COMPLEMENTARY 2-D FINITE ELEMENT PROCEDURES FOR THE MAGNETIC FIELD ANALYSIS USING A VECTOR HYSTERESIS MODEL

L. R. DUPRÉ¹, R. VAN KEER²* AND J. A. A. MELKEBEEK¹

¹ Department of Electrical Power Engineering, University of Gent, Sint-Pietersnieuwstraat 41, B-9000 Gent, Belgium
² Department of Mathematical Analysis, University of Gent, Galglaan 2, B-9000 Gent, Belgium

ABSTRACT

The main purpose of this paper is to incorporate a refined hysteresis model, viz., a vector Preisach model, in 2-D magnetic field computations. Two complementary formulations, based either on the scalar or on the vector potential, are considered. The governing Maxwell equations are rewritten in a suitable way, that allows to take into account the proper magnetic material parameters and, moreover, to pass to a variational formulation. The variational problems are solved numerically by a Finite Element approximation, using a quadratic mesh, followed by a time discretization method based upon a modified Crank–Nicholson algorithm. Finally, the effectiveness of the presented mathematical tools have been confirmed by several numerical experiments, comparing the complementarity of the two procedures. © 1998 John Wiley & Sons, Ltd.

KEY WORDS: 2-D magnetic field analysis; vector hysteresis model; complementary principles; FE–FD discretization

1. INTRODUCTION

In this paper we present the incorporation of the vector Preisach model, as described in detail by Mayergoyz,¹ in the magnetic field calculations in a 2-D domain $D$. In the conventional magnetic field analysis applied to rotating machines, the magnetic properties have mainly been modelled by using a single-valued material characteristic. This approach cannot accurately reflect the material behaviour.

In previous work by the authors² ³ the magnetic behaviour of the material has been described in terms of the uni-directional macroscopic fields, using a scalar Preisach model in order to take into account the hysteresis phenomena. In a 2-D domain, a vector hysteresis model is needed due to the local rotating magnetic flux excitations. These rotating flux excitations in electrical machines result from the complexity of the magnetic circuit and of the magnetic motoric force distributions.

We recall that in well-established hysteresis models, such as the Preisach model, the magnetic field constitutes the basic quantity, which favours a scalar potential formulation. In 2-D field calculations with single-valued material characteristics, the vector potential formulation is standard, due to the simplicity of the resulting algorithm.

* Correspondence to: R. Van Keer, Department of Mathematical Analysis, University of Gent, Galglaan 2, B-9000 Gent, Belgium. E-mail: rvk@cage.rug.ac.be

Contract/grant sponsor: Belgian Government

CCC 0029–5981/98/061005–19$17.50
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Received 6 August 1997
In this paper the two complementary formulations, described by Penman and Fraser\(^4\) and Ferrari and Pinchuk\(^5\) will be applied and compared, with respect to the incorporation of the vector Preisach model.

Throughout the paper we use standard notations and basic results from the vector calculus, see e.g. Chapter 7 in Reference 6.

2. ON THE MAXWELL EQUATIONS IN A DOMAIN WITH CONDUCTIVITY FREE-MAGNETIC MATERIALS

When neglecting capacity effects, the relevant Maxwell equations for the magnetic field \( \mathbf{H} = H_x \mathbf{i}_x + H_y \mathbf{i}_y \) and the magnetic induction \( \mathbf{B} = B_x \mathbf{i}_x + B_y \mathbf{i}_y \) in the domain \( D \) in the \((x, y)\)-plane read, see e.g. Reference 7,

\[
\nabla \times \mathbf{H} = \mathbf{J}, \quad (x, y) \in D, \quad t > 0
\]

\[
\nabla \cdot \mathbf{B} = 0, \quad (x, y) \in D, \quad t > 0
\]

which are completed with the constitutive laws, describing the material behaviour

\[
\frac{\partial \mathbf{B}}{\partial t} = \mathbf{\mu} \cdot \frac{\partial \mathbf{H}}{\partial t}, \quad (x, y) \in D, \quad t > 0
\]

or

\[
\frac{\partial \mathbf{H}}{\partial t} = \mathbf{\nu} \cdot \frac{\partial \mathbf{B}}{\partial t}, \quad (x, y) \in D, \quad t > 0
\]

Here \( \mathbf{J} \) represents the current density vector in the domain \( D \). The tensors \( \mathbf{\mu} \) and \( \mathbf{\nu} \) are the symmetric differential permeability and reluctivity tensors, respectively, given by

\[
\mathbf{\mu} = \begin{bmatrix}
\mu_{xx} & \mu_{xy} \\
\mu_{yx} & \mu_{yy}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial B_x}{\partial H_x} & \frac{\partial B_x}{\partial H_y} \\
\frac{\partial B_y}{\partial H_x} & \frac{\partial B_y}{\partial H_y}
\end{bmatrix}
\]

(5)

and

\[
\mathbf{\nu} = \begin{bmatrix}
\nu_{xx} & \nu_{xy} \\
\nu_{yx} & \nu_{yy}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial H_x}{\partial B_x} & \frac{\partial H_x}{\partial B_y} \\
\frac{\partial H_y}{\partial B_x} & \frac{\partial H_y}{\partial B_y}
\end{bmatrix}
\]

(6)

with, for physical reasons,

\[
det(\mathbf{\mu}) > 0, \quad det(\mathbf{\nu}) > 0, \quad \forall (x, y) \in D, \quad t > 0
\]

Let the boundary \( \partial D \) be divided into \( 2m \) open parts \( \partial D_1, \partial D_2, \ldots, \partial D_{2m} \), which alternatively represent flux walls and flux gates. We denote

\[
\partial D_1 = \bigcup_{s=1}^{m} \partial D_{2s}
\]

