\]

implying that the sequence \(\{\Phi_n(h(z))\sqrt{h'(z)}\}\) is orthonormal in \(\mathcal{H}_2(\Omega)\). By Corollary 1 and Lemma 1, it is seen that except for additional irrelevant phase factors \(e^{i\epsilon_n}\), these are precisely the orthonormal functions defined in (A6) and (A7). Hence, the basis is complete in \(\mathcal{H}_2(\Omega)\) if and only if the points \(\{h(\alpha_n)\}\) in \(U\) satisfy the Szász condition (A10).

REFERENCES


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