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Lie superalgebraic framework for generalization of quantum statistics

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Abstract

Para-Bose and para-Fermi statistics are known to be associated with representations of the Lie (super)algebras of class B. We develop a framework for the generalization of quantum statistics based on the Lie superalgebras $A(m|n)$, $B(m|n)$, $C(n)$ and $D(m|n)$.

1 Introduction

It has been known for more than 50 years that generalizations of ordinary Bose and Fermi quantum statistics are possible if one abandons the requirement for the commutator or anticommutator of two fields to be a c-number. The commutation (resp. anticommutation) relations between the Bose (resp. Fermi) creation and annihilation operators (CAOs) can be replaced by a weaker system of triple relations for the so-called para-Bose operators [1]

$$[\{B_j^\xi, B_k^\eta\}, B_l^\epsilon] = (\epsilon - \xi)\delta_{jl} B_k^\xi + (\epsilon - \eta)\delta_{kl} B_j^\eta, \\ \xi, \eta, \epsilon = \pm;\ j, k, l = 1, \ldots, n \tag{1}$$

and para-Fermi operators [1]

$$[[F_j^\xi, F_k^\eta], F_l^\epsilon] = \frac{1}{2}(\epsilon - \eta)^2 \delta_{kl} F_j^\xi - \frac{1}{2}(\epsilon - \xi)^2 \delta_{jl} F_k^\eta, \\ \xi, \eta, \epsilon = \pm \text{ or } \pm 1; \quad j, k, l = 1, \ldots, n. \tag{2}$$

It was shown by Kamefuchi and Takahashi [2], and by Ryan and Sudarshan [3], that the Lie algebra generated by the $2n$ elementss F_i^ξ subject to the relations (2) is the Lie algebra $so(2n+1) \equiv B_n$. Similarly Ganchev and Palev [4] discovered a new connection, namely between para-Bose statistics and the orthosymplectic

Lie superalgebra (LS) $osp(1|2n) \equiv B(0|n)$ [5]. The LS generated by $2n$ odd elements B_i^{ξ}, subject to the relations (1) is $osp(1|2n) \equiv B(0|n)$ [5]. Therefore para-statistics can be associated with representations of the Lie (super)algebras of class B. Alternative types of generalized quantum statistics in the framework of other classes of simple Lie algebras or superalgebras have been considered in particular by Palev [6]- [14]. Furthermore, a complete classification of all the classes of generalized quantum statistics for the classical Lie algebras A_n, B_n, C_n and D_n, by means of their algebraic relations, was given in [15]. In the present paper we make a similar classification for the basic classical Lie superalgebras.

2 Preliminaries, definition and classification method

Let G be a basic classical Lie superalgebra. G has a $\mathbb{Z}_2$-grading $G = G_{\bar{0}} \oplus G_{\bar{1}}$; an element x of $G_{\bar{0}}$ is an even element ($\deg(x) = 0$), an element y of $G_{\bar{1}}$ is an odd element ($\deg(y) = 1$). The Lie superalgebra bracket is denoted by $[\![x, y]\!]$. In the universal enveloping algebra of G, this stands for

$$[\![x, y]\!] = xy - (-1)^{\deg(x)\deg(y)} yx,$$

if x and y are homogeneous. So the bracket can be a commutator or an anti-commutator.

A generalized quantum statistics associated with G is determined by N creation operators x_i^+ and N annihilation operators x_i^-. Inspired by the para-statistics, Palev's statistics and [15], these $2N$ operators should generate the Lie superalgebra G, subject to certain triple relations. Let G_{+1} and G_{-1} be the subspaces of G spanned by the CAOs:

$$G_{+1} = \text{span}\{x_i^+;\ i = 1\ldots, N\}, \qquad G_{-1} = \text{span}\{x_i^-;\ i = 1\ldots, N\}. \quad (3)$$

We do not require that these subspaces are homogeneous. Putting $G_{\pm 2} = [\![G_{\pm 1}, G_{\pm 1}]\!]$ and $G_0 = [\![G_{+1}, G_{-1}]\!]$, the condition that G is generated by the $2N$ elements subject to triple relations only, leads to the requirement that $G = G_{-2} \oplus G_{-1} \oplus G_0 \oplus G_{+1} \oplus G_{+2}$, and this must be a $\mathbb{Z}$-grading of G. Since these subspaces are not necessarily homogeneous, this $\mathbb{Z}$-grading is in general not consistent with the $\mathbb{Z}_2$-grading.

