Triparametric Family of Cardinality-based Fuzzy Comparison Measures – Cheat Sheet

Definition:

$$\mathcal{M}^{\mathcal{H}}_{\varphi_1,\varphi_2}(A,B) = \frac{\varphi_1(\|A \diamondsuit_{\mathcal{H}} A\|, \|B \diamondsuit_{\mathcal{H}} B\|, \|A \diamondsuit_{\mathcal{H}} B\|)}{\varphi_2(\|A \diamondsuit_{\mathcal{H}} A\|, \|B \diamondsuit_{\mathcal{H}} B\|, \|A \diamondsuit_{\mathcal{H}} B\|)}$$

for all $A, B \in \mathcal{F}(U)$, with φ_1 and φ_2 two $[0,1]^3 \to \mathbb{R}$ mappings and $\diamondsuit_{\mathcal{H}}$ an infix notation for the pointwise extension of a binary aggregation operator $\mathcal{H}$, i.e., $(A \diamondsuit_{\mathcal{H}} B)(u) = \mathcal{H}(A(u), B(u))$ for all $u \in U$, with A and B fuzzy sets in U.

Potential properties:

$$\mathcal{M}^{\mathcal{H}}_{\varphi_1,\varphi_2} \text{ is } [0,1]\text{-valued} \iff (\forall A,B \in \mathcal{F}(U))(0 \le \mathcal{M}^{\mathcal{H}}_{\varphi_1,\varphi_2}(A,B) \le 1)$$

$$\mathcal{M}^{\mathcal{H}}_{\varphi_1,\varphi_2} \text{ is symmetric} \iff (\forall A,B \in \mathcal{F}(U))(\mathcal{M}^{\mathcal{H}}_{\varphi_1,\varphi_2}(A,B) = \mathcal{M}^{\mathcal{H}}_{\varphi_1,\varphi_2}(B,A))$$

$$\mathcal{M}^{\mathcal{H}}_{\varphi_1,\varphi_2} \text{ is reflexive} \iff (\forall A,B \in \mathcal{F}(U))(\mathcal{M}^{\mathcal{H}}_{\varphi_1,\varphi_2}(A,B) = 1 \Leftarrow A = B)$$

$$\mathcal{M}^{\mathcal{H}}_{\varphi_1,\varphi_2} \text{ is coreflexive} \iff (\forall A,B \in \mathcal{F}(U))(\mathcal{M}^{\mathcal{H}}_{\varphi_1,\varphi_2}(A,B) = 1 \Rightarrow A = B)$$

$$\mathcal{M}^{\mathcal{H}}_{\varphi_1,\varphi_2} \text{ is strong reflexive} \iff (\forall A,B \in \mathcal{F}(U))(\mathcal{M}^{\mathcal{H}}_{\varphi_1,\varphi_2}(A,B) = 1 \Leftarrow A \subseteq B \vee B \subseteq A)$$

$$\mathcal{M}^{\mathcal{H}}_{\varphi_1,\varphi_2} \text{ is weak coreflexive} \iff (\forall A,B \in \mathcal{F}(U))(\mathcal{M}^{\mathcal{H}}_{\varphi_1,\varphi_2}(A,B) = 1 \Rightarrow A \subseteq B \vee B \subseteq A)$$

$$\mathcal{M}^{\mathcal{H}}_{\varphi_1,\varphi_2} \text{ is inclusive} \iff (\forall A,B \in \mathcal{F}(U))(\mathcal{M}^{\mathcal{H}}_{\varphi_1,\varphi_2}(A,B) = 1 \Leftarrow A \subseteq B)$$

$$\mathcal{M}^{\mathcal{H}}_{\varphi_1,\varphi_2} \text{ is coinclusive} \iff (\forall A,B \in \mathcal{F}(U))(\mathcal{M}^{\mathcal{H}}_{\varphi_1,\varphi_2}(A,B) = 1 \Rightarrow A \subseteq B)$$

$$\mathcal{M}^{\mathcal{H}}_{\varphi_1,\varphi_2} \text{ is exclusive} \iff (\forall A,B \in \mathcal{F}(U))(\mathcal{M}^{\mathcal{H}}_{\varphi_1,\varphi_2}(A,B) = 0 \Leftarrow \operatorname{supp} A \cap \operatorname{supp} B = \emptyset)$$

$$\mathcal{M}^{\mathcal{H}}_{\varphi_1,\varphi_2} \text{ is coexclusive} \iff (\forall A,B \in \mathcal{F}(U))(\mathcal{M}^{\mathcal{H}}_{\varphi_1,\varphi_2}(A,B) = 0 \Rightarrow \operatorname{supp} A \cap \operatorname{supp} B = \emptyset)$$