(8)
and
\[ \partial D_1 = \bigcup_{s=1}^{m} \partial D_{2s-1} \]  
(9)

The boundary conditions (BCs) for the flux gates may be written as
\[ \phi_s = \int_{\partial D_{2s}} \mathbf{B} \cdot \mathbf{n} \, d\mathbf{l}, \quad s = 1, 2, 3, \ldots, m, \quad t > 0 \]  
(10)
and
\[ \mathbf{H} \times \mathbf{n} = 0, \quad \text{on} \ \partial D_{2s}, \quad s = 1, 2, \ldots, m, \quad t > 0 \]  
(11)

Here, \( \mathbf{n} \) stands for the unit outward normal vector to the boundary part \( \partial D_s \), and \( \phi_s \) represents the total flux through the \( s \)-th gate. The BCs for the flux walls read
\[ \mathbf{B} \cdot \mathbf{n} = 0 \quad \text{on} \ \partial D_{2s-1}, \quad s = 1, 2, \ldots, m, \quad t > 0 \]  
(12)

At \( t = 0 \), the material is assumed to be in the demagnetized state, reflected in the initial condition (IC):
\[ \mathbf{H}(x, y, t = 0) = \mathbf{B}(x, y, t = 0) = 0, \quad \forall (x, y) \in D \]  
(13)
Consequently, we assume that \( \mathbf{J} = 0 \) at \( t = 0 \).

2.1. Reduced scalar potential formulation

A source field \( \mathbf{H}_0 \), obeying \( \nabla \times \mathbf{H}_0 = \mathbf{J} \), is assumed known. Then, for a reduced scalar potential \( \varphi \), the magnetic field \( \mathbf{H} = \mathbf{H}_0 - \nabla \varphi \) automatically satisfies (1). From (2) the new unknown \( \varphi \) must obey
\[ \nabla \cdot \left( \mu \cdot \nabla \frac{\partial \varphi}{\partial t} \right) = \nabla \cdot \left( \mu \cdot \frac{\partial \mathbf{H}_0}{\partial t} \right) \]  
(14)

The BCs (10)–(12) may be rewritten in terms of the unknown \( \varphi \) as
\[ \frac{d\phi_s}{dt} = -\int_{\partial D_{2s}} \left( \mu \cdot \nabla \frac{\partial \varphi}{\partial t} \right) \cdot \mathbf{n} \, d\mathbf{l}, \quad s = 1, 2, 3, \ldots, m, \quad t > 0 \]  
(15)
\[ \varphi = C_{\varphi,s}(t) \quad \text{on} \ \partial D_{2s}, \quad s = 1, 2, 3, \ldots, m, \quad t > 0 \]  
(16)
and
\[ \left( \mu \cdot \nabla \frac{\partial \varphi}{\partial t} \right) \cdot \mathbf{n} = 0 \quad \text{on} \ \partial D_{2s-1}, \quad s = 1, 2, 3, \ldots, m, \quad t > 0 \]  
(17)
respectively. Here \( \mathbf{H}_0 \) is tacitly taken to be zero at the whole boundary \( \partial D \).

To remove the degree of freedom involved in the scalar potential \( \varphi \), we choose
\[ \varphi = 0 \quad \text{on} \ \partial D_{2m}, \quad t > 0 \]  
(18)
The problem for \( \varphi \) must be completed with the IC resulting from (13) and (18)
\[ \varphi = 0, \quad \forall (x, y) \in D, \quad t = 0 \]  
(19)
Here, tacitly we have chosen \( \mathbf{H}_0 = 0 \) at \( t = 0 \).
2.2. Vector potential formulation

Writing \( \mathbf{B} = \nabla \times \mathbf{A} \), where \( \mathbf{A} \) is an unknown vector potential, (2) is automatically satisfied. Clearly, from (1) the vector potential \( \mathbf{A} \) must obey

\[
\nabla \times \left( \mathbf{v} \cdot \nabla \frac{\partial \mathbf{A}}{\partial t} \right) = \frac{\partial \mathbf{J}}{\partial t} \tag{20}
\]

Evidently, \( \mathbf{J} \) and \( \mathbf{A} \) only show a \( z \)-component, i.e. \( \mathbf{J} = J_1 \mathbf{1}_z \), \( \mathbf{A} = A_1 \mathbf{1}_z \). Hence (20) may be simplified to

\[
\nabla \cdot \left( \mathbf{v}^* \cdot \nabla \frac{\partial \mathbf{A}}{\partial t} \right) = - \frac{\partial \mathbf{J}}{\partial t} \tag{21}
\]

with

\[
\mathbf{v}^* = - \frac{\mathbf{H}}{\det(\mathbf{\mu})} = \begin{bmatrix} v_{yy} & -v_{xy} \\ -v_{yx} & v_{xx} \end{bmatrix} \tag{22}
\]

Similarly, the BCs (10)–(12) may be rewritten in terms of \( \mathbf{A} \) as

\[
\frac{d}{dt} (A|_{\partial D_{s+1}} - A|_{\partial D_{s-1}}) = \frac{d\phi_s}{dt}, \quad s = 1, 2, 3, \ldots, m - 1, \quad \frac{d}{dt} (A|_{\partial D_1} - A|_{\partial D_{2m-1}}) = \frac{d\phi_m}{dt} \tag{23}
\]

\[
\left( \mathbf{v} \cdot \left( \nabla \times \frac{\partial \mathbf{A}}{\partial t} \right) \right) \times \mathbf{n} = \left( \mathbf{v}^* \cdot \nabla \frac{\partial \mathbf{A}}{\partial t} \right) \cdot \mathbf{n} = 0 \quad \text{on} \ \partial D_{2s}, \ s = 1, 2, 3, \ldots, m, \ t > 0 \tag{24}
\]

and

\[
A = C_{A,s}(t) \quad \text{on} \ \partial D_{2s-1}, \ s = 1, 2, 3, \ldots, m, \ t > 0 \tag{25}
\]

respectively.

