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Composite supersymmetric S-functions and characters of $\mathfrak{gl}(m|n)$ representations

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Abstract

It is shown how to associate to a highest weight Λ of the Lie superalgebra $\mathfrak{gl}(m|n)$ a composite partition $\overline{\nu};\mu$ with composite Young diagram $F(\overline{\nu};\mu)$. The corresponding supersymmetric Schur function $s_{\overline{\nu};\mu}(x/y)$ is defined. However, it is known that this S-function does not always coincide with the character of the irreducible representation V_Λ with highest weight Λ. Only for covariant, contravariant and typical representations the character and the S-function are known to coincide.

Here, the notions of critical composite partitions and critical highest weights are considered. It is shown that for critical composite partitions (subject to a technical restriction) the corresponding $\mathfrak{gl}(m|n)$ representation V_Λ is tame, so its character can be computed. Also for this class of representations the character coincides with the composite supersymmetric S-function $s_{\overline{\nu};\mu}(x/y)$. This extends considerably the classes of $\mathfrak{gl}(m|n)$ representations for which the character can be computed by means of S-functions.

1 Introduction

In Lie algebra theory the character of irreducible representations (irreps) of $\mathfrak{gl}(n)$ are given by ordinary Schur functions or S-functions $s_\lambda(x)$, and there is a simple relation between the highest weight of the representation and the partition λ.

For the superanalog, the Lie superalgebra $\mathfrak{gl}(m|n)$, the relation between characters of its irreps and "supersymmetric" S-functions is not so clear. In this context, composite supersymmetric S-functions $s_{\overline{\nu};\mu}(x/y)$ were introduced. These S-functions are labelled by a composite partition $\overline{\nu};\mu$ (two partitions μ and ν), and are functions of two sets of variables x and y.

In this paper we shall discuss some recent advances in understanding this relation. Many of the results of the present contribution are based upon the paper [19], so there is a great amount of overlap between [19] and the current contribution. The main difference is that now the results are presented for general $\mathfrak{gl}(m|n)$-standard composite partitions, and that we discuss some further properties illustrated by means of examples.

In the early days of Lie superalgebra representation theory, the notion of graded tensors was introduced [7], and it was believed [2, 3] that the standard methods of covariant, contravariant and mixed tensor representations with the corresponding Young techniques yield the characters of $\mathfrak{gl}(m|n)$ irreps in terms of supersymmetric S-functions $s_{\overline{\nu};\mu}(x/y)$. Although this is certainly true for the covariant and contravariant tensor representations [4, 7], it is not so for the mixed tensor representations, as already observed in [15, 20]. Despite this negative answer, it is still surprising how often $s_{\overline{\nu};\mu}(x/y)$ yields the correct character of a $\mathfrak{gl}(m|n)$ irrep. So far, there were no conditions known when this is actually the case, except the rule that "m and n should be sufficiently large compared to the number of boxes in $\overline{\nu};\mu$" [20]. In [19], we give a clear condition (criticality) under which $s_{\overline{\nu};\mu}(x/y)$ is actually the character of an irreducible $\mathfrak{gl}(m|n)$ representation. Note that also for typical representations, $s_{\overline{\nu};\mu}(x/y)$ yields the correct character (an unpublished result obtained by R.C. King). The fact that $s_{\overline{\nu};\mu}(x/y)$ yields the correct character also in the singly atypical case, follows in particular from the main theorem of this paper.

In this paper we describe the highest weight of a $\mathfrak{gl}(m|n)$ irrep by means of a $\mathfrak{gl}(m|n)$-standard composite partition $\overline{\nu};\mu$. The notion of a critical atypical irrep, introduced in [8], is described is Section 3. The following section is devoted to some examples, describing diagrammatic properties of typical and critical atypical irreps. Next, we use essentially the method of [18] to show (under the technical restriction of "no overlap") that these critical atypical representations are "tame", in the sense of Kac and Wakimoto [13]. Using their results, an explicit character formula for these irreps can be constructed, and we show how this formula can be rewritten in a determinantal form [19]. Using this determinantal form, the character can be shown to coincide with a composite supersymmetric S-function.

2 Composite Young diagrams and composite partitions

The composite Young diagram $F(\overline{\nu};\mu) = F(\ldots,-\nu_2,-\nu_1;\mu_1,\mu_2,\ldots)$, specified by the pair of partitions $\mu = (\mu_1,\mu_2,\ldots)$ and $\nu = (\nu_1,\nu_2,\ldots)$, consists of two conventional Young diagrams $F(\mu)$ and $F(\nu)$. The former is composed of boxes arranged in left-adjusted rows of lengths $\mu_1,\mu_2,\ldots$ (from top to bottom), and the latter of boxes arranged in right-adjusted rows of lengths $\nu_1,\nu_2,\ldots$ (from bottom to top). A manner of junxtaposition of $F(\mu)$ and $F(\nu)$ to form $F(\overline{\nu};\mu)$ was given in [6]; we shall refer to this as the traditional corner representation. To some extent this is a refining of the back-to-back notation of [1] and [14]. By way of illustration, for $\overline{\nu};\mu = (\overline{3},\overline{8});(5,3,1)$ the composite Young diagram is displayed in Figure 1(a). Note that in $(\overline{3},\overline{8})$ we have used the convention of putting the minus-signs on top of the integers; so in this example $\mu = (5,3,1)$ and $\nu = (8,3)$. We shall refer to $\overline{\nu};\mu$ as being a "composite partition".

Let m and n be fixed. In the process of associating to a weight of $\mathfrak{gl}(m|n)$ a composite partition $\overline{\nu};\mu$, there is another way to visualize $\overline{\nu};\mu$ by putting them together in a $(m \times n)$-rectangle. The partition μ is now composed of boxes arranged in left-adjusted rows of lengths $\mu_1, \mu_2, \ldots$ starting at the top left-hand corner of this rectangle, and the partition ν of boxes arranged in right-adjusted rows of lengths $\nu_1, \nu_2, \ldots$ starting at the bottom right-hand corner of the rectangle. For $\overline{\nu};\mu = (\overline{1},\overline{1},\overline{2},\overline{5},\overline{5},\overline{9});(5,4,4,1)$ and $(m|n) = (5|7)$ this is illustrated in Figure 1(b). Observe that in this second visualisation, there can be overlap between the two diagrams (and parts of the diagram might actually fall outside of the $(m \times n)$-rectangle).