We impose two further requirements: first of all, the generating elements $x_i^{\pm}$ must be root vectors of G. Secondly, $\omega(x_i^+) = x_i^-$, where ω is the standard antilinear anti-involutive mapping of G (in terms of root vectors e_α, ω satisfies $\omega(e_\alpha) = e_{-\alpha}$). This leads to the following definition:

Definition 1 *Let G be a basic classical Lie superalgebra, with antilinear anti-involutive mapping ω. A set of $2N$ root vectors $x_i^{\pm}$ ($i = 1, \ldots, N$) is called a set of creation and annihilation operators for G if:*

- $\omega(x_i^{\pm}) = x_i^{\mp}$,
- $G = G_{-2} \oplus G_{-1} \oplus G_0 \oplus G_{+1} \oplus G_{+2}$ *is a $\mathbb{Z}$-grading of G, with $G_{\pm 1} =$ span$\{x_i^{\pm},\ i = 1\ldots, N\}$ and $G_{j+k} = [\![G_j, G_k]\!]$.*

The algebraic relations $\mathcal{R}$ satisfied by the operators $x_i^\pm$ are the relations of a generalized quantum statistics (GQS) associated with G.

A consequence of this definition is that the algebraic relations $\mathcal{R}$ consist of quadratic and triple relations only. Another consequence is that G_0 is a subalgebra of G spanned by root vectors of G, i.e. G_0 is a regular subalgebra of G. By the adjoint action, the remaining G_i's are G_0-modules. Thus the following technique can be used in order to obtain a complete classification of all GQS associated with G:

1. Determine all regular subalgebras G_0 of G.

2. For each regular subalgebra G_0, determine the decomposition of G into simple G_0-modules g_k ($k = 1, 2, \ldots$).

3. Investigate whether there exists a $\mathbb{Z}$-grading of G of the form

$$G = G_{-2} \oplus G_{-1} \oplus G_0 \oplus G_{+1} \oplus G_{+2}, \tag{4}$$

   where each G_i is either directly a module g_k or else a sum of such modules $g_1 \oplus g_2 \oplus \cdots$, such that $\omega(G_{+i}) = G_{-i}$.

If the $\mathbb{Z}$-grading is of the form (4) with $G_{\pm 2} \neq 0$, we shall say that it has *length* 5; if $G_{+2} = 0$ (then $G_{-2} = 0$, but $G_{\pm 1} \neq 0$), then the $\mathbb{Z}$-grading is of length 3.

In the following section we shall give a summary of the classification process for the basic classical Lie superalgebras $A(m|n)$, $B(m|n)$, $B(0|n)$, $D(m|n)$ and $C(n)$. For more details on the classification techniques, see [16].

3 Classification

3.1 The Lie superalgebra $A(m|n)$

Let G be the special linear Lie superalgebra $A(m|n) \equiv sl(m+1|n+1)$, consisting of traceless $(m+n+2) \times (m+n+2)$ matrices. The root vectors of G are known to be the elements e_{jk} ($j \neq k = 1, \ldots, m+n+2$), where e_{jk} is a matrix with zeros everywhere except a 1 on the intersection of row j and column k. The $\mathbb{Z}_2$-grading is such that $\deg(e_{jk}) = \theta_{jk} = \theta_j + \theta_k$, where

$$\theta_j = \begin{cases} 0 & \text{if} \quad j = 1, \cdots, m+1 \\ 1 & \text{if} \quad j = m+2, \cdots, m+n+2. \end{cases} \tag{5}$$