Again, to remove the degree of freedom involved in the vector potential \( A \), we choose

\[
A = C_{A,m} = 0 \quad \text{on} \ \partial D_{2m-1}, \ t > 0 \tag{26}
\]

(13) then implies the IC

\[
A = 0, \quad \forall (x, y) \in D, \ t = 0 \tag{27}
\]

2.3. Hysteresis models

(1) **Scalar Preisach model**: The BH-relation can be described by a scalar Preisach model if \( \mathbf{H} \) and \( \mathbf{B} \) are uni-directional.

In the Preisach model, the material is assumed to consist of small dipoles, each being characterized by a rectangular hysteresis loop as shown in Figure 1. The magnetization of the dipole \( M_d \) takes the value \(-1\) or \(+1\). The characteristic parameters \( x \) and \( \beta \) are distributed statistically according to a Preisach function \( P_s(x, \beta) \). This distribution function \( P_s(x, \beta) \) is a material parameter, which can be identified directly using a well-defined measurement technique.

The BH-relation is given by

\[
B(H, H_{\text{past}}) = \int_{-H_n}^{H_n} \int_{-H_n}^{H_n} \frac{d\beta}{\int_{-H_n}^{H_n} \text{d}\beta} \eta_s(x, y; \mathbf{z}, \mathbf{\beta}, t) P_s(x, \beta) \tag{28}
\]
Here $\eta(x, y, x, \beta, t)$ takes the time-dependent value of the magnetization $M_d$ for the dipole with parameters $x$ and $\beta$. Consequently, the induction $B(x, y, t)$ depends upon the magnetic field $H(x, y, t)$ and its history, denoted by $H_{past}(x, y, t)$.

(2) *Vector Preisach model*: In the models (14)–(19) and (21)–(27) the magnetic field $H$ and the induction $B$ are allowed to rotate in a plane parallel to the $(x, y)$-plane. Therefore, we must pass to a vector hysteresis model.

In the vector Preisach model, as described by Mayergoyz, the vector $H$ is projected on an axis $d$, which encloses an angle $\theta$ with the fixed $x$-axis, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, see Figure 2. The corresponding value $H_d = H_x \cos \theta + H_y \sin \theta$ is taken to be the input of a scalar Preisach model on the $d$ axis.

The BH-relation is now given by

$$B(H, H_{past}) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} d\theta B_\theta(H_\theta, H_{past, \theta}) \mathbf{1}_\theta$$

(29)
with

$$B_0(H_0, H_{\text{past},0}) = \int_{-H_n}^{H_n} dx \int_{-H_n}^{H_n} d\beta \, \eta_r(0, x, y, \alpha, \beta, t) P_r(\alpha, \beta)$$

(30)

\(\eta_r(0, x, y, \alpha, \beta, t)\) is obtained from the component \(H_0\), and thus depends on \(H(x, y, t)\) and \(H_{\text{past}}(x, y, t)\). The Preisach function \(P_r\) used in this rotational model can be obtained from the function \(P_s\), entering (28).

With a view to the 2-D models (14)–(19) or (21)–(27), the entries of the tensors \(\mu\) and \(v\) must be related to the magnetic field \(H(x, y, t)\). For the vector Preisach model one simply has

\[
\begin{align*}
\mu_{xx} &= \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \mu_r(H, H_{\text{past}}) \cos^2 \gamma \, d\gamma, \\
\mu_{xy} &= \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \mu_r(H, H_{\text{past}}) \cos \gamma \sin \gamma \, d\gamma, \\
\mu_{yy} &= \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \mu_r(H, H_{\text{past}}) \sin^2 \gamma \, d\gamma
\end{align*}
\]

(31)

where \(\mu_r\) represents the differential permeability of the scalar Preisach model along the axis that encloses an angle \(\gamma\) with the \(x\)-axis.

Taking into account the history effects at time point \(t = 0\), the ICs (19) and (27) must be rewritten as

\[
\begin{align*}
\varphi(x, y, t = 0) &= 0, \\
\eta_r(\gamma, x, y, x, \beta, t = 0) &= +1 \quad \text{when } x + \beta < 0, \\
\eta_r(\gamma, x, y, x, \beta, t = 0) &= -1 \quad \text{when } x + \beta > 0
\end{align*}
\]

(32)

and

\[
\begin{align*}
A(x, y, t = 0) &= 0, \\
\eta_r(\gamma, x, y, x, \beta, t = 0) &= +1 \quad \text{when } x + \beta < 0, \\
\eta_r(\gamma, x, y, x, \beta, t = 0) &= -1 \quad \text{when } x + \beta > 0
\end{align*}
\]

(33)

\[-\frac{\pi}{2} < \gamma < \frac{\pi}{2}, \forall (x, y) \in D\]

2.4. Source conditions

In both complementary formulations, two types of source terms are present.

(a) The first source term for the magnetic fields corresponds to the current vector \(J\) in (1). In the scalar potential formulation, this source term is replaced by the source field \(H_0\), appearing in (14), while for the vector potential formulation the current density enters the RHS of (21).

(b) The second source term for the magnetic fields follows from the fluxes enforced through the flux gates of the boundary \(\partial D\). In case of the scalar potential formulation, this source term is reflected in the BCs (15)–(17). Here, the functions \(\phi_s(t)\), \(s = 1, \ldots, m\) are known, while the functions \(C_{p,s}(t)\), \(s = 1, \ldots, m\) must be determined as part of the problem. When using the vector potential formulation, the gate fluxes are enforced by means of the BCs (23)–(25). Here, the functions \(C_{d,s}(t)\) are known through \(\phi_s\), \(s = 1, \ldots, m\), see (23) and (26). On account of (2), we must require that

\[
\sum_{s=1}^{m} \phi_s(t) = 0, \quad t > 0
\]

(34)
Due to the BC (23), this condition is automatically satisfied in case of the vector potential formulation.

