AN EXPONENTIAL TRUNCATION BOUND FOR ANALYTIC FOURIER SERIES

Luc Knockaert and Daniel De Zutter

Abstract

It is shown that analytic Fourier series admit a uniform exponential truncation bound. As a consequence, an explicit expression is obtained for the number of principal harmonics consistent with a preassigned admissible uniform error level.

1 INTRODUCTION

The problem of the truncation of infinite sums and series, and in particular Fourier series, is a critical issue which is frequently overlooked, since from a purely mathematical point of view the main point is often the mere convergence of the series, and not the asymptotical rate of convergence. From a practical computational point of view however, the rate of convergence is of utmost importance, since it can frequently be translated into useful truncation and stopping rules.

For a general Fourier series it is a priori difficult to know how or when to truncate, i.e. to estimate the number of principal harmonics, without making use of statistical information theoretic criteria such as MDL [1] or MAPME [2]. On the other hand, in important electrical engineering areas such as frequency modulation [3] and electromagnetic scattering theory [4] one often deals with analytic Fourier series, i.e. Fourier series associated with periodic analytic functions. In this correspondence we show that all analytic Fourier series exhibit a uniform exponential truncation bound. In terms of uniform approximation this means that for a given admissible uniform error, an explicit expression for the truncation threshold can be obtained.

1EDICS 3.1.1–3.8.1
2Dept. of Information Technology INTEC, St. Pietersnieuwstraat 41, B-9000 Gent, Belgium. Tel: +32 9 264 33 16, Fax: +32 9 264 42 99, e-mail: knokaert@intec.rug.ac.be
2 PROBLEM STATEMENT AND MAIN RESULT

Let \( f(x) \) be a complex analytic periodic function with period \( 2\pi \) defined over the real axis. Then \( f(x) \) admits the Fourier series

\[
f(x) = \sum_{n=-\infty}^{\infty} a_n e^{-inx}
\]

which converges absolutely and uniformly. By the principle of analytic continuation [5] p. 306, the series (1) can be continued in the complex plane to yield absolute and uniform convergence within a strip \( |\Im(z)| < t \). The above Fourier series can be written as \( f_N(x) + S_N(x) \), where

\[
f_N(x) = \sum_{n=-N}^{N} a_n e^{-inx}
\]

is the truncated series and

\[
S_N(x) = \sum_{n=N+1}^{\infty} \left( a_n e^{-inx} + a_{-n} e^{inx} \right)
\]

is the residual. Clearly, on the real axis we have

\[
|S_N(x)| \leq R_N = \sum_{n=N+1}^{\infty} (|a_n| + |a_{-n}|).
\]

Although we know that \( \lim_{N \to \infty} R_N = 0 \), it is important to have more explicit information concerning the convergence rate. This is the purpose of the following theorem.

**Theorem:**

\[
R_N \leq 4eM(\delta) \frac{2+\delta}{\delta^2} \sqrt{N\delta} e^{-N\delta} \quad \text{for} \quad \frac{1}{N} \leq \delta < t,
\]

where

\[
M(\delta) = \max_{|\Im(z)| \leq \delta} |f(z)| = \max_{x} \max(|f(x + i\delta)|, |f(x - i\delta)|).
\]

**Proof:** Since

\[
a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{inx} dx
\]

it is easily proved by successive partial integrations that

\[
|a_n||n|^m \leq \mu_m = \max_{x} |f^{(m)}(x)| \quad n \neq 0 \quad m \geq 0.
\]

Hence

\[
R_N \leq 2\mu_m \sum_{n=N+1}^{\infty} n^{-m} \leq 2\mu_m \int_{N}^{\infty} x^{-m} dx = 2\frac{\mu_m}{m-1} N^{-m+1} \quad m \geq 2.
\]
Let $C$ be the sausage-like curve consisting of the two line-segments $y = \pm \delta$, $-\pi \leq x \leq \pi$ and the two closing semi-circles of radius $\delta$ centered on $\pi$ and $-\pi$. The curve $C$ has the interesting property that the minimum distance from any point of $C$ to the real line-segment $[-\pi, \pi]$ is equal to $\delta$. Application of Cauchy’s theorem leads to

$$|f^{(m)}(x)| \leq \frac{m!}{2\pi} \int_C \frac{|f(z)|}{|z-x|^{m+1}} ds.$$  \hfill (10)