The root corresponding to e_{jk} ($j, k = 1, \ldots, m+1$) is given by $\epsilon_j - \epsilon_k$; for $e_{m+1+j,m+1+k}$ ($j, k = 1, \ldots, n+1$) it is $\delta_j - \delta_k$; and for $e_{j,m+1+k}$, resp. $e_{m+1+k,j}$, ($j = 1, \ldots, m+1$, $k = 1, \ldots, n+1$) it is $\epsilon_j - \delta_k$, resp. $\delta_k - \epsilon_j$. The anti-involution is such that $\omega(e_{jk}) = e_{kj}$.

In order to find regular subalgebras of $G = A(m|n)$, one should delete nodes from the Dynkin diagrams of $A(m|n)$ (first the ordinary, and then the extended). **Step 1.** Delete node i from the distinguished Dynkin diagram. Then $A(m|n) = G_{-1} \oplus G_0 \oplus G_{+1}$, with $G_0 = sl(i) \oplus sl(m+1-i|n+1)$ for $i = 1, \ldots, m+1$ and

$G_0 = sl(m+1|i-m-1) \oplus sl(n+m+2-i)$ for $i = m+2, \dots, m+n+1$; $G_{-1} = \mathrm{span}\{e_{kl};\ k = 1, \dots, i,\ l = i+1, \dots, m+n+2\}$; $G_{+1} = \mathrm{span}\{e_{lk};\ k = 1, \dots, i,\ l = i+1, \dots, m+n+2\}$ and $N = i(m+n+2-i)$.

For $i = 1$, $N = m+n+1$. Putting $a_j^- = e_{1,j+1}, \quad a_j^+ = e_{j+1,1}, \qquad j = 1, \cdots, m+n+1$, the relations $\mathcal{R}$ are:

$$\begin{aligned}
&[\![a_j^+, a_k^+]\!] = [\![a_j^-, a_k^-]\!] = 0, \\
&[\![[\![a_j^+, a_k^-]\!], a_l^+]\!] = (-1)^{\theta_{j+1}} \delta_{jk} a_l^+ + \delta_{kl} a_j^+, \qquad (6)\\
&[\![[\![a_j^+, a_k^-]\!], a_l^-]\!] = -(-1)^{\theta_{j+1}} \delta_{jk} a_l^- - (-1)^{\theta_{j+1,k+1}\theta_{l+1}} \delta_{jl} a_k^-.
\end{aligned}$$

For $m = 0$, these are the relations of A-superstatistics [10], [14]. Also for general m and n, these relations have been considered in another context [13].

For $i = 2$, $N = 2(m+n)$. One puts

$$\begin{aligned}
&a_{-,j}^- = e_{1,j+2}, \qquad a_{+,j}^- = e_{2,j+2}, \qquad j = 1, \dots, m+n, \\
&a_{-,j}^+ = e_{j+2,1}, \qquad a_{+,j}^+ = e_{j+2,2}, \qquad j = 1, \dots, m+n.
\end{aligned}$$

Then the corresponding relations read ($\xi, \eta, \epsilon = \pm$; $j, k, l = 1, \dots, m+n$):