3. WEAK FORMULATION OF THE COMPLEMENTARY BOUNDARY VALUE PROBLEMS

To derive a suitable variational form of this problem, we introduce the two function spaces

\[ V_\phi = \{ v \in W^1_2(D); \ v|_{\partial D_0} \ \text{is a constant depending on} \ s, \ s = 1, 2, 3, \ldots, m \} \quad (35) \]

and

\[ V_A = \{ v \in W^1_2(D); \ v|_{\partial D_{m-1}} \ \text{is a constant depending on} \ s, \ s = 1, 2, 3, \ldots, m \} \quad (36) \]

Here, \( W^1_2(D) \) is the usual first order Sobolev space on \( D \) and the condition \( \cdot |_{\partial D} \) is constant must be understood in the sense of traces.\(^{12}\)

In the scalar potential formulation, when multiplying both sides of (14) with a test function \( v(x, y) \in V_\phi \), next integrating over \( D \) and applying Green’s theorem (see e.g. Reference 12) and finally invoking the BC (17), the problem (14)–(19) is found to be (formally) equivalent with the following variational problem:

Find a function \( \phi(x, y, t) \), obeying \( \phi \in V_\phi \) and \( \partial \phi/\partial t \in L_2(D) \) for every \( t > 0 \), such that

\[ \int_{D} (\mu \cdot \nabla \partial \phi/\partial t) \cdot \nabla v \ dx \ dy = \int_{\partial D_1} \left( \mu \cdot \nabla \partial \phi/\partial t \right) \cdot n \ v \ ds - \int_{D} \nabla (\mu \cdot \partial H_0/\partial t) \cdot v \ dx \ dy \]

\[ \forall v \in V_\phi, \ t > 0 \]

(37)

along with

\[ \frac{d \phi_s}{dt} = - \int_{\partial D_0} \left( \mu \cdot \nabla \partial \phi/\partial t \right) \cdot n \ dl, \ s = 1, 2, 3, \ldots, m, \ t > 0 \]

(38)

and

\[ \phi = 0, \ \text{for} \ t = 0, \ \text{in} \ D \]

(39)

Notice that by the requirement \( \phi \in V_\phi \) for every \( t > 0 \), (16) is automatically taken into account.

Similarly, the vector potential formulation (21)–(27) is found to be (formally) equivalent to the following variational problem:

Find a function \( \partial A/\partial t(x, y, t) \), obeying \( A \in V_A \) and \( \partial A/\partial t \in L_2(D) \) for every \( t > 0 \), such that

\[ \int_{D} (v^* \cdot \nabla \partial A/\partial t) \cdot \nabla v \ dx \ dy = \int_{D} \partial J/\partial t \ v \ dx \ dy + \int_{\partial D_1} \left( v^* \cdot \nabla \partial A/\partial t \right) \cdot n \ v \ ds \ \forall v \in V_A, \ t > 0, \]

(40)

along with

\[ \frac{d A_s}{dt}(A|_{\partial D_{m-1}} - A|_{\partial D_{m-1}}) = \frac{d \phi_s}{dt}, \ s = 1, 2, 3, \ldots, m - 1, \]

\[ \frac{d}{dt}(A|_{\partial D_1} - A|_{\partial D_{m-1}}) = \frac{d \phi_m}{dt}, \ t > 0 \]

(41)

and

\[ A = 0, \ \text{for} \ t = 0, \ \text{in} \ D \]

(42)
Notice again that, by the requirement $A \in V_2$ for every $t > 0$, (25) is automatically taken into account. (39) and (42) must be interpreted in the sense of (32) and (33), respectively.

4. NUMERICAL APPROXIMATION BY FE–FD SCHEMES

The problems (37)–(39) and (40)–(41) are solved numerically. We combine (a) a finite element method with quadratic interpolation functions for the discretization in space, (b) a modified Crank–Nicholson finite difference scheme for the time discretization, (c) a numerical quadrature formula with equidistant nodes for the integral (29).

A major computational difficulty in the discretization arises from the hysteresis behaviour of the material, reflected in the dependency of the functions $\mu_{xx}, \ldots, \mu_{yy}$ or the functions $v_{xx}, \ldots, v_{yy}$ on the vector $H_{\text{pass}}(x, y, t)$ (as well as on $H(x, y, t)$ itself).

4.1. Space discretization

Let $\mathcal{T}_h$ be a regular triangulation of the domain $D$, with mesh parameter $h$. We consider a quadratic finite element mesh, corresponding to $\mathcal{T}_h$.

By $N_j(x, y)$, $(j = 1, \ldots, J)$, we denote the standard cardinal basis functions, associated to the nodes $(x_j, y_j)$, $(j = 1, \ldots, J)$, $J$ being the total number of nodes. Let $C^0_0(D)$ be the space of continuous functions on $D$ and let $P_2(T)$ be the space of polynomials of degree $\leq 2$ on $T$. We then have,

$$X_h = \{ v \in C^0_0(D) ; v|_T \in P_2(T), \forall T \in \mathcal{T}_h \} = \text{span}(N_j)_{j=1}^J \quad (43)$$

We now pass to the two complementary formulations.

