ON THE COMPLETENESS OF EIGENMODES IN A PARALLEL PLATE WAVEGUIDE WITH A PERFECTLY MATCHED LAYER TERMINATION

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Abstract — An explicit proof of the completeness of the eigenmodes of a grounded parallel plate waveguide with a perfectly matched layer termination is given. The proof is based on a general theorem governing the completeness of sets of complex exponentials. The Green’s function of the structure is then obtained by means of a biorthogonalization procedure. A comparison with the Green’s function of a grounded halfspace indicates that a perfectly matched layer termination simulates the open halfspace efficiently.

Index Terms — Perfectly matched layer, parallel plate waveguide, Green’s function, eigenmodes, completeness.

1 INTRODUCTION

In [1] Berenger proposed the perfectly matched layer (PML) to truncate computational domains for use in the numerical solution of Maxwell’s equations, without introducing reflections. The original split-field approach of Berenger was reformulated by Chew and Weedon [2] in terms of complex coordinate stretching and by Sacks et al. [3] in terms of perfectly matched anisotropic absorbers. It was shown by the authors [4] that the complex coordinate stretching and diagonal anisotropy formulations are equivalent in a general orthogonal coordinate system setting.

Recently a new method [5] was proposed to derive an eigenmode series expansion for the two-dimensional Green’s function of a planar substrate, by using a PML to turn the originally open configuration into a closed parallel plate waveguiding system. However, the proposed technique

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presents conceptual and theoretical difficulties, mainly due to the fact that complex stretching results in losing the self-adjoint nature of the underlying Sturm-Liouville problem [6].

In this paper we give an explicit proof of the completeness of the eigenmodes of a grounded parallel plate waveguide with a PML termination. The proof is based on a general theorem governing the completeness of sets of complex exponentials [7]. The resulting Green’s function can then be obtained by utilizing a biorthogonal expansion for the Dirac distribution. Comparison with the Green’s function of a grounded halfspace demonstrates that the PML termination judiciously simulates the open halfspace.

2 PARALLEL PLATE WAVEGUIDE WITH PML TERMINATION

Consider the Helmholtz equation governing the determination of the scalar Green’s function in a homogeneous parallel plate waveguide with propagation in the z-direction. We have

\[
\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial z^2} + k^2 G = \delta(x - x')\delta(z - z') \quad 0 \leq x, x' \leq a \quad -\infty \leq z, z' \leq \infty
\]  

(1)

The Green’s function \(G(x, x'; z, z')\) is of the form \(G(x, x'; z - z')\) and hence with the Fourier transform

\[
\tilde{G}(x, x'; \gamma) = \int_{-\infty}^{\infty} e^{i\gamma(z - z')} G(x, x'; z - z') dz
\]

(2)

the defining equation (1) becomes

\[
\frac{\partial^2 \tilde{G}}{\partial x^2} + \beta^2 \tilde{G} = \delta(x - x') \quad 0 \leq x, x' \leq a
\]

(3)

with \(\beta = \sqrt{k^2 - \gamma^2}\). Given appropriate boundary conditions, the Green’s function in the Fourier domain can be readily obtained. For a parallel plate waveguide with Dirichlet boundary conditions at \(x = 0\) and \(x = a\), the Green’s function in the Fourier domain is

\[
\tilde{G}_p(x, x'; \gamma) = \frac{\sin \beta x_\gamma \sin \beta (x_\gamma - a)}{\beta \sin \beta a}
\]

(4)

with \(x_\gamma = \min(x, x')\), \(x_\gamma = \max(x, x')\). Hence

\[
G_p(x, x'; z - z') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\gamma(z - z')} \tilde{G}_p(x, x'; \gamma) d\gamma
\]