When $\nu = 0$, the (ordinary) partition μ labels a covariant representation of $\mathfrak{gl}(m|n)$ if $\mu_{m+1} \leq n$; and when $\mu = 0$, $\overline{\nu}$ labels (under similar conditions) a contravariant representation of $\mathfrak{gl}(m|n)$ [4]. In both cases, the partition determines a certain highest weight Λ of the corresponding irreducible representation (or simple module) V_Λ. In [22], it was shown how to determine the highest weight Λ for the given partition μ or $\overline{\nu}$. Such a partition μ determines a (covariant) highest weight Λ_μ if $\mu_{m+1} \leq n$ (in this case, the partition μ is said to be $\mathfrak{gl}(m|n)$-standard). Graphically, this means that the Young diagram of μ should fit inside the so-called $(m \times n)$-hook, see Figure 2. In this example, $(m,n) = (5,8)$ and $\mu = (11,9,4,3,2,2,2,1)$. For such a partition, the corresponding $\mathfrak{gl}(m|n)$ highest weight Λ_μ, in the standard ϵ-δ-basis [19], is determined as follows:

$$\Lambda_\mu = \sum_{i=1}^{m} \mu_i \epsilon_i + \sum_{j=1}^{n} \langle \mu'_j - m \rangle \delta_j \tag{1}$$

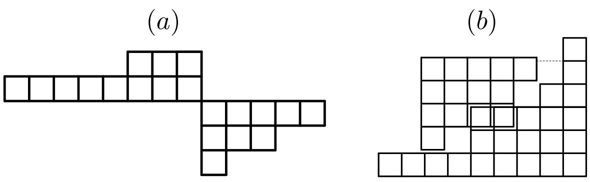

Figure 1. (a) The Young diagram $F(\overline{\nu};\mu)$ of a composite partition in its traditional corner presentation. (b) The Young diagram of another composite partition $\overline{\nu};\mu$ positioned in the $(m \times n)$-rectangle.

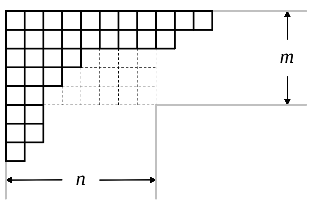

Figure 2. The Young diagram $F(\mu)$ inside the $(m \times n)$-hook.

where $\langle a \rangle = \max(0,a)$. Thus, for the above example,

$$\Lambda_\mu = (11,9,4,3,2;\ 3,2,0,0,0,0,0,0),$$

the coordinates being written in the standard ϵ-δ-basis of the weight space of $\mathfrak{gl}(m|n)$.

The relation between a composite partition $\overline{\nu};\mu$ and a certain $\mathfrak{gl}(m|n)$ weight $\Lambda_{\overline{\nu};\mu}$ is more complicated. This relation has been given in [6, §3] or [5]. Just as for ordinary partitions, there is a condition to be satisfied:

Definition 2.1 *A composite partition $\overline{\nu};\mu$ is said to be $\mathfrak{gl}(m|n)$-standard if and only if there exist J and L such that*

$$\begin{aligned} &J = \min\{j|\mu'_{j+1} + \nu'_{n-j+1} \leq m\} \qquad \textit{with} \qquad 0 \leq J \leq n, \\ &L = \min\{l|\mu_{m-l+1} + \nu_{l+1} \leq n\} \qquad \textit{with} \qquad 0 \leq L \leq m. \end{aligned} \tag{2}$$

In that case, let $I = m - L$ and $K = n - J$.

The notions of this definition are illustrated in Figure 3. Graphically, $\mathfrak{gl}(m|n)$-standardness means that the diagram $F(\overline{\nu};\mu)$, in its traditional corner representation, should fit inside the $(m\times n)$-cross, as illustrated in Figure 3. Furthermore, it should be shifted as far as possible to the right and to the top inside this cross. In this position, the coordinates of the corresponding $\mathfrak{gl}(m|n)$ highest weight $\Lambda_{\overline{\nu};\mu}$ in the standard ϵ-δ-basis can be determined:

$$\begin{aligned} \Lambda_{\overline{\nu};\mu} \quad = \quad &(\mu_1,\mu_2,\ldots,\mu_I, n-\nu_L,\ldots,n-\nu_1; \\ &\mu'_1 - m,\ldots,\mu'_J - m, -\nu'_K,\ldots,-\nu'_1). \end{aligned} \tag{3}$$

This yields a unique correspondence between integral highest weights of $\mathfrak{gl}(m|n)$ and $\mathfrak{gl}(m|n)$-standard composite partitions. Note that by (3) it is very

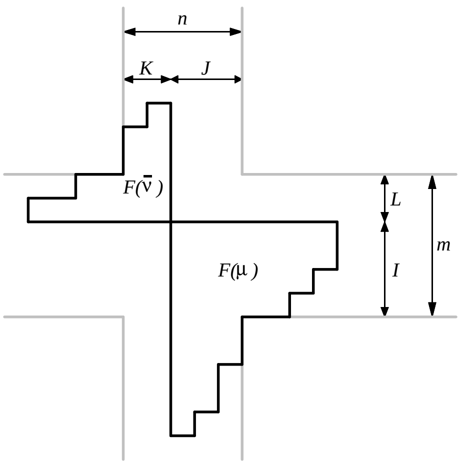

Figure 3. The Young diagram $F(\overline{\nu};\mu)$ of a composite partition $\overline{\nu};\mu$ inside the $(m \times n)$-cross, and the meaning of the indices (I,J,K,L)

easy to go from a given $\mathfrak{gl}(m|n)$-standard composite partition $\overline{\nu};\mu$ to its corresponding highest weight $\Lambda_{\overline{\nu};\mu}$. The converse process, going from an integral highest weight Λ to a $\mathfrak{gl}(m|n)$-standard composite partition, is not so easy; even though it is still a unique process [5, 6]. For more explicit examples, see Section 4.

At this point it is convenient to say something about the connection between representations of $\mathfrak{gl}(m|n)$ and of $\mathfrak{sl}(m|n)$, which is similar to that between $\mathfrak{gl}(m)$ and $\mathfrak{sl}(m)$. Recall that $\mathfrak{sl}(m|n)$ consists of those elements of $\mathfrak{gl}(m|n)$ with zero supertrace. Define the element σ in the standard ϵ-δ-basis by

$$\sigma = \sum_{i=1}^{m} \epsilon_i - \sum_{j=1}^{n} \delta_j, \tag{4}$$

or in coordinates $\sigma = (1, 1, \ldots, 1;\ -1, -1, \ldots, -1)$. Then $\sigma = 0$ in the weight space of $\mathfrak{sl}(m|n)$ (but not in the weight space of $\mathfrak{gl}(m|n)$). So two highest weights Λ and $\Lambda + j\sigma$ of $\mathfrak{gl}(m|n)$ stand for the same highest weight in $\mathfrak{sl}(m|n)$. This implies that the corresponding highest weight representations V_Λ and $V_{\Lambda+j\sigma}$ must have the same character as $\mathfrak{sl}(m|n)$ representations. Then their $\mathfrak{gl}(m|n)$ characters are also the same, up to a factor. More explicitly,

$$\mathrm{ch}\, V_{\Lambda+j\sigma} = (e^\sigma)^j\ \mathrm{ch}\, V_\Lambda, \tag{5}$$

with e the formal exponential (see next section).

3 Typical, atypical and critical representations

Let $\mathfrak{g}$ be the Lie superalgebra $\mathfrak{gl}(m|n)$ and $\mathfrak{h}$ its Cartan subalgebra. The weight space of $\mathfrak{g}$ is the dual space $\mathfrak{h}^*$ with standard basis $\{\epsilon_1, \ldots, \epsilon_m, \delta_1, \ldots, \delta_n\}$ In the so-called *distinguished choice* [11] for a triangular decomposition of $\mathfrak{g}$, the simple root system is given by

$$\Pi = \{\epsilon_1 - \epsilon_2, \ldots, \epsilon_{m-1} - \epsilon_m, \epsilon_m - \delta_1, \delta_1 - \delta_2, \ldots, \delta_{n-1} - \delta_n\}. \tag{6}$$

In that case, the positive even roots are given by $\Delta_{0,+} = \{\epsilon_i - \epsilon_j | 1 \leq i < j \leq m\} \cup \{\delta_i - \delta_j | 1 \leq i < j \leq n\}$, and the positive odd roots by $\Delta_{1,+} = \{\epsilon_i - \delta_j | 1 \leq i \leq m,\ 1 \leq j \leq n\}$.