Since the length of the curve $C$ is given by $4\pi + 2\pi\delta$ we obtain the following upper bound for $\mu_m$:

$$\mu_m \leq \frac{m!}{\delta^{m+1}}(\delta + 2) \max_{z \in C} |f(z)|.$$  \hfill (11)

By the periodicity of $f(z)$ and the maximum principle, the maximum on $C$ coincides with the maximum on the boundary of the strip $|y| \leq \delta$. Hence $R_N$ is bounded by

$$R_N \leq 2N^{-m+1} \frac{\delta + 2}{m-1} \frac{m!}{\delta^{m+1}} M(\delta) \quad 0 < \delta < t \quad m \geq 2.$$  \hfill (12)

Since $m/(m-1) \leq 2$ for $m \geq 2$, formula (12) can be slightly coarsened to yield

$$R_N \leq 4M(\delta) \frac{\delta + 2}{\delta^2} \frac{m!}{(N\delta)^m} \quad 0 < \delta < t \quad m \geq 1.$$  \hfill (13)

The above formula contains the free parameter $m$, and hence we can minimize the r.h.s. of (13) with respect to $m$. Following [6] p. 167 this minimum is attained for $m = \lfloor N\delta \rfloor$, where $\lfloor x \rfloor$ stands for the greatest integer $\leq x$. Consideration of the following easily proved bound variant of Stirling’s formula

$$x! \leq (x/e)^{x} e^{1-x} \quad x \geq 1,$$  \hfill (14)

implies that the proof is complete.

**Corollary:** Let $\epsilon > 0$ and $N$ be such that

$$N \geq \min_{0 < \delta < t} \Lambda(\delta)$$  \hfill (15)

where

$$\Lambda(\delta) = \frac{1}{\delta} \phi \left( \frac{4\epsilon(2+\delta)M(\delta)}{e\delta^2} \right)$$  \hfill (16)

and $\phi(x)$ is defined as

$$\phi(x) = \begin{cases} 1 & \text{for } x \leq \epsilon \\ \ln x + \ln \ln x & \text{for } x \geq \epsilon. \end{cases}$$  \hfill (17)
Then $R_N \leq \epsilon$.

**Proof:** From the premises of the theorem we infer that what we need is an inversion of the inequality
\[ \sqrt{x} e^{-x} \leq e^{-a} \quad x \geq 1, \]  
(18)
where
\[ e^a = \frac{4e(2 + \delta)M(\delta)}{e\delta^2} \]  
(19)
and $x = N\delta$. For $a \leq 1$ we can clearly take $x = 1$. When $a > 1$ we need to find a small enough $x \geq 1$ such that (18) is satisfied. Since $\sqrt{x} e^{-x}$ is a decreasing function from $x = \frac{1}{2}$ on, we ought to find the solution of
\[ \frac{1}{2} \ln x - x = -a. \]  
(20)
This transcendental equation has no solution in terms of 'simple' functions. Therefore we set our sights lower and put forward a solution of the form $x = a + \gamma \ln a$. It is easy to show that the lowest possible positive value of $\gamma$ such that
\[ a + \gamma \ln a - \frac{1}{2} \ln(a + \gamma \ln a) \geq a \quad \text{for} \quad a \geq 1 \]  
(21)
is $\gamma = 1$, whence the corollary follows.