$$\begin{aligned}
&[\![a_{\xi j}^+, a_{\eta k}^+]\!] = [\![a_{\xi j}^-, a_{\eta k}^-]\!] = 0, \\
&[\![a_{\xi j}^+, a_{-\xi k}^-]\!] = 0, \qquad [\![a_{-j}^+, a_{-k}^-]\!] = [\![a_{+j}^+, a_{+k}^-]\!], \qquad j \neq k, \qquad (7)\\
&[\![a_{+j}^+, a_{-j}^-]\!] = [\![a_{+k}^+, a_{-k}^-]\!], \qquad \text{for } \theta_j = \theta_k, \\
&[\![a_{-j}^+, a_{+j}^-]\!] = [\![a_{-k}^+, a_{+k}^-]\!], \qquad \text{for } \theta_j = \theta_k, \\
&[\![[\![a_{\xi j}^+, a_{\eta k}^-]\!], a_{\epsilon l}^+]\!] = (-1)^{\deg(a_{\xi j}^+)\deg(a_{\eta k}^-) + \delta_{\xi,-\eta}\theta_{12}\deg(a_{\epsilon l}^+)} \delta_{\eta\epsilon}\delta_{jk} a_{\xi l}^+ \\
&\qquad\qquad + \delta_{\xi\eta}\delta_{kl} a_{\epsilon j}^+, \\
&[\![[\![a_{\xi j}^+, a_{\eta k}^-]\!], a_{\epsilon l}^-]\!] = -(-1)^{\deg(a_{\xi j}^+)\deg(a_{\eta k}^-)} \delta_{\xi\epsilon}\delta_{jk} a_{\eta l}^- \\
&\qquad\qquad - (-1)^{\theta_{j+2,k+2}\deg(a_{\epsilon l}^-)} \delta_{\xi\eta}\delta_{jl} a_{\epsilon k}^-.
\end{aligned}$$

Step 2. Delete node i and j from the distinguished Dynkin diagram. We have $G_0 = H + sl(i) \oplus sl(j-i) \oplus sl(m+1-j|n+1)$ for $1 \leq i < j \leq m+1$, $G_0 = H + sl(i) \oplus sl(m+1-i|j-m-1) \oplus sl(m+n+2-j)$ for $1 \leq i \leq m+1$, $m+2 \leq j \leq m+n+1$ and $G_0 = H + sl(m+1|i-m-1) \oplus sl(j-i) \oplus sl(m+n+2-j)$ for $m+2 \leq i < j \leq m+n+1$. There are six simple G_0-modules. All the possible combinations of these modules give rise to gradings of length 5. There are three different ways in which these G_0-modules can be combined. To characterize these three cases, it is sufficient to give only G_{-1}:

$$\begin{aligned}
G_{-1} &= \mathrm{span}\{e_{kl}, e_{lp};\ k = 1, \dots, i,\ p = j+1, \dots, m+n+2, \\
&\qquad l = i+1, \dots, j\}, \quad \text{with } N = (j-i)(m+n+2-j+i); \qquad (8)\\
G_{-1} &= \mathrm{span}\{e_{kl}, e_{pk};\ k = 1, \dots, i,\ p = j+1, \dots, m+n+2\}, \\
&\qquad l = i+1, \dots, j,\ \text{with } N = i(m+n+2-i); \qquad (9)\\
G_{-1} &= \mathrm{span}\{e_{kl}, e_{lp};\ k = 1, \dots, i,\ l = j+1, \dots, m+n+2\}, \\
&\qquad p = i+1, \dots, j,\ \text{with } N = j(m+n+2-j). \qquad (10)
\end{aligned}$$

For $j-i=1$ one can label the CAOs as follows: $a_k^-=e_{k,i+1},\ a_k^+=e_{i+1,k},\ k=1,\ldots,i;\ \ a_k^-=e_{i+1,k+1},\ \ a_k^+=e_{k+1,i+1},\ \ k=i+1,\ldots,m+n+1$. Using

$$\langle k\rangle=\left\{\begin{array}{lll} 0 & \text{if} & k=1,\ldots,i,\\ 1 & \text{if} & k=i+1,\ldots,m+n+1,\end{array}\right. \tag{11}$$

the quadratic and triple relations now read:

$$\begin{aligned}
&[\![a_k^+,a_l^+]\!]=[\![a_k^-,a_l^-]\!]=0,\ k,l=1,\ldots,i \text{ or } k,l=i+1,\ldots,m+n+1,\\
&[\![a_k^-,a_l^+]\!]=[\![a_k^+,a_l^-]\!]=0,\ k=1,\ldots,i,\ l=i+1,\ldots,m+n+1,\\
&[\![[\![a_k^+,a_l^-]\!],a_p^+]\!]=(-1)^{\langle l\rangle+\langle p\rangle+\langle k\rangle\theta_{k+1,i+1}}\delta_{kl}a_p^+\\
&\qquad\qquad +(-1)^{\langle l\rangle+\langle p\rangle+(1-\langle l\rangle)\theta_{l,i+1}(\theta_{lk}+\theta_{k,i+1})}\delta_{lp}a_k^+,\\
&\qquad\qquad k,l=1,\cdots,i,\ \text{or } k,l=i+1,\ldots,m+n+1,\\
&[\![[\![a_k^+,a_l^-]\!],a_p^-]\!]=-(-1)^{\langle l\rangle+\langle p\rangle+\deg(a_k^+)[\langle k\rangle\theta_{k+1,l+1}+(1-\langle l\rangle)\theta_{l,i+1}]}\delta_{kp}a_l^-\\
&\qquad\qquad -(-1)^{\langle l\rangle+\langle p\rangle+\langle k\rangle\theta_{k+1,i+1}}\delta_{kl}a_p^-,\\
&\qquad\qquad k,l=1,\cdots,i,\ \text{or } k,l=i+1,\ldots,m+n+1,\\
&[\![[\![a_k^\xi,a_l^\xi]\!],a_p^{-\xi}]\!]=-(-1)^{\frac12\theta_{p,i+1}[(1+\xi)\theta_{l+1,i+1}+(1-\xi)\theta_{k,l+1}]}\delta_{kp}a_l^\xi\\
&\qquad\qquad +(-1)^{\frac12(1+\xi)\theta_{l+1,i+1}(\theta_{k,i+1}+\theta_{k,l+1})}\delta_{lp}a_k^\xi,\\
&\qquad\qquad k=1,\ldots,i,\ l=i+1,\ldots,m+n+1,\\
&[\![[\![a_k^\xi,a_l^\xi]\!],a_p^\xi]\!]=0,\qquad \xi=\pm;\ k,l,p=1,\ldots,m+n+1.
\end{aligned} \tag{12}$$

Step 3. If we delete three or more nodes from the distinguished Dynkin diagram, the resulting $\mathbb{Z}$-gradings of $A(m|n)$ are no longer of the required form.

Step 4. If we delete node i from the extended distinguished Dynkin diagram, the remaining diagram is again (a non-distinguished Dynkin diagram) of type $A(m|n)$, so $G_0=G$, and there are no CAOs.

Step 5. Delete node i and j ($i<j$) from the extended distinguished Dynkin diagram. Then $A(m|n)=G_{-1}\oplus G_0\oplus G_{+1}$ with $G_0=H+sl(m|n+1)$ or $H+sl(m+1|n)$ when the nodes are adjacent, and $G_0=H+sl(k|l)\oplus sl(p|q)$ with $k+p=m+1$ and $l+q=n+1$ when the nodes are nonadjacent.

$$G_{-1}=\text{span}\{e_{kl};\ k=i+1\ldots,j,\ l\neq i+1,\ldots,j\}$$

and $N=(j-i)(n+m+2-j+i)$.

Step 6. Delete nodes i, j and k from the extended distinguished Dynkin diagram ($i<j<k$). For three adjacent nodes $G_0=H+sl(m-1|n+1)$, $H+sl(m|n)$ or $H+sl(m+1|n-1)$. For two adjacent and one nonadjacent nodes $G_0=H+sl(l|p)\oplus sl(q|r)$ with $l+q=m$, $p+r=n+1$ or $l+q=m+1$, $p+r=n$. If all three nodes are nonadjacent $G_0=H+sl(l|p)\oplus sl(q|r)\oplus sl(s|t)$ with $l+q+s=m+1$, $p+r+t=n+1$. One or two of these three Lie superalgebras is $sl(r|0)=sl(0|r)=sl(r)$. There are three different ways in

which the corresponding G_0-modules can be combined. We give here only G_{-1}:

$$\begin{aligned}
G_{-1} &= \text{span}\{e_{ps}, e_{sq};\ p = 1, \ldots, i, k+1, \ldots, n+m+2,\\
&\quad s = i+1, \ldots, j,\ q = j+1, \ldots, k\},\\
&\quad \text{with } N = (j-i)(n+m+2-j+i);\\
G_{-1} &= \text{span}\{e_{ps}, e_{qp};\ p = 1, \ldots, i, k+1, \ldots, n+m+2,\\
&\quad s = i+1, \ldots, j,\ q = j+1, \ldots, k\},\\
&\quad \text{with } N = (k-i)(n+m+2+i-k);\\
G_{-1} &= \text{span}\{e_{pq}, e_{qs};\ p = 1, \ldots, i, k+1, \ldots, n+m+2,\\
&\quad s = i+1, \ldots, j,\ q = j+1, \ldots, k\},\\
&\quad \text{with } N = (k-j)(n+m+2+j-k).
\end{aligned}$$

Step 7. If we delete four or more nodes from the extended distinguished Dynkin diagram the $\mathbb{Z}$-grading of $A(m|n)$ satisfies no longer the required properties.

Step 8. Next, one should repeat the process for all nondistinguished Dynkin diagrams of G and their extensions. The only new result corresponds to Step 6 deleting three nonadjacent nodes from the extended Dynkin diagram. We have $G_0 = H + sl(l|p) \oplus sl(q|r) \oplus sl(s|t)$ with $l+q+s = m+1$, $p+r+t = n+1$ and in some cases none of the three algebras is $sl(r|0) = sl(0|r) = sl(r)$.

3.2 The Lie superalgebras $B(m|n)$

We summarize the classification process for the Lie superalgebras $B(m|n)$ giving for all nonisomorphic GQS the subalgebra G_0 (each G_0 contains the complete Cartan subalgebra H, so we only list the remaining part of $G_0 = H + \cdots$); the length ℓ of the $\mathbb{Z}$-grading and the number N of annihilation operators:

$G_0 = H + \cdots$	ℓ	N
$sl(k\|l) \oplus B(m-k\|n-l)$ ($k = 0, \ldots, m$; $l = 0, \ldots, n$; $(k,l) \notin \{(0,0),(1,0)\}$)	5	$(k+l)(2m-2k+2n-2l+1)$
$B(m-1\|n)$ $[(k,l) = (1,0)]$	3	$2m+2n-1$

The most interesting case is with $k = m, l = n$. Then $G_0 = sl(m|n)$, $N = n+m$ and the CAOs:

$$\begin{aligned}
&b_j^- \equiv B_j^- = -\sqrt{2}(e_{2m+1,2m+1+n+j} + e_{2m+1+j,2m+1}),\\
&b_j^+ \equiv B_j^+ = \sqrt{2}(e_{2m+1,2m+1+j} - e_{2m+1+n+j,2m+1}),\\
&b_{n+k}^- \equiv F_k^- = \sqrt{2}(e_{k,2m+1} - e_{2m+1,m+k}),\\
&b_{n+k}^+ \equiv F_k^+ = \sqrt{2}(e_{2m+1,k} - e_{m+k,2m+1}),\\
&\quad j = 1, \ldots, n;\ \ k = 1, \ldots, m,
\end{aligned}$$

with

$$\deg(b_j^{\pm}) = \langle j \rangle = \begin{cases} 1 & \text{if} \quad j = 1, \ldots, n \\ 0 & \text{if} \quad j = n+1, \ldots, n+m \end{cases}$$

satisfy only triple relations:

$$[\![[\![b_j^{\xi}, b_k^{\eta}]\!], b_l^{\epsilon}]\!] = -2\delta_{jl}\delta_{\epsilon,-\xi}\epsilon^{\langle l \rangle}(-1)^{\langle k \rangle\langle l \rangle} b_k^{\eta} + 2\epsilon^{\langle l \rangle}\delta_{kl}\delta_{\epsilon,-\eta} b_j^{\xi},$$
$$\xi, \eta, \epsilon = \pm \text{ or } \pm 1; \quad j, k, l = 1, \ldots, n+m.$$

Note that $B_j^{\pm}, j = 1, \ldots, n$ (resp. $F_k^{\pm}, k = 1, \ldots, m$) are para-Bose (1) (resp. para-Fermi (2)) CAOs. The fact that $B(m|n)$ can be generated by n pairs of para-Bose and m pairs of para-Fermi operators has been discovered by Palev [17].