In the distinguished basis there is only one simple root which is odd. As usual, we put

$$\rho_0 = \frac{1}{2}\Big(\sum_{\alpha\in\Delta_{0,+}} \alpha\Big), \qquad \rho_1 = \frac{1}{2}\Big(\sum_{\alpha\in\Delta_{1,+}} \alpha\Big), \qquad \rho = \rho_0 - \rho_1. \tag{7}$$

There is a symmetric form $(\ ,\ )$ on $\mathfrak{h}^*$ induced by the invariant symmetric form on $\mathfrak{g}$, and in the natural basis it takes the values $(\epsilon_i, \epsilon_j) = \delta_{ij}$, $(\epsilon_i, \delta_j) = 0$ and $(\delta_i, \delta_j) = -\delta_{ij}$.

The Weyl group of $\mathfrak{g}$ is the Weyl group W of $\mathfrak{g}_0$, hence it is the direct product of symmetric groups $S_m \times S_n$. For $w \in W$, we denote by $\varepsilon(w)$ its signature.

Let V_Λ be a finite-dimensional irreducible representation of $\mathfrak{g}$ with highest weight Λ. Such representations are $\mathfrak{h}$-diagonalizable with weight decomposition $V = \oplus_\mu V(\mu)$, and the character is defined to be $\mathrm{ch}\, V = \sum_\mu \dim V(\mu)\, e^\mu$, where e^μ ($\mu \in \mathfrak{h}^*$) is the formal exponential. To express such characters, we shall use $x_i = e^{\epsilon_i}$ and $y_j = e^{\delta_j}$.

It is known that the notion of typical and atypical representations of $\mathfrak{gl}(m|n)$ plays an important role [12]. A representation V_Λ with highest weight Λ is *typical* if $(\Lambda + \rho, \beta) \neq 0$ for all positive odd roots $\beta \in \Delta_{1,+}$; otherwise Λ and V_Λ are called *atypical*. Since $\Delta_{1,+}$ consists of the roots $\beta_{ij} = \epsilon_i - \delta_j$ ($1 \leq i \leq m$, $1 \leq j \leq n$), it is natural to construct the *atypicality matrix* $A(\Lambda)$. This is an $(m \times n)$-matrix consisting of the numbers $A(\Lambda)_{ij} = (\Lambda + \rho, \beta_{ij})$ [22,23]. So if no zeros appear in $A(\Lambda)$, V_Λ is typical. If a zeros appear in $A(\Lambda)$, Λ and V_Λ are a-fold atypical (or the atypicality of Λ is a). When dealing with highest weights related to (composite) partitions, it is often convenient to put the entries of the atypicality matrix in an $(m \times n)$-rectangle, together with the corresponding composite Young diagram. This is illustrated for $\mathfrak{gl}(5|7)$ and $\overline{\nu}; \mu = (\overline{4}, \overline{6}, \overline{6}, \overline{6}); (3, 3, 2, 2)$ in Figure 4. Note that for this example, $\Lambda \equiv \Lambda_{\overline{\nu};\mu}$ is 3-fold atypical, since there are three zeros in the atypicality matrix.

For a given atypical weight Λ with atypicality a, let $\{\gamma_1, \ldots, \gamma_a\}$ be the sets of odd roots $\gamma_s = \beta_{i_s, j_s}$ such that $(\Lambda + \rho, \beta_{i_s, j_s}) = 0$, where $j_1 < j_2 < \cdots < j_a$ (in this order). In the example of Figure 4, $a = 3$ and $(\gamma_1, \gamma_2, \gamma_3) = (\beta_{5,1}, \beta_{3,2}, \beta_{1,4})$. Notice that $\gamma_1, \ldots, \gamma_a$ are ordered from the bottom left-hand corner to the top right-hand corner.

With the notations of [8], we distinguish between *normal, critical and quasicritical* related roots of this set $\{\gamma_1, \ldots, \gamma_a\}$. Let x_{pq} with $1 \leq p < q \leq a$ be the entry in $A(\Lambda)$ at the intersection of the column containing the γ_p zero with the row containing the γ_q zero. Let h_{pq} be the hook length between the zeros corresponding to γ_p and γ_q, i.e. the number of steps needed to go from the γ_p zero of $A(\Lambda)$ via x_{pq} to the γ_q zero, where the zeros themselves are included in the count. In the example of Figure 4, with $(\gamma_1, \gamma_2, \gamma_3) = (\beta_{5,1}, \beta_{3,2}, \beta_{1,4})$, the

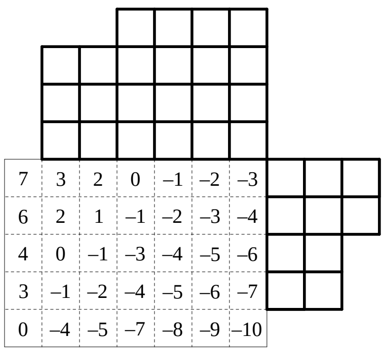

Figure 4. The Young diagram $F(\overline{\nu}; \mu)$ of a composite partition $\overline{\nu}; \mu$ in its traditional position. Here $\overline{\nu}; \mu = (\overline{4}, \overline{6}, \overline{6}, \overline{6}); (3, 3, 2, 2)$. For $\mathfrak{gl}(m|n) = \mathfrak{gl}(5|7)$ also the atypicality matrix of $\Lambda_{\overline{\nu};\mu}$ is given.

hook lengths are $h_{12} = 4$, $h_{13} = 8$ and $h_{23} = 5$, and the x_{pq} values can be read from the atypicality matrix: $x_{12} = 4$, $x_{13} = 7$ and $x_{23} = 3$.

Definition 3.1 *Let Λ be a highest weight of $\mathfrak{gl}(m|n)$ with atypicality a and atypical roots $\{\gamma_1, \ldots, \gamma_a\}$. Then for every $1 \leq p < q \leq a$: γ_p and γ_q are normally related if and only if $x_{pq} + 1 > h_{pq}$; γ_p and γ_q are quasicritically related if and only if $x_{pq}+1 = h_{pq}$; γ_p and γ_q are critically related if and only if $x_{pq}+1 < h_{pq}$.*

Thus in the example of Figure 4, γ_1 and γ_2 are normally related, γ_1 and γ_3 are quasicritically related and γ_2 and γ_3 are critically related.

If each couple (γ_i, γ_{i+1}) ($i = 1, 2, \ldots, a-1$) is critically related, then all elements of $\{\gamma_1, \ldots, \gamma_a\}$ are critically related. Then the highest weight Λ and the representation V_Λ are called *critical*. If $\Lambda \equiv \Lambda_{\overline{\nu};\mu}$ is originating from a composite partition $\overline{\nu};\mu$, we shall also refer to $\overline{\nu};\mu$ as a critical composite partition. Criticality coincides with the notion of *totally connected*, as described in [21, 24].

For an alternative combinatorial way to check criticality, see [19].

4 Some examples

In this section, we shall give some examples of composite partitions, their Young diagrams (both in the $(m \times n)$-cross and in the $(m \times n)$-rectangle), their atypicality matrix, and some related composite partitions.