(5)
Unfortunately, the Fourier integral (5) is not known in closed form. Nevertheless the Green’s function can be obtained by means of an eigenmode expansion. Since the eigenmode sequence \( \{\sin(n\pi x/a)\}, \quad n = 1, 2, \ldots \) forms a complete orthogonal basis in \( L_2[0,a] \) we may write

\[
\tilde{G}_p(x, x'; \gamma) = \frac{2}{a} \sum_{n=1}^{\infty} \frac{\sin(n\pi x/a) \sin(n\pi x'/a)}{\beta^2 - (n\pi/a)^2}
\]  

(6)

yielding

\[
G_p(x, x'; z - z') = i \sum_{n=1}^{\infty} \frac{\sin(n\pi x/a) \sin(n\pi x'/a) e^{-i\sqrt{k^2 - (n\pi/a)^2}|z - z'|}}{\sqrt{k^2 - (n\pi/a)^2}}
\]  

(7)

Now if, while keeping the Dirichlet condition at \( x = 0 \), we add a perfectly matched layer in the interval \([a, d]\), \( d > a \) and a Dirichlet condition at \( x = d \), the equations in the Fourier domain become

\[
\frac{\partial^2 \tilde{G}}{\partial X^2} + \beta^2 \tilde{G} = \delta(x - x') \quad 0 \leq x, x' \leq a 
\]  

(8)

\[
\frac{\partial^2 \tilde{G}}{\partial X^2} + \beta^2 \tilde{G} = 0 \quad a \leq x \leq d 
\]  

(9)

where \( X(x) \) is a stretched PML coordinate [4] satisfying

\[
X(x) = x \quad \text{for } 0 \leq x \leq a, \quad X'(a+) = 1, \quad X''(a+) = 0
\]  

(10)

and which is complex and twice continuously differentiable in \([a, d]\). It is easy to show that the eigenmode sequence is now \( \{\sin(n\pi x/b)\}, \quad n = 1, 2, \ldots , \) with \( b = X(d) \). It would therefore seem that formulas (6) and (7) remain valid by replacing \( a \) with \( b \), but nothing is less true. The point is that, due to the complex stretching, the equations (8) and (9) taken together do not form a self-adjoint Sturm-Liouville problem [6], which implies that completeness and orthogonality are far from guaranteed. However in our case we have the following

**Theorem :** The sequences \( \{\sin(n\pi x/b)\}, \quad n = 1, 2, \ldots \) and \( \{\cos(n\pi x/b)\}, \quad n = 0, 1, \ldots \) with \( b \) complex are complete over \( L_p[0,a] \) provided \( a \leq |b| \).

**Proof :** By Theorem 2 of the Appendix the sequence \( \{e^{n\pi i x/b}\} \) is complete over \( L_p[-a,a] \) if \( a \leq |b| \). Now the even and odd functions in \( L_p[-a,a] \) are clearly spanned by the sequences \( \{\cos(n\pi x/b)\}, \quad n = 0, 1, \ldots \) and \( \{\sin(n\pi x/b)\}, \quad n = 1, 2, \ldots \) respectively. Also, any function in \( L_p[0,a] \) has a unique even and odd extension in \( L_p[-a,a] \), and this completes the proof.
Next even when completeness is guaranteed, it is not possible, in the Hilbert space case $p = 2$, to write
\[
\delta(x - x') = \frac{2}{b} \sum_{n=1}^{\infty} \sin(n\pi x/b) \sin(n\pi x'/b) \quad 0 \leq x, x' \leq a
\] (11)
since the sequence \( \{ \sqrt{2/b} \sin(n\pi x/b) \} \), \( n = 1, 2, \ldots \) does not form a complete orthonormal basis for the Hilbert space \( L_2[0, a] \) with scalar product \( \langle f | g \rangle = \int_{0}^{a} f(x)g(x)dx \). However, given a complete sequence \( \{ \phi_n(x) \} \) we can find the biorthogonal sequence \( \{ \psi_n(x) \} \) by the rule \( \langle \phi_n | \psi_m \rangle = \delta_{n,m} \).