$$\mathcal{M}^{\mathcal{H}}_{\varphi_1,\varphi_2} \text{ is left-restrictable} \iff (\forall A,B \in \mathcal{F}(U))(\mathcal{M}^{\mathcal{H}}_{\varphi_1,\varphi_2}(A,B) = \mathcal{M}^{\mathcal{H}}_{\varphi_1,\varphi_2}(A/_{\operatorname{supp} A}, B/_{\operatorname{supp} A}))$$

$$\mathcal{M}^{\mathcal{H}}_{\varphi_1,\varphi_2} \text{ is right-restrictable} \iff (\forall A,B \in \mathcal{F}(U))(\mathcal{M}^{\mathcal{H}}_{\varphi_1,\varphi_2}(A,B) = \mathcal{M}^{\mathcal{H}}_{\varphi_1,\varphi_2}(A/_{\operatorname{supp} B}, B/_{\operatorname{supp} B}))$$

$$\mathcal{M}^{\mathcal{H}}_{\varphi_1,\varphi_2} \text{ is weak left-restrictable} \iff (\forall A,B \in \mathcal{F}(U))(\mathcal{M}^{\mathcal{H}}_{\varphi_1,\varphi_2}(A,B) \le \mathcal{M}^{\mathcal{H}}_{\varphi_1,\varphi_2}(A/_{\operatorname{supp} A}, B/_{\operatorname{supp} A}))$$

$$\mathcal{M}^{\mathcal{H}}_{\varphi_1,\varphi_2} \text{ is weak right-restrictable} \iff (\forall A,B \in \mathcal{F}(U))(\mathcal{M}^{\mathcal{H}}_{\varphi_1,\varphi_2}(A,B) \le \mathcal{M}^{\mathcal{H}}_{\varphi_1,\varphi_2}(A/_{\operatorname{supp} B}, B/_{\operatorname{supp} B}))$$

General constraints:

$$\mathcal{M}^{\mathcal{H}}_{\varphi_1,\varphi_2} \text{ is } [0,1]\text{-valued} \Leftarrow (\forall x,y,z \in [0,1])(x+y-1 \le z \le x+y \Rightarrow 0 \le \varphi_1(x,y,z) \le \varphi_2(x,y,z))$$

$$\mathcal{M}^{\mathcal{H}}_{\varphi_1,\varphi_2} \text{ is reflexive} \Leftarrow (\forall x \in [0,1])(\varphi_1(x,x,x) = \varphi_2(x,x,x))$$

Constraints for the case $\mathcal{H} = \mathcal{C}$, with $\mathcal{C}$ a commutative conjunctor:

$$\mathcal{M}^{\mathcal{C}}_{\varphi_1,\varphi_2} \text{ is symmetric} \Leftarrow (\forall x,y,z \in [0,1])(\varphi_1(x,y,z) = \varphi_1(y,x,z) \wedge \varphi_2(x,y,z) = \varphi_2(y,x,z))$$

$$\mathcal{M}^{\mathcal{C}}_{\varphi_1,\varphi_2} \text{ is strong reflexive} \Leftarrow (\forall x,y,z \in [0,1])(\min(x,y) \le z \le \max(x,y) \Rightarrow \varphi_1(x,y,z) = \varphi_2(x,y,z))$$

$$\mathcal{M}^{\mathcal{C}}_{\varphi_1,\varphi_2} \text{ is inclusive} \Leftarrow (\forall x,y,z \in [0,1])(x \le z \le y \Rightarrow \varphi_1(x,y,z) = \varphi_2(x,y,z))$$

$$\mathcal{M}^{\mathcal{C}}_{\varphi_1,\varphi_2} \text{ is exclusive} \Leftarrow (\forall x,y \in [0,1])(\varphi_1(x,y,0) = 0)$$

$$\mathcal{M}^{\mathcal{C}}_{\varphi_1,\varphi_2} \text{ is left-restrictable} \Leftarrow (\forall x,z \in [0,1])(\forall u,v \in [0,1])(\varphi_1(x,u,z) = \varphi_1(x,v,z) \wedge \varphi_2(x,u,z) = \varphi_2(x,v,z))$$

$$\mathcal{M}^{\mathcal{C}}_{\varphi_1,\varphi_2} \text{ is right-restrictable} \Leftarrow (\forall y,z \in [0,1])(\forall u,v \in [0,1])(\varphi_1(u,y,z) = \varphi_1(v,y,z) \wedge \varphi_2(u,y,z) = \varphi_2(v,y,z))$$