Let us take $\mathfrak{gl}(m|n) = \mathfrak{gl}(5|7)$, and consider as first example the composite partition

$$\overline{\nu};\mu = (\overline{3}, \overline{3}, \overline{4}, \overline{6}, \overline{7}); (5, 5, 5, 4, 2, 1, 1, 1). \tag{8}$$

The Young diagram of $\overline{\nu};\mu$ – in its proper corner position in the $(m \times n)$-cross – is given in Figure 5(a). So in this case, $(I, J, K, L) = (0, 5, 2, 5)$, and hence we find from (3) that the corresponding weight $\Lambda = \Lambda_{\overline{\nu};\mu}$ is given by

$$\Lambda = (4, 4, 3, 1, 0;\ 3, 0, -1, -1, -2, -5, -5). \tag{9}$$

The Young diagram of $\overline{\nu};\mu$ is also given in Figure 5(b), where it is represented in the $(m \times n)$-rectangle. Notice that in this case, there is overlap between the two diagrams (that of μ given in black and that of $\overline{\nu}$ given in gray). Furthermore, in the last figure we also give the atypicality matrix $A(\Lambda)$, in the appropriate positions of the $(m \times n)$-rectangle. Notice that there are no zeros in this matrix, so Λ is typical.

We can now consider the closely related weight

$$\widetilde{\Lambda} = \Lambda + \sigma = (5, 5, 4, 2, 1;\ 2, -1, -2, -2, -3, -6, -6). \tag{10}$$

Using (3), it is easy to work out the composite partition corresponding to $\widetilde{\Lambda}$. One finds

$$\overline{\widetilde{\nu}};\widetilde{\mu} = (\overline{2}, \overline{2}, \overline{2}, \overline{3}, \overline{5}, \overline{6}); (5, 5, 4, 2, 1, 1, 1), \tag{11}$$

with $(I,J,K,L) = (5,1,6,0)$. Now we can consider the Young diagram of $\overline{\overline{\widetilde{\nu}}};\widetilde{\mu}$, once in its corner position in the $(m \times n)$-cross – given here in Figure 6(a); and once represented in the $(m \times n)$-rectangle – given in Figure 6(b). Also the atypicality matrix is once again given, and obviously $A(\Lambda) = A(\widetilde{\Lambda})$, since $(\sigma, \beta_{ij}) = 0$ for all odd roots β_{ij}. Notice that in the $(m \times n)$-rectangle (Figure 6(b)), the Young diagrams of $\widetilde{\mu}$ and $\overline{\overline{\widetilde{\nu}}}$ have no overlap and just "touch" each other along their boundaries. All positive entries in the atypicality matrix are inside the diagram of $\widetilde{\mu}$, whereas all negative entries of $A(\widetilde{\Lambda})$ are inside $\overline{\overline{\widetilde{\nu}}}$. This is no coincidence. One can show that this is a general feature of typical weights. More explicitly, let $\overline{\nu};\mu$ be a composite partition with corresponding

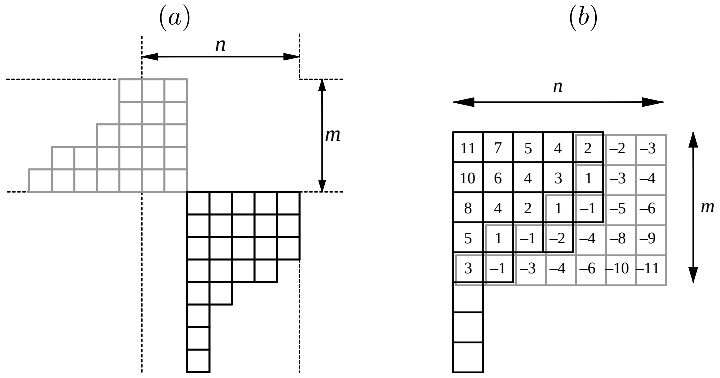

Figure 5. Young diagram of a composite partition $\overline{\nu};\mu$ in (a) the $(m\times n)$-cross and (b) the $(m \times n)$-rectangle, together with its atypicality matrix. Here, $\mu = (5,5,5,4,2,1,1,1)$ and $\nu = (7,6,4,3,3)$.

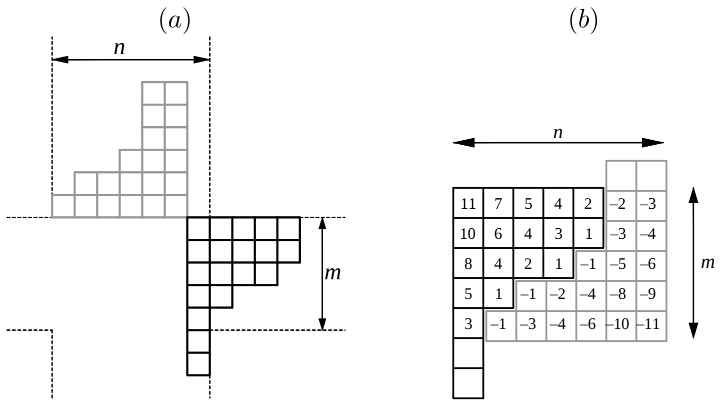

Figure 6. Young diagram of a composite partition $\overline{\nu};\mu$ in (a) the $(m \times n)$-cross and (b) the $(m \times n)$-rectangle, together with its atypicality matrix. Here, $\mu = (5,5,4,2,1,1,1)$ and $\nu = (6,5,3,2,2,2)$.

weight $\Lambda = \Lambda_{\overline{\nu};\mu}$ and suppose Λ is typical. Then there is a unique integer j such that $\widetilde{\Lambda} = \Lambda + j\sigma$, for which the corresponding composite partition is $\overline{\widetilde{\nu}};\widetilde{\mu}$, satisfies the following properties:

- the Young diagrams of $\widetilde{\mu}$ and $\overline{\widetilde{\nu}}$ have no overlap (no intersection) in the $(m \times n)$-rectangle;
- each box in the $(m \times n)$-rectangle is either part of the Young diagram of $\widetilde{\mu}$ or else of the Young diagram of $\overline{\widetilde{\nu}}$;
- all positive entries in the atypicality matrix are inside the Young diagram of $\widetilde{\mu}$, and all negative entries are inside the Young diagram of $\overline{\widetilde{\nu}}$.

As a second example in $\mathfrak{gl}(5|7)$, let us take the composite partition

$$\overline{\nu};\mu = (\overline{2},\overline{2},\overline{3},\overline{6},\overline{7});(4,3,3,1,1). \tag{12}$$

The Young diagram of $\overline{\nu};\mu$, properly situated in the $(m \times n)$-cross, is given in Figure 7(a). Note that $(I,J,K,L) = (5,0,7,0)$, and we find from (3) that the corresponding weight $\Lambda = \Lambda_{\overline{\nu};\mu}$ is given by

$$\Lambda = (4,3,3,1,1;\ -1,-2,-2,-2,-3,-5,-5). \tag{13}$$

The Young diagram of $\overline{\nu};\mu$ is also given in Figure 7(b), represented in the $(m \times n)$-rectangle. Notice the overlap between the two diagrams. As for the previous example, we also give the atypicality matrix $A(\Lambda)$, in the appropriate positions of the $(m \times n)$-rectangle in Figure 7(b). Notice that there are two zeros in this matrix, so Λ is atypical. By the entry "4" in the hook connecting the two zeros (in the terminology of Definition 3.1, $x_{12} = 4$ and $h_{12} = 6$), it follows that Λ is critical.