The reproducing kernel
\[
K(x, x') = \sum_n \phi_n(x)\overline{\psi_n(x')}
\] (12)
exhibits the sieve property \( \int_{0}^{a} K(x, x')\phi_n(x')dx' = \phi_n(x) \), which makes it a representation of the Dirac distribution \( \delta(x - x') \) with respect to this complete set. Hence if we take \( \phi_n(x) = \{ \sin(n\pi x/b) \} \), \( n = 1, 2, \ldots \), we have
\[
G_{pml}(X(x), x'; z - z') = \frac{i}{2} \sum_{n=1}^{\infty} \phi_n(X(x))\overline{\psi_n(x')}e^{-i\sqrt{k^2 - (n\pi/b)^2}|z - z'|} (13)
\]
It is easily seen that the PML Green’s function satisfies the Dirichlet boundary conditions at \( x = 0 \) and \( x = d \). Note that besides the condition for completeness \( a \leq |b| \) we must also require \( \Re(b)\Im(b) < 0 \) in order to obtain decaying exponentials for large values of \( n \) in (13) The relation between the biorthogonal sequences is easy to find: if we restrict ourselves to the finite segment \( \phi_n(x), n = 1, 2, \ldots, N \), we can express the \( \psi_n \) as functions of \( \phi_n \) as
\[
\psi_n(x) = \sum_{k=1}^{N} A_{n,k}\phi_k(x)
\] (14)
and it is an easy matter to show that the matrix \( A = H^{-1} \), the inverse of the Grammian \( H \) with entries \( H_{n,m} = \langle \phi_n | \phi_m \rangle \). Note that the Grammian \( H \) is ill-conditioned in general, and it is therefore better, numerically speaking, to take \( A = H^\dagger \), the Moore-Penrose inverse \cite{8}.

**REMARK**: An interesting point is that, since we have \( a < d \), where \( d \) is the real thickness of the PML waveguide, and since we need \( a \leq |b| \), where \( |b| \) is the modulus of the complex thickness \( b \) of the PML waveguide, we can take \( |b| = d \) and \( a = d(1 - \epsilon) \), where \( \epsilon \) may be an arbitrarily small positive number. This can be understood from the following argument:

Let \( X(x) = x + \eta(x - a)^3 \) in \([a, d]\), with \( \eta \) complex. It is seen that \( X(x) \) satisfies all the required continuity conditions. We have \( b = X(d) \) and if we put \( d = |b| \) and \( a = |b|(1 - \epsilon) \) we obtain the
following expression for \( \eta \) as a function of \( b \) and \( \epsilon \):

\[
\eta = \frac{b - |b|}{|b|^3 \epsilon^3} \quad (15)
\]

In this way we can make the PML layer as thin as we want except nil, without changing its properties. Of course, as we will see in the next section, it is compulsory to take \( \Im b < 0 \) and \( \Re b > a/2 \). In practice we obtain good results with the formula\( b = d(\sqrt{1 - \tau^2} - i\tau) \), where \( \tau \) is a small positive number. Also, when \( a = |b|(1 - \epsilon) \), with \( \epsilon \approx 0 \), and \( |b - d| = |b - |b|| \) is sufficiently small, we may expect that

\[
\psi_n(x) \approx 2 \frac{\sin(n\pi x/b)}{b} = 2 \frac{\phi_n(x)}{b} 
\]

which follows from the “real” orthonormality relations

\[
2 \int_0^b \frac{\sin(n\pi x/b)}{b} \sin(m\pi x/b) \, dx = \delta_{n,m} \quad (17)
\]

In Figure 1 and Figure 2 we respectively plotted the magnitudes and arguments of \( \psi_n(x) \) and \( 2 \phi_n(x)/b \) for \( d = 1 \), \( \tau = 0.01 \), \( N = 40 \). This indicates that the approach of [5], where the sequence \( \{2\phi_n(x)/b\} \) instead of the exact biorthogonal sequence \( \{\psi_n(x)\} \) was utilized, still can be used as a first approximation, without having to calculate the Grammian \( H \) and its Moore-Penrose inverse.