### 3 EXAMPLES

- As a first example we consider the basic entire periodic function $e^{imz}$, where $m$ is an integer. The maximum modulus on the strip is $M(\delta) = e^{|m|\delta}$ and the strip width is clearly $t = \infty$. It is not too hard to show that $\Lambda(\delta)$ is decreasing. Hence
\[ N \geq \lim_{\delta \to \infty} \Lambda(\delta) = |m|. \]  
(22)
Of course, this had to be the answer. We would surely have expected a flaw in the theorem had it been otherwise, since $e^{imz}$ cannot possibly be expressed as a sum of lower order harmonics.
- As a second example we consider the entire periodic function $e^{i\beta \sin z}$, with $\beta$ real, which is frequently encountered in electromagnetic scattering problems in cylindrical coordinates [4] and frequency modulation theory [3]. The relevant Fourier series is [7] p. 7
\[ e^{i\beta \sin z} = \sum_{n=-\infty}^{\infty} e^{inz} J_n(\beta), \]  
(23)
where \( J_n(\beta) \) is the Bessel function of order \( n \). In the context of frequency modulation the classical truncation criterion, based on the location of the zeros of the Bessel functions, is Carson’s rule [3] which states that \( N > |\beta| \) for \( |\beta| \) sufficiently large. We shall now show that Carson’s rule is asymptotically correct. If we take \( \beta > 0 \) then the maximum modulus on the strip is simply \( M(\delta) = e^{|\beta|\sinh \delta} \). For \( \beta \sim \infty \) we have for \( \Lambda(\delta; \beta, \epsilon) \), omitting the \( \ln \ln \) part:

\[
\Lambda(\delta; \beta, \epsilon) \sim \frac{1}{\delta} \ln \left( \frac{4e(2 + \delta)}{e\delta^2} \right) + \frac{\beta \sinh \delta}{\delta}.
\]

Expression (24) diverges to \( \infty \) for \( \delta \to 0 \) and \( \delta \to \infty \) and hence a minimum exists. An approximate choice for the abscissa of the minimum is \( \delta = 1 \) leading to

\[
\Lambda(1; \beta, \epsilon) \sim \ln \left( \frac{12e}{\epsilon} \right) + \sinh(1) + \beta \sinh(1) + \ln \beta.
\]

Asymptotically this represents a straight line i.e.

\[
\lim_{\beta \to \infty} \frac{1}{\beta} \Lambda(1; \beta, \epsilon) = \sinh(1) = 1.1752.
\]

The minimum of expression (24) is obtained when

\[
\beta = \frac{\ln(4e/\epsilon) + 2 + \ln(2 + \delta) - 2 \ln \delta - \delta/(2 + \delta)}{\delta \cosh \delta - \sinh \delta}.
\]

For \( \beta \to \infty \) we have \( \delta \to 0 \). Hence for \( \delta \sim 0 \) we have as a first approximation

\[
\beta = -6\delta^{-3} \ln(\delta/\delta_0) \quad \delta < \delta_0.
\]

where \( \delta_0 = \sqrt{8e^3/\epsilon} \). Following [8] p. 25 this equation can be asymptotically inversed yielding

\[
\delta(\beta) = \left\{ 2 \ln \left( \beta \delta_0^3/2 \right) / \beta \right\}^{\frac{1}{3}} \beta > 2e\delta_0^{-3}.
\]

Utilizing this function it is not too hard to prove that Carsons rule is asymptotically correct i.e.

\[
\lim_{\beta \to \infty} \frac{1}{\beta} \Lambda(\delta(\beta); \beta, \epsilon) = 1.
\]

In Fig. 1 we plotted \( \Lambda(1; \beta, \epsilon) \) and \( \Lambda(\delta(\beta); \beta, \epsilon) \) as a function of \( \beta \) for \( \epsilon = 0.01 \). It is seen that both curves very closely resemble straight lines.
4 CONCLUSION

We have proposed a highly pertinent truncation bound for analytic Fourier series based on the maximum modulus of an analytic function inside a strip. This resulted in an asymptotic proof of Carson’s rule in frequency modulation theory. Since the truncation bound and the effective bandwidth are related by $BW = 4\pi N$, the results obtained make it possible to calculate the effective bandwidth of virtually every periodic analytic function by performing a maximization over a strip in the complex plane, followed by a minimization over the positive real axis.

References