Let us consider the closely related weight

$$\widetilde{\Lambda} = \Lambda - \sigma = (3,2,2,0,0;\ 0,-1,-1,-1,-2,-4,-4). \tag{14}$$

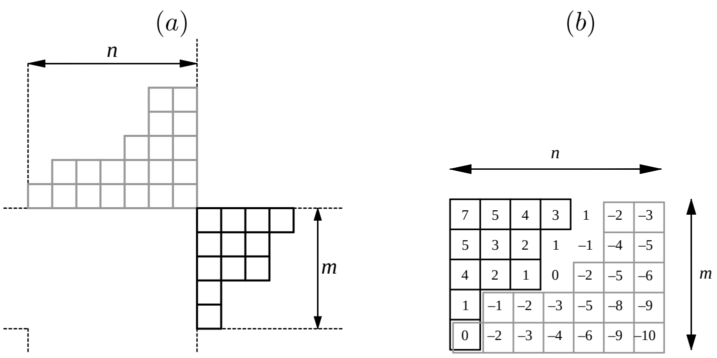

Figure 7. Young diagram of a composite partition $\overline{\nu};\mu$ in (a) the $(m \times n)$-cross and (b) the $(m \times n)$-rectangle, together with its atypicality matrix. Here, $\mu = (4,3,3,1,1)$ and $\nu = (7,6,3,2,2)$.

Using (3), the composite partition corresponding to $\widetilde{\Lambda}$ is

$$\overline{\widetilde{\nu}};\widetilde{\mu} = (\overline{2},\overline{2},\overline{3},\overline{6});(3,2,2), \tag{15}$$

with again $(I,J,K,L) = (5,0,7,0)$. The Young diagram of $\overline{\widetilde{\nu}};\widetilde{\mu}$, properly positioned in the $(m\times n)$-cross, is given in Figure 8(a); and in Figure 8(b) it is once again given but now positioned in the $(m\times n)$-rectangle, together with the atypicality matrix (again $A(\Lambda) = A(\widetilde{\Lambda})$). Notice that in the $(m\times n)$-rectangle (Figure 8(b)), the Young diagrams of $\widetilde{\mu}$ and $\overline{\widetilde{\nu}}$ have no overlap, and the zeros of the atypicality matrix are positioned in the "gap" between the two diagrams. This is a general property of critical atypical weights. More explicitly, let $\overline{\nu};\mu$ be a composite partition with corresponding weight $\Lambda = \Lambda_{\overline{\nu};\mu}$ and suppose Λ is atypical and critical. Then there is a unique integer j such that $\widetilde{\Lambda} = \Lambda + j\sigma$, for which the corresponding composite partition $\overline{\widetilde{\nu}};\widetilde{\mu}$ satisfies the following properties

- the Young diagrams of $\widetilde{\mu}$ and $\overline{\widetilde{\nu}}$, positioned in the $(m\times n)$-rectangle, do not cover the complete rectangle but leave a connected "gap";
- all the zeros of the atypicality matrix appear in this connected gap.

5 Tame representations in $\mathfrak{gl}(m|n)$

Let V be an irreducible representation of $\mathfrak{gl}(m|n)$ with highest weight Λ in the standard (distinguished) simple root basis Π. The atypicality of V and of Λ is the number of zeros in the atypicality matrix $A(\Lambda)$, where $A(\Lambda)_{ij} = (\Lambda+\rho,\epsilon_i-\delta_j)$. Note that all the roots $\epsilon_i-\delta_j$ from $\Delta_{1,+}$ are isotropic: $(\epsilon_i-\delta_j,\epsilon_i-\delta_j) = 0$. So the determination of the atypicality of V is performed with the highest weight of V with respect to the distinguished set of simple roots (6), and the corresponding set Δ_+. But one can give a definition of atypicality that is

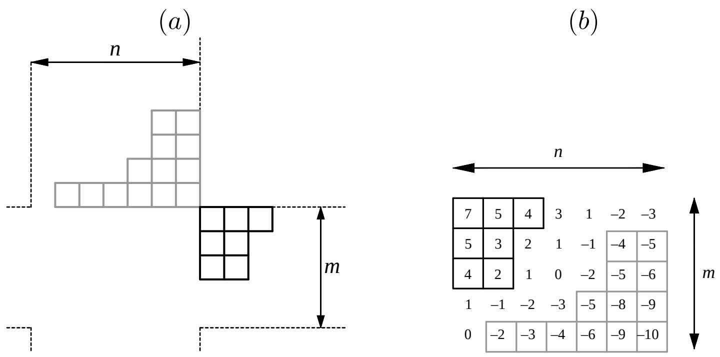

Figure 8. Young diagram of a composite partition $\overline{\nu};\mu$ in (a) the $(m\times n)$-cross and (b) the $(m\times n)$-rectangle, together with its atypicality matrix. Here, $\mu = (3,2,2)$ and $\nu = (6,3,2,2)$.

independent from this choice of simple roots. Let $\Lambda \in \mathfrak{h}^*$; the *atypicality* of Λ, denoted by $a = \text{atyp}(\Lambda)$, is the maximal number of linearly independent roots β_i such that $(\beta_i, \beta_j) = 0$ and $(\Lambda + \rho, \beta_i) = 0$ for all i and j [13]. Such a set $\{\beta_i\}$ is called a Λ-maximal isotropic subset of Δ.

Let the highest weight of an irreducible representation V be given by Λ in the distinguished simple root system, with atypicality a. With respect to another set of simple roots Π' (with the corresponding ρ'), V has a different highest weight Λ'. Then it was shown that $\text{atyp}(\Lambda')$ is also equal to a. In other words, one can speak of the atypicality of the irrep V: atypicality is independent of the choice of simple root system that it is computed in [13].

The purpose of the following is to show that for an atypical critical representation $V = V_\Lambda$ with highest weight Λ in the distinguished basis, there exists another basis Π' in which V has highest weight Λ', in such a way that the Λ'-maximal isotropic subset of Δ' is actually a subset of Π'. In this case, V is called *tame*, and a character formula can be given.

In order to go from Π to Π', we shall follow the technique of simple odd reflections, described in [18].

Let Λ be determined by some composite partition $\overline{\nu};\mu$, so $\Lambda = \Lambda_{\overline{\nu};\mu}$. We also need the notion of the (m,n)-index k of $\overline{\nu};\mu$; this is the number

$$k = \min\Bigg(\bigg\{ i \in \{1,\ldots,m\} | \exists j \in \{1,\ldots,n\} : \\ \mu_i + \langle \mu'_{n-j+1} - m \rangle + (m-i) = \nu'_j + \langle \nu_{m-i+1} - n \rangle + (n-j) \bigg\} \cup \{m+1\} \Bigg) \tag{16}$$

In what follows, k will always denote this number. In the special case where $\nu = 0$, this definition coincides with the one given in [18]. When the representation is typical k will be equal to $m+1$; otherwise k corresponds to the smallest row number in the atypicality matrix in which there occurs a zero. Thus in the following we shall assume that $k \leq m$.