4.1.1. Scalar potential formulation. Let $J_{2x-1}$ and $J_{2x}$ be the number of nodes on $\partial D_{2s-1}$ (open) and $\partial D_{2s}$ (closed), respectively, $s = 1, 2, \ldots, m$. We put $I_1 = \sum_{s=1}^m J_{2s-1}$. The $J$ nodes in the closed domain $D$ are numbered such that the first $I_1$ of them, $J > I + I_1$, belong either to the open domain $D$ ($I$ nodes) or to the boundaries $\partial D_{2s-1}$, $s = 1, 2, \ldots, m$. The next $J_2$ nodes belong to $\partial D_2$, the next $J_3$ nodes belong to $\partial D_3$, etc. We then have,

$$X_{0,0}^{s} = \{ v \in X_h | \text{ on } \partial D_{2s}, s = 1, 2, 3, \ldots, m \} = \text{span}(N_j)_{j=1}^{I+I_1} \quad (44)$$

Next, let

$$P_s = \sum_{k=1}^{s-1} J_{2k}, \quad s = 1, 2, \ldots, m \quad (P_1 = 0) \quad (45)$$

We introduce the interpolation functions

$$\psi_{t+I_1+s}(x, y) = \sum_{j=1}^{I+P_s+1} N_j(x, y), \quad s = 1, 2, 3, \ldots, m \quad (46)$$

Evidently, the functions $\phi_{t+I_1+s}$, $s = 1, 2, \ldots, m$ belong to the space $X_h$, (43).

On a side $\zeta$ of the triangle $T \in \mathcal{T}_h$, for which $\zeta \subset \partial D_{2s}$, we have $\psi_{t+I_1+s}|_{\zeta} = 1$, as clearly $\psi_{t+I_1+s}|_{\zeta}$ is a quadratic function of one local variable showing the value 1 in the 3 nodes on $\zeta$. Consequently,

$$\psi_{t+I_1+s} = 1 \text{ on } \partial D_{2s}, s = 1, 2, 3, \ldots, m \quad (47)$$
Moreover, \( \psi_{I+I+s} \) is readily understood to vanish throughout \( D \) apart from the triangles \( T \in \tau_h \) adjacent to \( \partial D_{2s} \), \( s = 1, 2, \ldots, m \).

Writing, for convenience, \( \psi_j \equiv N_j, \) \( 1 \leq j \leq I + I_1 \), we finally define the function space \( V_{h, \varphi} \):

\[
V_{h, \varphi} \equiv \text{span} (\psi_j)_{j=1}^{I+I_1+m} = X_{0h, \varphi} \oplus \text{span} (\psi_{I+I_1+s})_{s=1}^{m}
\]

This space suits for a conforming finite element approximation, as, by construction, \( V_{h, \varphi} \subset V \).

The finite element approximation \( \varphi_h(x, y; t) \in V_{h, \varphi} \) of the unknown reduced scalar potential \( \varphi(x, y, t) \) is defined by a system similar to (37)–(39), now with the space of test functions \( V \) replaced by the finite-dimensional subspace \( V_{h, \varphi} \). Here, we approximate the space dependency of the tensor \( \mathbf{u} \), by passing to \( \mathbf{u}^1 \equiv \mathbf{u} \), defined by

\[
\mathbf{u}^1(x, y, t, \mathbf{H}_0(x, y; t), \varphi_h(x, y; t), \varphi_h^{(\text{past})}(x, y; t))
\]

\[
= \mathbf{u}(x_T, y_T, t, \mathbf{H}_0(x_T^c, y_T^c; t), \varphi_h(x_T^c, y_T^c; t), \varphi_h^{(\text{past})}(x_T^c, y_T^c; t))
\]

\[
(x, y) \in T, \quad \forall T \in \tau_h, \quad t > 0
\]

where \( (x_T^c, y_T^c) \) is the centre of gravity of \( T \). This allows us to take properly into account the non-linear and hysteresis effects, resulting in the complicated form of the differential permeability tensor \( \mathbf{u} \). Here, \( \mathbf{u} \) now depends upon the finite element approximation \( H_h(x, y; t) = \mathbf{H}_0 - \nabla \varphi_h \) and \( H_h^{(\text{past})}(x, y; t) = H_0^{(\text{past})} - \nabla \varphi_h^{(\text{past})} \) of the magnetic field \( H(x, y, t) \) and its history \( H^{(\text{past})}(x, y, t) \), respectively.

Explicitly, recalling (18) and decomposing \( \varphi_h \) as

\[
\varphi_h(x, y; t) = \sum_{j=1}^{I+I_1+m} \varphi_j(t) \psi_j(x, y), \quad (x, y) \in D, \quad t > 0
\]

we first have that \( \varphi_j(t) = \varphi_h(x_j, y_j; t) \), \( 1 \leq j \leq I + I_1 \), and moreover that \( \varphi_{I+I+s}(t) = \varphi_h(x, y; t)|_{\partial D_{2s}} \), \( s = 1, 2, \ldots, m \), due to the proper choice of the basis functions of \( V_{h, \varphi} \), (48).

The resulting system of first-order ODEs for these unknown coefficient functions takes the form

\[
\frac{dC}{dt} = F, \quad t > 0
\]

along with the ICs, cf. (19),

\[
C(0) = 0
\]

and

\[
\begin{cases}
\eta_{1}(\gamma, x, y, \alpha, \beta, t = 0) = +1: \quad \alpha + \beta < 0 \\
\eta_{2}(\gamma, x, y, \alpha, \beta, t = 0) = -1: \quad \alpha + \beta > 0, \quad -\frac{\pi}{2} < \gamma < \frac{\pi}{2}, \quad \forall (x, y) \in D
\end{cases}
\]

The second IC corresponds to the demagnetised state of the material at \( t = 0 \).