3 COMPARISON WITH THE GREEN’S FUNCTION OF A GROUNDED HALFSPACE

Since we would expect that the Green’s function (13) might be sufficiently close to the Green’s function of a halfspace grounded at \( x = 0 \) we write it down here for comparison purposes

\[
G_h(x, x'; z - z') = \frac{i}{4} \left\{ H_0^{(2)}\left( k\sqrt{(x - x')^2 + (z - z')^2} \right) - H_0^{(2)}\left( k\sqrt{(x + x')^2 + (z - z')^2} \right) \right\} \quad (18)
\]

The function \( H_0^{(2)}(z) \) is the Hankel function of the second kind of order zero. In the Fourier domain this can be written as

\[
\tilde{G}_h(x, x'; \gamma) = -\frac{1}{\beta} e^{-i\beta x} \sin \beta x < \quad (19)
\]

The Green’s function for the PML terminated waveguide in the Fourier domain can also be written as

\[
\tilde{G}_{pml}(x, x'; \gamma) = \frac{\sin \beta x < \sin \beta(x_>-b)}{\beta \sin \beta b} \quad (20)
\]
by the usual Lagrange procedure of variation of parameters. Hence, for \( \tilde{G}_h \) and \( \tilde{G}_{pml} \) to be close to one another, we should require

\[
\frac{\sin \beta (x - b)}{\sin \beta b} \approx -e^{-i\beta x} \quad 0 \leq x \leq a \quad \beta \in \Gamma
\]

where \( \Gamma \) is the branch cut line formed by the negative imaginary axis and the interval \([0, k]\). A minimal asymptotic requirement is to have

\[
\lim_{\beta \to -i\infty} \frac{\sin \beta (x - b)}{\sin \beta b} = 0 \quad 0 < x \leq a
\]

This is the case when \( \Re b > a/2 \). Of course, we then also need \( \Im b < 0 \). As has been said before, we obtain good results with the formula \( b = |b|(\sqrt{1 - \tau^2} - i\tau) \), where \( \tau \) is a small positive number. From \( \Re b > a/2 \) and \( a = |b|(1 - \epsilon) \) it is in any case necessary to have \( \tau < \sqrt{3}/2 \). In Figure 3 we plotted the dB error \( 20 \log_{10} |G_h(x, x'; z) - G_{pml}(x, x'; z)| \) for \( d = 1, \tau = 0.01, N = 40, x = 0.1, x' = 0.5 \). It is seen that the error is strictly decreasing with \( z \).

**APPENDIX: COMPLETENESS OF COMPLEX EXPONENTIALS**

Preliminaries: \( \lfloor x \rfloor \) stands for the integer part of \( x \), i.e. the largest integer \( m \) such that \( m \leq x \).

A complex-valued function \( f(x) \) is \( L^p \) over a closed interval \([a, b]\) if

\[
\int_a^b |f(t)|^p dt < \infty
\]

In Redheffer’s important survey article [7] the following theorem due to Levinson is proved:

**Theorem 1**: The sequence \( \{e^{i\lambda_n x}\} \) with \( \{\lambda_n\} \) complex is complete \( L^p \) \((p \geq 1)\) on an interval of length \( 2\pi D \) if

\[
\limsup_{r \to \infty} T(r) > -\infty
\]

where

\[
T(r) = \int_1^r (\Lambda(t) - 2Dt) \frac{dt}{t} + \frac{1}{q} \log r
\]

\( 1/p + 1/q = 1 \) and \( \Lambda(t) \) is the number of \( \lambda_n \) satisfying \( |\lambda_n| \leq t \).