Denote $\Lambda^{(1)} = \Lambda$, $\rho^{(1)} = \rho$ and $\Pi^{(1)} = \Pi$. Now we perform a sequence of simple odd $\alpha^{(i)}$-reflections [18]; each of these reflections preserve $\Delta_{0,+}$ but may change $\Lambda^{(i)} + \rho^{(i)}$ and $\Pi^{(i)}$. Denote the sequence of reflections by:

$$\Lambda^{(1)} + \rho^{(1)},\, \Pi^{(1)} \quad \xrightarrow{\alpha^{(1)}} \quad \Lambda^{(2)} + \rho^{(2)},\, \Pi^{(2)} \quad \xrightarrow{\alpha^{(2)}} \quad \cdots \\ \cdots \quad \xrightarrow{\alpha^{(f)}} \quad \Lambda' + \rho',\, \Pi' \tag{17}$$

where, at each stage, $\alpha^{(i)}$ is an odd root from $\Pi^{(i)}$. For given $\overline{\nu};\mu$, consider the following sequence of odd roots (with positions on row m, row $m-1$, ..., row k):

$$\begin{array}{ll} \text{row } m: & \beta_{m,1}, \beta_{m,2}, \ldots, \beta_{m,\min\{n,\mu_k-k+m\}} \\ \text{row } m-1: & \beta_{m-1,1}, \beta_{m-1,2}, \ldots, \beta_{m-1,\min\{n,\mu_k-k+m-1\}} \\ \vdots & \vdots \\ \text{row } k: & \beta_{k,1}, \beta_{k,2}, \ldots, \beta_{k,\mu_k} \end{array} \tag{18}$$

in this particular order (i.e. starting with $\beta_{m,1}$ and ending with β_{k,μ_k}). Then we have:

Lemma 5.1 *Let $\overline{\nu};\mu$ be $\mathfrak{gl}(m|n)$-standard and critical in $\mathfrak{gl}(m|n)$ and suppose that the diagrams of ν and μ do not overlap in the $(m\times n)$-rectangle. Then the sequence (18) is a proper sequence of simple odd reflections for $\Lambda_{\overline{\nu};\mu}$, i.e. $\alpha^{(i)}$ is a simple odd root from $\Pi^{(i)}$. At the end of the sequence, one finds:*

$$\begin{aligned} \Pi' \;=\; & \{\epsilon_1-\epsilon_2, \epsilon_2-\epsilon_3, \ldots, \epsilon_{k-2}-\epsilon_{k-1}, \epsilon_{k-1}-\delta_1, \\ & \delta_1-\delta_2, \delta_2-\delta_3, \ldots, \delta_{\mu_k-1}-\delta_{\mu_k}, \delta_{\mu_k}-\epsilon_k, \\ & \epsilon_k-\delta_{\mu_k+1}, \delta_{\mu_k+1}-\epsilon_{k+1}, \epsilon_{k+1}-\delta_{\mu_k+2}, \ldots, \\ & \delta_{\mu_k+m-k}-\epsilon_m, \epsilon_m-\delta_{\mu_k+m+1-k}, \\ & \delta_{\mu_k+m+1-k}-\delta_{\mu_k+m+2-k}, \ldots, \delta_{n-1}-\delta_n\}. \end{aligned} \tag{19}$$

Furthermore,

$$\Lambda'+\rho' = \Lambda_\lambda+\rho+\sum_{i=k+1}^{k+a-1}\sum_{j=\mu_i+1}^{\mu_k-k+i}\beta_{i,j}+\sum_{i=k+a}^{m}\sum_{j=\mu_i+1}^{\max\{0,n-\nu_{m-i+1}\}}\beta_{i,j}. \tag{20}$$

This proof is similar to the proof of Lemma 2.3 in [18], and has been given in [19]. In particular, note that criticality is necessary in this process: if Λ is not critical, the sequence of odd reflections can be performed, but the atypicality matrix of Λ' would not have its zeros in the right places so as to find a proper Λ'-maximal isotropic subset.

Note that the technical restriction of “no overlap” (meaning that the Young diagrams of μ and ν do not overlap in the $(m\times n)$-rectangle) is no real restriction for critical representations. Indeed, by the conclusion of the previous section, one can perform a shift $\widetilde{\Lambda}=\Lambda+j\sigma$ such that there is no overlap for the related corresponding partition $\overline{\widetilde{\nu}};\widetilde{\mu}$, with essentially the same character, see (5). From now on, we shall assume that $\overline{\nu};\mu$ is $\mathfrak{gl}(m|n)$-standard, critical and with no overlap.

Corollary 5.2 *The critical representation $V_{\Lambda_{\overline{\nu};\mu}}\equiv V_{\overline{\nu};\mu}$ is tame.*

Proof. Having performed the simple odd reflections (18), one can compute the atypicality matrix for $\Lambda'+\rho'$ using (20). This gives:

$$(\Lambda'+\rho',\beta_{ij})=0 \text{ for all } (i,j) \text{ with } k\le i\le k+a-1, \mu_k+1\le j\le\mu_k+a. \tag{21}$$

Therefore the set

$$S_{\Lambda'} = \{\epsilon_k - \delta_{\mu_k+1}, \epsilon_{k+1} - \delta_{\mu_k+2}, \ldots, \epsilon_m - \delta_{\mu_k+a}\} \tag{22}$$

is a $(\Lambda' + \rho')$-maximal isotropic subset. Furthermore, $S_{\Lambda'} \subset \Pi'$, see (19). This implies that $V_{\overline{\nu};\mu}$ is tame [18]. If $\overline{\nu};\mu$ is not critical, (21) does not hold, and there is no Λ'-maximal isotropic subset that is also a subset of Π'. $\square$

Let us illustrate some of these notions for $\overline{\nu};\mu = (\overline{3},\overline{3});(9,5,3,3,2,2,1)$ in $\mathfrak{gl}(5|7)$. In Figure 9(a), the atypicality matrix associated with $\overline{\nu};\mu$ is given. In Figure 9(b) the positions marked with "i" refer to the $(\Lambda'+\rho')$-maximal isotropic set (22). For convenience, let us refer to these positions as "the isotropic diagonal." The positions of the odd roots that have been used for the sequence of reflections to go from $\Lambda_{\overline{\nu};\mu}$ and Π to Λ' and Π' are marked by "x" in Figure 9(b). So, they are simply all positions to the left of the isotropic diagonal. Finally, Figure 9(c) shows the positions of those β_{ij} that appear on the right hand side of (20); they are marked by "*". These are all positions to the left of the isotropic diagonal that are not inside $F(\overline{\nu};\mu)$. One can see from this example and others that the (m,n)-index k determines all other necessary ingredients.

6 Character formulas and $s_{\overline{\nu};\mu}(x/y)$.

We have just seen that the representation $V_{\overline{\nu};\mu}$ is tame when $\overline{\nu};\mu$ is $\mathfrak{gl}(m|n)$-standard, critical, and with no overlap in the $(m \times n)$-rectangle.