Here, the matrices involved read as follows. \( C \) and \( C^{(\text{past})} \) stand for the column matrices,

\[
C(t) = [\varphi_1(t), \varphi_2(t), \ldots, \varphi_{I+I_1+m-1}(t)]^T
\]

and

\[
C^{(\text{past})}(t) = [\varphi_1^{(\text{past})}(t), \varphi_2^{(\text{past})}(t), \ldots, \varphi_{I+I_1+m-1}^{(\text{past})}(t)]^T
\]
respectively, while \( M \) is the mass matrix given by

\[
M(t, C(t), C^{(\text{past})}(t)) = (M_{i,m})_{1 \leq i, m \leq I_1 + m - 1}
\]  

with

\[
M_{i,m} = \int_D (\mu_1 \cdot \nabla \psi_i) \cdot \nabla \psi_m \, dx \, dy
\]

Finally, the \((I + I_1 + m - 1) \times 1\)-force matrix \( F \), corresponding to the RHS of (37), reads

\[
F(t) = [F_1(t), F_2(t), \ldots, F_{I_1 + I_1 + 1}(t), F_{I_1 + I_1 + m - 1}(t) + G_{m-1}(t), \ldots, F_{I_1 + I_1 + m - 1}(t) + G_{m-1}(t)]^T
\]

where

\[
F_i(t) = -\int_D \nabla \cdot \left( \mu \cdot \frac{\partial H_0}{\partial t} \right) \psi_i \, dx \, dy, \quad i = 1, 2, \ldots, I_1 + m - 1
\]

and

\[
G_j(t) = -\frac{d\phi_j}{dr}, \quad j = 1, \ldots, m - 1
\]

### 4.1.2. Vector potential formulation

Let now \( J_{2s-1} \) and \( J_{2s} \) be the number of nodes on \( \partial D_{2s-1} \) (closed) and \( \partial D_{2s} \) (open), respectively. Put \( I_2 = \sum_{s=1}^{m} J_{2s} \). The \( J \) nodes in \( D \) are numbered such that the first \( I_1 + I_2 \) of them, \( J > I_1 + I_2 \), belong to the open domain \( D \) (1 nodes) or to the boundaries \( \partial D_{2s}, s = 1, 2, \ldots, m \), the next \( J_1 \) nodes belong to \( \partial D_1 \), the next \( J_3 \) nodes belong to \( \partial D_3 \), etc. We now have

\[
X_{0h,A} = \{ v \in X_h \mid v = 0 \text{ on } \partial D_{2s-1}, \ s = 1, 2, 3, \ldots, m \} = \text{span}(N_{j})_{j=1}^{I_1 + I_1}
\]

Similarly, as in Section 4.1.1, we put

\[
P_s = \sum_{k=1}^{s-1} J_{2k-1}, \quad s = 1, 2, \ldots, m, \quad (P_1 = 0)
\]

and we introduce the interpolation functions

\[
\psi_{I_1 + I_2 + s}(x, y) = \sum_{j=I_1 + I_2 + P_{i-1}}^{I_1 + I_1 + P_s} N_j(x, y) \quad s = 1, 2, 3, \ldots, m
\]

Now, the functions \( \psi_{I_1 + I_2 + s}, s = 1, 2, \ldots, m \), belong to the space \( X_h \), (43).

Writing, for convenience, \( \psi_j \equiv N_j \), \( 1 \leq j \leq I_1 + I_2 \), we finally define the function space \( V_{h,A} \):

\[
V_{h,A} = \text{span}(\psi)_{j=1}^{I_1 + I_1 + m} = X_{0h,A} \oplus \text{span}(\psi_{I_1 + I_2 + s})_{s=1}^{m}
\]

As, by construction, \( V_{h,A} \subset V \), the space \( V_{h,A} \) suits for a conforming FEM.

The finite element approximation \( A_h(x, y; t) \in V_{h,A} \) of the unknown vector potential \( A(x, y; t) \) is defined by a system similar to (40)–(42), now with the space of trial and test functions \( V \) replace
by $V_{h,A}$. Here, we approximate the space dependency of the tensor $\mathbf{v}^*$, by passing to $\mathbf{v}^{*,1} \simeq \mathbf{v}^*$, defined by

$$\mathbf{v}^{*,1} = \frac{\mathbf{\mu}^1}{\det(\mathbf{\mu}^1)}$$  (65)

Recalling (26) and decomposing $A_h$ as

$$A_h(x, y; t) = \sum_{j=1}^{l_1+I_2+m-1} A_j(t) \psi_j(x, y), \quad (x, y) \in D, \ t > 0$$  (66)

we first have that $A_j(t) = A_h(x_j, y_j; t)$, $1 \leq j \leq I_1 + I_2$, and next that $A_{I_1+I_2+s}(t) = A_h(x, y; t)|_{\partial D_{s-1}}$, $s = 1, 2, \ldots, m$, due to the proper choice of the basis functions of $V_{h,A}$, (64).

The resulting initial value problem for the unknown coefficient functions $A_j(t)$, $1 \leq j \leq I_1 + I_2$, take a similar form (51)–(53). Now, the matrices involved read as follows. $C$ and $C^{(\text{past})}$ are the column matrices,

$$C(t) = [A_1(t), A_2(t), \ldots, A_{I_1+I_2}(t)]^T$$  (67)

and

$$C^{(\text{past})}(t) = [A_1^{(\text{past})}(t), A_2^{(\text{past})}(t), \ldots, A_{I_1+I_2}^{(\text{past})}(t)]^T$$  (68)

respectively, while $M$ is the mass matrix given by

$$M(t, C(t), C^{(\text{past})}(t)) = (M_{l,m})_{1 \leq l, m \leq I_1 + I_2}$$  (69)

with

$$M_{l,m} = \int_D (\mathbf{v}^{*,1} \cdot \nabla \psi_l) \cdot \nabla \psi_m \, dx \, dy$$  (70)

Finally, the $(I_1 + I_2) \times 1$-force matrix $F$, corresponding to the RHS of (40), reads

$$F(t) = [F_1(t), F_2(t), \ldots, F_{I_1+I_2}(t)]^T$$  (71)

where

$$F_i(t) = \int_D \frac{\partial J}{\partial t} \psi_i \, dx \, dy, \quad i = 1, 2, \ldots, I_1 + I_2$$  (72)

4.2. Time discretization

The IVP (51)–(53) is solved numerically by a modified Cranks–Nicholson method. The analysis proceeds similarly as in previous work,\textsuperscript{14} to which we refer for details.