Theorem 1 allows the following specialization:

**Theorem 2**: The sequence \( \{e^{i\alpha x}\} \) with \( \alpha \neq 0 \) complex, \( n \in \mathbb{Z} \) is complete \( L^p \) \((p \geq 1)\) on an interval of length smaller than or equal to \( 2\pi/|\alpha| \).
Proof: Defining the step function $\Upsilon(t)$ as $\Upsilon(t) = 0, t < 0$, $\Upsilon(t) = 1, t \geq 0$ we can write with $\rho = |\alpha|$

$$\Lambda(t) = \Upsilon(t) + 2 \sum_{n=1}^{\infty} \Upsilon(t - n\rho) \quad (A3)$$

Defining $\xi(r, a)$ as

$$\xi(r, a) = \int_1^r \Upsilon(t - a) / t \, dt = \Upsilon(r - \max(a, 1)) \log(r / \max(a, 1)) \quad r \geq 1 \quad (A4)$$

we can write

$$T(r) = (1 + 1/q) \log r - 2D(r - 1) + 2 \sum_{n=1}^{\infty} \xi(r, n\rho) \quad (A5)$$

For $r > \rho$ we distinguish two cases:

- $\rho \geq 1$

We have

$$T(r) = (1 + 1/q) \log r - 2D(r - 1) + 2 \sum_{n=1}^{\lfloor r/\rho \rfloor} \log(r/n\rho) \quad (A6)$$

$$= (1 + 1/q) \log(\rho(m + \theta)) - 2D(\rho(m + \theta) - 1) + 2m \log(m + \theta) - 2 \sum_{n=1}^{m} \log n$$

where $m = \lfloor r/\rho \rfloor$ and $\theta$ are respectively the integer and fractional parts of $r/\rho$. Note that $0 \leq \theta < 1$. From the asymptotic Stirling formula

$$\log m! = \sum_{n=1}^{m} \log n = m \log m - m - \frac{1}{2} \log m + \frac{1}{2} \log(2\pi) + O\left(\frac{1}{m}\right) \quad (A7)$$

we obtain

$$T(r) = 2m(1 - D\rho) + (1 + 1/q) \log(m + \theta) + \log m + C + O\left(\frac{1}{m}\right) \quad (A8)$$

For $m \to \infty$ $T(r)$ tends to $\infty$ for $D \leq 1/\rho$ and to $-\infty$ for $D > 1/\rho$ and this proves the theorem.

- $\rho < 1$

Let $s = \lfloor 1/\rho \rfloor$. We can write

$$T(r) = (1 + 1/q + 2s) \log r - 2D(r - 1) + 2 \sum_{n=s+1}^{\lfloor r/\rho \rfloor} \log(r/n\rho) \quad (A9)$$

$$= (1 + 1/q + 2s) \log(\rho(m + \theta)) - 2D(\rho(m + \theta) - 1) + 2(m - s) \log(m + \theta) - 2 \sum_{n=s+1}^{m} \log n$$

This completes the proof, since the asymptotics of an infinite sum is not altered by deleting a finite number of terms.
References


Figure Captions

Fig. 1: Magnitude of $\psi_5(x)$ versus magnitude of $2\phi_5(x)/\tilde{b}$

Fig. 2: Argument of $\psi_5(x)$ versus argument of $2\phi_5(x)/\tilde{b}$

Fig. 3: The dB error $20\log_{10}|G_h(x, x'; z) - G_{pm}(x, x'; z)|$ for $d = 1$, $\tau = 0.01$, $N = 40$, $x = 0.1$, $x' = 0.5$. 
$|\psi_5(x)|$

$|2(\phi_5(x)/b)^*|$
$\text{Arg } \psi_5(x)$

$\text{Arg } 2(\phi_5(x)/b)^*$
$|G_h(x,x';z) - G_{pml}(x,x';z)|$ (dB)

$x = 0.1$, $x' = 0.5$