For tame representations, a character formula is known due to Kac and Wakimoto [13]. It reads, in terms of Λ':

$$\operatorname{ch} V_{\overline{\nu};\mu} = j_{\Lambda'}^{-1} e^{-\rho'} R'^{-1} \sum_{w\in W} \varepsilon(w) w\Big(e^{\Lambda'+\rho'} \prod_{\beta\in S_{\Lambda'}} (1+e^{-\beta})^{-1}\Big), \tag{23}$$

where

$$R' = \prod_{\alpha\in\Delta_{0,+}} (1-e^{-\alpha}) \Big/ \prod_{\alpha\in\Delta'_{1,+}} (1+e^{-\alpha}) \tag{24}$$

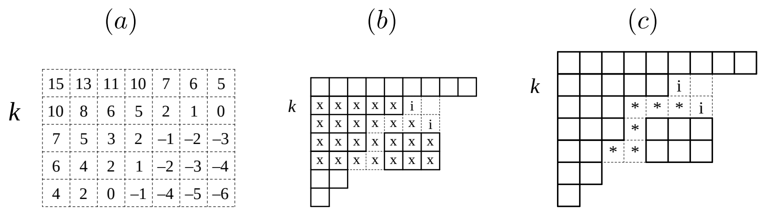

Figure 9. Young diagram of $\overline{\nu};\mu$ in the $(m \times n)$-rectangle, for $\mu = (9,5,3,3,2,2,1)$ and $\nu = (3,3)$. In (a), the atypicality matrix and (m,n)-index k are determined; in (b) and (c) the ingredients used in the sequence of odd reflections are indicated.

and $j_{\Lambda'}$ is a normalization coefficient to make sure that the coefficient of $e^{\Lambda'}$ on the right hand side of (23) is 1. By definition of ρ and R

$$e^{-\rho'}R'^{-1} = e^{-\rho}R^{-1}.$$

As usual in this context we put

$$x_i = e^{\epsilon_i}, \qquad y_j = e^{\delta_j} \qquad (1 \le i \le m, 1 \le j \le n). \tag{25}$$

Now we have

$$\mathrm{ch}\, V_{\overline{\nu};\mu} = j^{-1}_{\Lambda'_{\overline{\nu};\mu}} D^{-1} \sum_{w \in W} \varepsilon(w) w(t_{\overline{\nu};\mu}),$$

with

$$D = \frac{\prod_{1\le i<j\le m}(x_i - x_j) \prod_{1\le i<j\le n}(y_i - y_j)}{\prod_{i=1}^m \prod_{j=1}^n (x_i + y_j)} \tag{26}$$

and

$$\begin{aligned} t_{\overline{\nu};\mu} &= \prod_{i=1}^{k-1} x_i^{\mu_i+m-i-n} \prod_{j=1}^{l-1} y_j^{\mu'_j+n-j-m} \prod_{i=k}^{k+a-1} \frac{y^r_{i-k+l}}{x_i^r(x_i + y_{i-k+l})} \\ &\quad \times \prod_{i=k+a}^{n} x_i^{m-i-\nu_{m-i+1}} \prod_{j=l+a}^{n} y_j^{n-j-\nu'_{n-j+1}} \end{aligned} \tag{27}$$

where $l = \mu_k + 1$ and $r = n - m + k - l$ and $j_{\Lambda'_{\overline{\nu};\mu}} = a!$ (due to symmetry there are $a!$ elements of $S_m \times S_n$ that leave $t_{\overline{\nu};\mu}$ invariant).

This expression can be written in a nicer form:

Theorem 6.1 *Let $t_{\overline{\nu};\mu}$ be given by (27) and $r = n - m + k - \mu_k - 1$. Then*

$$\frac{1}{a!} \sum_{w \in S_m \times S_n} \varepsilon(w) w(t_{\overline{\nu};\mu}) = (-1)^{(m-a)(l-1)+n(m-a-k+1)} \det(C), \tag{28}$$

where C is the following square matrix of order $n + m - a$:

$$C = \begin{pmatrix} 0 & Y_{\mu'} & 0 \\ X_\mu & R^{(r)} & X_\nu \\ 0 & Y_{\nu'} & 0 \end{pmatrix} \qquad \textit{with} \quad R^{(r)} = \left(\frac{y_j^r}{x_i^r(x_i + y_j)} \right)_{1\le i\le m,\ 1\le j\le n} \tag{29}$$

and with

$$\begin{aligned} X_\mu &= \left(x_i^{\mu_j+m-n-j} \right)_{1\le i\le m,\ 1\le j\le k-1}, \\ X_\nu &= \left(x_i^{m-j-\nu_{m-j+1}} \right)_{1\le i\le m,\ k+a\le j\le m}, \\ Y_{\mu'} &= \left(y_j^{\mu'_i+n-m-i} \right)_{1\le i\le l-1,\ 1\le j\le n}, \\ Y_{\nu'} &= \left(y_j^{n-i-\nu'_{n-i+1}} \right)_{l+a\le i\le n,\ 1\le j\le n}. \end{aligned}$$

Proof. The proof is similar to that of [18][Lemma 3.1]. Apply Laplace's theorem for the expansion of $\det(C)$ with respect to columns $1, 2, \ldots, k-1, k+n, k+n+1, \ldots, n+m-a$. Keeping track of the zero blocks, one finds

$$\begin{aligned} \det(C) &= (-1)^{\frac{(m-a)(m-a+1)}{2}} \sum_{1\leq i_1<\cdots<i_{m-a}\leq m} (-1)^{i_1+\cdots+i_{m-a}+(m-a)(l-1)} \\ &\quad \times \det(C_x)\det(C_y), \end{aligned} \tag{30}$$

where C_x is the $(m-a)\times(m-a)$-matrix consisting of rows $i_1, i_2, \ldots, i_{m-a}$ of the matrix $\begin{pmatrix} X_\mu & X_\nu \end{pmatrix}$, and C_y is the $n\times n$-matrix

$$\begin{pmatrix} Y_{\mu'} \\ \tilde{R}^{(r)} \\ Y_{\nu'} \end{pmatrix},$$

where $\tilde{R}^{(r)}$ is obtained by removing rows $i_1, i_2, \ldots, i_{m-a}$ in $R^{(r)}$. The number of terms on the rhs of (30) is $\binom{m}{m-a}(m-a)!n! = m!n!/a!$; due to symmetry considerations this is the same as the number of distinct terms on the lhs of (28). For $(i_1, \ldots, i_{m-a}) = (1, \ldots, k-1, k+n, ..., n+m-a)$, and the diagonal term in $\det C_x$ and $\det C_y$, the contribution on the rhs of (30) is now easily seen to be $(-1)^{(m-a)(l-1)+n(m-a-k+1)}t_{\overline{\nu};\mu}$. But by definition of the determinant, every term on the rhs of (30) is (up to the overall sign factor $(-1)^{(m-a)(l-1)}$) of the form $\varepsilon(w)w(t_{\overline{\nu};\mu})$ with $w\in S_m\times S_n$. Conversely, every term of the form $\varepsilon(w)w(t_{\overline{\nu};\mu})$ appears as a term on the rhs of (30). It follows that (28) holds. $\square$

With the same notation, one finds

Corollary 6.2 *The character of a critical representation labelled by a $\mathfrak{gl}(m|n)$-standard composite partition $\overline{\nu};\mu$ (without overlap) has the following determinantal form:*

$$\mathrm{ch}\, V_{\overline{\nu};\mu} = (-1)^{(m-a)(l-1)+n(m-a-k+1)}D^{-1}\det(C).$$