Let $\Delta t$ be a time step and $t_k = k \cdot \Delta t$, $k = 0, 1, 2, \ldots$ be the corresponding equidistant time points. We define an approximation $C^{(k)} = [\varphi_1^{(k)}, \varphi_2^{(k)}, \ldots, \varphi_p^{(k)}]^T$ of $C(t_k) = [\varphi_1(t_k), \varphi_2(t_k), \ldots, \varphi_p(t_k)]^T$ ($k = 1, 2, \ldots$), by the following recurrent algebraic system:

$$\begin{equation}
M^{(k)} C^{(k)} - C^{(k-1)} = F(t_k) + F(t_{k-1}), \quad k = 1, 2, \ldots
\end{equation}$$  (73)
starting from, see (52),

\[ C^{(0)} = 0 \] (74)

In case of the scalar potential formulation, we construct an approximation \( \phi^{(k)}(x, y) \) of \( \phi_h(x, y, t_k) \), (50), by means of \( C^{(k)} \), viz.,

\[ \phi_h^{(k)}(x, y) = \sum_{j=1}^{l-2} C_j \phi_j(x, y), \quad (x, y) \in D \] (75)

As the matrix \( \tilde{M}^{(k)} \), appearing in (73) and specified below, depends on the unknown \( C^{(k)} \), we set-up an iterative procedure to solve the non-linear system at every time point \( t_k \), the number of iterations being denoted by \( n_k \). The approximation of \( C^{(k)} \) at the \( l \)th iteration level is denoted by \( C^{(k)};l \). The corresponding approximation of (75) is written as \( \phi^{(k)};l(h(x, y)) \).

In the iterative procedure the matrix \( \tilde{M}^{(k)} \) is generated from the matrix \( M \) by a suitable averaging procedure over the interval \( [t_{k-1}, t_k] \). Here, at the \( l \)th iteration level the matrix \( \tilde{M}^{(k)};l \) is updated using \( \phi_h^{(k)};l-1(x, y) \).

4.3. Discretization Preisach model

Let \( N > 0 \) be an integer. The BH-relation (29) is discretized as

\[ B(H, H_{past}) = \frac{1}{N} \sum_{l=1}^{N} B_{i_l}(H_{0_i}, H_{past, i}) \mathbf{1}_{i_l} \] (76)

Here \( H_{0_i} \) and \( H_{past, i} \) are the projection of \( H \) and \( H_{past} \), respectively, on the axis \( d \) enclosing an angle \( \theta_i \) with the x-axis. (31) leads to \( \mu_{xx}, \mu_{xy}, \mu_{yx}, \mu_{yy} \) as a function of \( \mu_{\gamma_i} \), \( s = 1, \ldots, N \), where \( \mu_{\gamma_i} \) is the differential permeability in the scalar Preisach model on the \( l \)-axis, but using the Preisach function \( P_r \) belonging to the vector Preisach model. Thus,

\[
\begin{align*}
\mu_{xx} &= \frac{1}{N} \sum_{l=1}^{N} \mu_{\gamma_i} \cos^2 \gamma_i, \\
\mu_{xy} &= \frac{1}{N} \sum_{l=1}^{N} \mu_{\gamma_i} \cos \gamma_i \sin \gamma_i, \\
\mu_{yx} &= \frac{1}{N} \sum_{l=1}^{N} \mu_{\gamma_i} \sin \gamma_i \cos \gamma_i, \\
\mu_{yy} &= \frac{1}{N} \sum_{l=1}^{N} \mu_{\gamma_i} \sin^2 \gamma_i
\end{align*}
\] (77)

The IC (53) may then be discretized by replacing \( \gamma_i \) by \( \gamma_l \), \( l = 1, \ldots, N \).

4.4. Solving the resulting algebraic system of equations

The \( k \times k \) matrix \( M \) entering (51), is of the form (56)–(57) (scalar potential formulation, \( k = I + I_1 + m - 1 \)) or of the form (69)–(70) (vector potential formulation, \( k = I + I_2 \)). Clearly, the matrix \( M \) is symmetric. Moreover, the matrix is positive definite. In other words, for any real column matrix \( \zeta = [\zeta_1, \zeta_2, \ldots, \zeta_k]^T \), we have that \( \zeta^T \cdot M \cdot \zeta \geq 0 \), and that \( \zeta^T \cdot M \cdot \zeta \) is zero only if \( \zeta \) is the zero matrix. To this end, notice that

\[ \zeta^T \cdot M \cdot \zeta = \int_D (\mathbf{1} \cdot \zeta) \cdot \zeta \, dx \, dy \] (78)
or
\[
\zeta^T \cdot M \cdot \zeta = \int_D (\psi^{(s,1)} \cdot \xi) \cdot \xi \, dx \, dy \tag{79}
\]
where
\[
\xi = \sum_{i=1}^{k} \zeta_i \nabla \psi_i \tag{80}
\]
From a classical result (see e.g. Reference 6, p. 68), in both formulations the integrand in (79) or in (78) is strictly positive when \( \zeta \neq 0 \) as the two eigenvalues of \( \mu \) or \( \nu \) are strictly positive throughout \( D \) at each time point \( t \). The latter easily follows from (7).

Hence, to solve the system of linear equations (73), we may use a suitable conjugate gradient method, see e.g. References 15 and 16.