$$\mathcal{M}^{\mathcal{C}}_{\varphi_1,\varphi_2} \text{ is weak left-restr.} \Leftarrow (\forall x,z \in [0,1])(\forall u,v \in [0,1])(\varphi_1(x,u,z) = \varphi_1(x,v,z) \wedge (u \le v \Rightarrow \varphi_2(x,u,z) \le \varphi_2(x,v,z)))$$

$$\mathcal{M}^{\mathcal{C}}_{\varphi_1,\varphi_2} \text{ is weak right-restr.} \Leftarrow (\forall y,z \in [0,1])(\forall u,v \in [0,1])(\varphi_1(u,y,z) = \varphi_1(v,y,z) \wedge (u \le v \Rightarrow \varphi_2(u,y,z) \le \varphi_2(v,y,z)))$$

Constraints for the case $\mathcal{H} = T_{\mathrm{L}}$, with T_{L} the Lukasiewicz t-norm:

$$\mathcal{M}^{T_{\mathrm{L}}}_{\varphi_1,\varphi_2} \text{ is } [0,1]\text{-valued} \Leftarrow (\forall x,y,z \in [0,1])(x+y-1 \le z \le (x+y)/2 \Rightarrow 0 \le \varphi_1(x,y,z) \le \varphi_2(x,y,z))$$

Constraints for the case $\mathcal{H} = T_{\mathrm{P}}$, with T_{P} the product:

$$\mathcal{M}^{T_{\mathrm{P}}}_{\varphi_1,\varphi_2} \text{ is } [0,1]\text{-valued} \Leftarrow (\forall x,y,z \in [0,1])(x+y-1 \le z \le \sqrt{x \cdot y} \Rightarrow 0 \le \varphi_1(x,y,z) \le \varphi_2(x,y,z))$$

$$\mathcal{M}^{T_{\mathrm{P}}}_{\varphi_1,\varphi_2} \text{ is coexclusive} \Leftarrow (\forall x,y,z \in]0,1])(x+y-1 \le z \le \sqrt{x \cdot y} \Rightarrow \varphi_1(x,y,z) > 0)$$

Constraints for the case $\mathcal{H} = T_{\mathrm{M}}$, with T_{M} the minimum:

$$\mathcal{M}^{T_{\mathrm{M}}}_{\varphi_1,\varphi_2} \text{ is } [0,1]\text{-valued} \Leftarrow (\forall x,y,z \in [0,1])(x+y-1 \le z \le \min(x,y) \Rightarrow 0 \le \varphi_1(x,y,z) \le \varphi_2(x,y,z))$$

$$\mathcal{M}^{T_{\mathrm{M}}}_{\varphi_1,\varphi_2} \text{ is coreflexive} \Leftarrow (\forall x,y,z \in [0,1])(z < \max(x,y) \wedge z \le \min(x,y) \Rightarrow \varphi_1(x,y,z) < \varphi_2(x,y,z))$$

$$\mathcal{M}^{T_{\mathrm{M}}}_{\varphi_1,\varphi_2} \text{ is strong reflexive} \Leftarrow (\forall x,y \in [0,1])(\varphi_1(x,y,\min(x,y)) = \varphi_2(x,y,\min(x,y)))$$

$$\mathcal{M}^{T_{\mathrm{M}}}_{\varphi_1,\varphi_2} \text{ is weak coreflexive} \Leftarrow (\forall x,y,z \in [0,1])(z < \min(x,y) \Rightarrow \varphi_1(x,y,z) < \varphi_2(x,y,z))$$

$$\mathcal{M}^{T_{\mathrm{M}}}_{\varphi_1,\varphi_2} \text{ is inclusive} \Leftarrow (\forall x,y \in [0,1])(x \le y \Rightarrow \varphi_1(x,y,x) = \varphi_2(x,y,x))$$

$$\mathcal{M}^{T_{\mathrm{M}}}_{\varphi_1,\varphi_2} \text{ is coinclusive} \Leftarrow (\forall x,y,z \in [0,1])(z < x \wedge z \le y \Rightarrow \varphi_1(x,y,z) < \varphi_2(x,y,z))$$

$$\mathcal{M}^{T_{\mathrm{M}}}_{\varphi_1,\varphi_2} \text{ is coexclusive} \Leftarrow (\forall x,y,z \in]0,1])(x+y-1 \le z \le \min(x,y) \Rightarrow \varphi_1(x,y,z) > 0)$$