As an example, let $m=4$, $n=5$ and $\overline{\nu};\mu = (\overline{1},\overline{1},\overline{4});(3,1)$. One finds

6	4	3	2	–1
3	1	0	–1	–4
1	–1	–2	–3	–6
0	–2	–3	–4	–7

$$\begin{aligned} k &= 2 \\ l &= \mu_k+1=2 \\ a &= 2 \\ &\Rightarrow\ r = n-m+k-l=1 \\ &\Rightarrow\ n+m-a=7 \end{aligned}$$

Thus, according to formula (29), $\mathrm{ch}\, V_{(\overline{1},\overline{1},\overline{4});(3,1)} = D^{-1}\ \det(C)$ with C given by the matrix

$$\begin{pmatrix} 0 & y_1^2 & y_2^2 & y_3^2 & y_4^2 & y_5^2 & 0 \\ x_1 & \frac{y_1}{x_1(x_1+y_1)} & \frac{y_2}{x_1(x_1+y_2)} & \frac{y_3}{x_1(x_1+y_3)} & \frac{y_4}{x_1(x_1+y_4)} & \frac{y_5}{x_1(x_1+y_5)} & x_1^{-4} \\ x_2 & \frac{y_1}{x_2(x_2+y_1)} & \frac{y_2}{x_2(x_2+y_2)} & \frac{y_3}{x_2(x_2+y_3)} & \frac{y_4}{x_2(x_2+y_4)} & \frac{y_5}{x_2(x_2+y_5)} & x_2^{-4} \\ x_3 & \frac{y_1}{x_3(x_3+y_1)} & \frac{y_2}{x_3(x_3+y_2)} & \frac{y_3}{x_3(x_3+y_3)} & \frac{y_4}{x_3(x_3+y_4)} & \frac{y_5}{x_3(x_3+y_5)} & x_3^{-4} \\ x_4 & \frac{y_1}{x_4(x_4+y_1)} & \frac{y_2}{x_4(x_4+y_2)} & \frac{y_3}{x_4(x_4+y_3)} & \frac{y_4}{x_4(x_4+y_4)} & \frac{y_5}{x_4(x_4+y_5)} & x_4^{-4} \\ 0 & y_1^0 & y_2^0 & y_3^0 & y_4^0 & y_5^0 & 0 \\ 0 & y_1^{-3} & y_2^{-3} & y_3^{-3} & y_4^{-3} & y_5^{-3} & 0 \end{pmatrix}.$$

Thus the determinantal formula is very explicit.

The main goal of this determinantal formula however is that it allows us to make the link with another explicit formula that is even more useful, namely a composite supersymmetric S-function. In order to make this connection, we need to introduce some notations and properties of (supersymmetric) S-functions. We shall assume that the reader is familiar with notations of ordinary S-functions [17], such as $s_\lambda(x)$, $s_{\lambda/\mu}(x)$, $c^\nu_{\lambda\mu}$ for Littlewood-Richardson coefficients, etc.

The "contravariant" S-functions are usually defined in terms of the ordinary (or "covariant") S-functions. Suppose we have a set of variables $x = (x_1,\ldots,x_m)$. For a partition λ, let $\overline{\lambda} = (-\lambda_1,-\lambda_2,\ldots)$ and denote by $\overline{x_i} = \frac{1}{x_i}$, for all $i = 1,\ldots,m$, then,

$$s_{\overline{\lambda}}(x) = s_\lambda(\overline{x}). \tag{31}$$

Similarly, $s_{\overline{\lambda/\mu}}(x) = s_{\lambda/\mu}(\overline{x})$. Using the contravariant S-functions, the composite or "mixed" S-functions are defined [5] by

$$s_{\overline{\nu};\mu}(x) = \sum_\zeta (-1)^{|\zeta|} s_{\overline{\nu/\zeta}}(x) s_{\mu/\zeta'}(x). \tag{32}$$

The product of a covariant and a contravariant S-function is given by

$$s_{\overline{\nu}}(x)s_\mu(x) = s_\nu(\overline{x})s_\mu(x) = \sum_\eta s_{\overline{\nu/\eta};\mu/\eta}(x) \tag{33}$$

where

$$s_{\overline{\nu/\eta};\mu/\eta}(x) = \sum_{\varphi,\psi} c^\nu_{\varphi\eta} c^\mu_{\psi,\eta} s_{\overline{\varphi};\psi}(x). \tag{34}$$

The composite S-functions can also be written in terms of a decomposition [15] of $x = x' + x''$, namely

$$s_{\overline{\nu};\mu}(x) = \sum_{\rho,\sigma,\tau} s_{\overline{\nu/\sigma};\mu/\tau}(x') s_{\overline{\sigma/\rho};\tau/\rho}(x''). \tag{35}$$

In [6] the composite supersymmetric S-functions are defined in terms of ordinary composite S-functions, namely:

$$s_{\overline{\nu};\mu}(x/y) = \sum_{\rho,\zeta,\xi} s_{\overline{\nu/\xi};\mu/\zeta}(x) s_{\overline{\xi'/\rho'};\zeta'/\rho'}(y) = \sum_{\rho,\varphi,\psi} s_{\overline{\nu/\varphi\rho};\mu/\psi\rho}(x) s_{\overline{\varphi'};\psi'}(y). \tag{36}$$

and it can be generalized to composite skew partitions:

$$s_{\overline{\nu/\eta};\lambda/\mu}(x/y) = \sum_{\rho,\sigma,\tau} s_{\overline{\nu/\eta\sigma};\lambda/\mu\tau}(x) s_{\overline{(\sigma/\rho)'};(\tau/\rho)'}(y). \tag{37}$$

The functions $s_{\overline{\nu};\mu}(x/y)$ have many properties similar to ordinary Schur functions [5, 6, 9, 10, 16]. For example [6],

$$s_{\overline{\nu};\mu}(x/y) = \det \begin{pmatrix} h_{\nu_l+k-l}(\overline{x}/\overline{y}) & h_{\mu_j-k-j+1}(x/y) \\ h_{\nu_l-i-l+1}(\overline{x}/\overline{y}) & h_{\mu_j+i-j}(x/y) \end{pmatrix} \tag{38}$$

where i, j, k resp. l runs from top to bottom, from left to right, from bottom to top, resp. from right to left, and $h_r(x/y)$ are the complete supersymmetric polynomials defined by $h_r(x/y) = \sum_{k=0}^{r} h_{r-k}(x)e_k(y)$.

We can now formulate our main result:

Theorem 6.3 *Let $\overline{\nu};\mu$ be a $\mathfrak{gl}(m|n)$-standard and critical composite partition with no overlap. The character $\mathrm{ch}\, V_{\overline{\nu};\mu}$ is equal to $s_{\overline{\nu};\mu}(x/y)$.*

The proof is similar to the proof of [18, Theorem 5.5]. However, there are many technical details which need to be reinvestigated, see the Appendix of [19].

7 Conclusions

The determination of characters for $\mathfrak{gl}(m|n)$ irreps has a long history, see [21] (where a complete solution is given) and references therein. In our work we emphasize the relation between characters and supersymmetric S-functions. It was known for a long time that the characters of typical, of (typical or atypical) covariant and of (typical or atypical) contravariant representations are given by supersymmetric S-functions. In the current paper a special class of atypical representations, namely the critical irreps, was described. Also for this class of representations does the character coincide with a supersymmetric S-function.

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