5. APPLICATION OF THE COMPLEMENTARY FORMULATIONS TO A TOOTH REGION OF AN ASYNCHRONOUS MACHINE

The effectiveness of the variational approximation method for the problems (14)–(19) and (21)–(27), as outlined in the previous sections, has been confirmed by numerical experiments dealing with a tooth region of an asynchronous machine. The numerical results obtained with the two complementary models are compared.

5.1. Formulation of the problem

As a test problem we consider a 3 kW four-pole induction machine. The stator has a single-layer winding with three slots per pole and per phase. The rotor has 32 unskewed open slots. The 3 kW induction machine, delta-connected, has been tested under no-load (slip \( \leq 0.004 \)).

One tooth region of an asynchronous machine, as shown in Figure 3, is chosen to be the domain \( D \). The enforced (realistic) flux patterns through the flux gates \( \partial D_2 \) and \( \partial D_6 \) are measured using pick-up coils. They may be written as
\[
\phi_s(t) = a_{s1} \cos(2\pi ft + \gamma_{s1}) + a_{s15} \cos(30\pi ft + \gamma_{s15}) + a_{s17} \cos(34\pi ft + \gamma_{s17}),
\]
\[
s = 1, 2, \quad t > 0
\]

On account of (34), we have for the flux through \( \partial D_6 \)
\[
\phi_3(t) = -\phi_1(t) - \phi_2(t), \quad t > 0 \tag{81}
\]

The amplitudes and phases are given in Table I (\( f = 50 \text{Hz} \)). The current vector \( \mathbf{J} \) is taken to be zero throughout the domain \( D \) at every time \( t > 0 \) and for simplicity, we choose the source field \( \mathbf{H}_0 \) also equal to zero.

5.2. Comparison of the two formulations and adaptive mesh refinement using the complementary principles

The initial mesh to perform the finite element calculations, using the two formulations, is shown in Figure 4. This mesh is constructed by a geometrical discretization and by first adaptive refinement after computations with a single-valued material characteristic.\(^{17}\) Figures 5 and 6 show the
Figure 3. One tooth region of the asynchronous machine considered.

Table I. Amplitudes (T) and angles (deg) of the excitation

<table>
<thead>
<tr>
<th></th>
<th>$a_{x,1}$</th>
<th>$a_{x,15}$</th>
<th>$a_{x,17}$</th>
<th>$\gamma_{x,1}$</th>
<th>$\gamma_{x,15}$</th>
<th>$\gamma_{x,17}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi_1$</td>
<td>1.262</td>
<td>0.0178</td>
<td>0.0105</td>
<td>25</td>
<td>109</td>
<td>-36</td>
</tr>
<tr>
<td>$\phi_2$</td>
<td>1.268</td>
<td>0.0067</td>
<td>0.0050</td>
<td>5.9</td>
<td>-155</td>
<td>27</td>
</tr>
</tbody>
</table>

The time variation of the $x$-component and the $y$-component of the magnetic induction $B$ in point 1 of the domain $D$. We observe a discrepancy between the waveforms obtained with the scalar potential formulation (dotted line) and the one obtained with the vector potential formulation (dot-dashed line). The corresponding BH-loops are given in Figures 7 and 8.

To develop a method for an adaptive mesh refinement, we define the following error function in terms of the local $B_x$, $B_y$-values, obtained with the vector potential formulation and the scalar potential formulation:

$$E(x,y) = \frac{(B_{x,A}(x,y) - B_{x,\sigma}(x,y))^2}{(B_{x,A}(x,y) + B_{x,\sigma}(x,y))^2} + \frac{(B_{y,A}(x,y) - B_{y,\sigma}(x,y))^2}{(B_{y,A}(x,y) + B_{y,\sigma}(x,y))^2}, \quad \forall (x,y) \in D$$

The resulting finite element mesh is given in Figure 9. The $B$-waveforms and the corresponding BH-loops for the point 1 of the domain $D$ are given in Figures 5–8, in full line and in dashed line for the scalar and vector potential formulation, respectively. Due to the local mesh refinement a good correspondence between the results of the two formulations is obtained.

Figure 4. Initial finite element mesh

Figure 5. $B_x$-waveforms
Figure 6. $B_y$-waveforms

Figure 7. $B_x,H_x$-loops
Figure 8. $B_y H_y$-loops

Figure 9. Finite element mesh after refinement
6. CONCLUSIONS

In this paper we outlined a numerical approximation method for the evaluation of local field patterns in a 2-D-region applying two complementary formulations, and based upon a proper reformulation of the governing Maxwell-equations. The formulations mentioned are a reduced scalar potential formulation and a vector potential formulation, respectively. This allowed us to incorporate a refined hysteresis model, viz., a vector Preisach model, in the magnetic field computations, resulting in the specific form of the differential tensor $\mu$ and differential reluctivity tensor $v$. These tensors depend on $H$ as well as on $H_{\text{past}}$, reflecting the memory property of the material.

After a finite element method with respect to the space variable, we are left with an initial-value problem for a system of first-order non-linear differential equations with respect to time. We presented a modified Crank–Nicholson algorithm for the discretization in time of this much involved problem. The resulting algebraic system of equations is shown to have a symmetric positive-definite matrix. Consequently, we may resort to a proper conjugate gradient method in order to solve the system.

When comparing the two formulations, differences were found for the local $B$-patterns, when evaluated with an initial mesh, due to the different approximation methods. When using an adaptive mesh refinement, taking properly into account the complementary local $B$-patterns and the complex vector hysteresis behaviour, the results provided by the two formulations were found to be in good agreement.

ACKNOWLEDGEMENTS

We gratefully acknowledge the financial support by the Belgium Government in the frame of the Inter-University Attraction Poles for fundamental research. The first author is a postdoctoral researcher of the Fund of Scientific Research—Flanders.

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