In the next subsections we summarize the classification process for the Lie superalgebras $B(0|n)$, $D(m|n)$ and $C(n)$.

3.3 The Lie superalgebras $B(0|n)$

$G_0 = H + \cdots$	ℓ	N
$sl(i) \oplus B(0\|n-i)$ ($i = 1, \ldots, n$)	5	$i(2n - 2i + 1)$

The most interesting case corresponds to $i = n$. Then $N = n$; the CAOs

$$B_j^- = -\sqrt{2}(e_{1,1+n+j} + e_{1+j,1}), \quad j = 1, \ldots, n,$$
$$B_j^+ = \sqrt{2}(e_{1,1+j} - e_{1+n+j,1}), \quad j = 1, \ldots, n$$

are all odd generators of $B(0|n)$ and the relations $\mathcal{R}$ consists of the triple para-Bose relations (1).

3.4 The Lie superalgebras $D(m|n)$

$G_0 = H + \cdots$	ℓ	N
$sl(k\|l) \oplus D(m-k\|n-l)$ ($k = 0, 1, \ldots, m$; $l = 0, 1, \ldots, n$; $(k, l) \notin \{(0,0), (1,0), (m-1,n), (m,n)\}$)	5	$2(k+l)(m+n-k-l)$
$D(m-1\|n)$ $\quad [(k,l) = (1,0)]$	3	$2(m+n-1)$
$sl(m\|n)$ $\quad [(k,l) = [m,n)]$	3	$\frac{(m+n)(m+n+1)}{2} - m$
$sl(m-1\|n)$ $\quad [(k,l) = (m-1,n)]$	5	$\frac{(m+n)(m+n+1)}{2} - m$
$sl(m-1\|n)$ $\quad [(k,l) = (m-1,n)]$	5	$2(m+n-1)$

3.5 The Lie superalgebras $C(n)$

$G_0 = H + \cdots$	ℓ	N
$sl(k\|l) \oplus D(1-k\|n-1-l)$ $(k=0,1;\, l=1,\ldots,n-2)$	5	$2(k+l)(n-k-l)$
$C_{n-1} \quad [(k,l)=(1,0)]$	3	$2(n-1)$
$sl(1\|n-1) \quad [(k,l)=(1,n-1)]$	3	$n(n+1)/2-1$
$sl(n-1) \quad [(k,l)=(0,n-1)]$	5	$n(n+1)/2-1$
$sl(n-1) \quad [(k,l)=(0,n-1)]$	5	$2(n-1)$

4 Conclusions and possible applications

We have obtained a complete classification of all GQS associated with the basic classical Lie superalgebras. The familiar cases (para-Bose, para-Fermi and A-(super)statistics) appear as simple examples in our classification. In order to talk about a quantum statistics in the physical sense, one should take into account additional requirements for the CAOs, related to certain quantization postulates. These conditions are related to the existence of state spaces, in which the CAOs act in such a way that the corresponding observables are Hermitian operators. We hope that some cases of our classification will yield interesting GQS also from this point of view.

As a second application, we mension the problem of finding solutions of the compatibility conditions (CCs) of a Wigner quantum oscillator system [18]. These compatibility conditions take the form of certain triple relations for operators. So formally the CCs appear as special triple relations among operators which resemble the creation and annihilation operators of a generalized quantum statistics. One can thus investigate which formal GQSs also provide solutions of the CCs. It turns out that the classification presented here, with CAOs consisting of odd generators only, yields new solutions of these compatibility conditions corresponding to each basic classical Lie superalgebra [19].

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