The finite group of the Kummer solutions

S. Lievens, K. Srinivasa Rao[†] and J. Van der Jeugt

Department of Applied Mathematics and Computer Science,
University of Ghent, Krijgslaan 281-S9, B-9000 Gent, Belgium, and
[†] The Institute of Mathematical Sciences, Chennai-600113, India.

Stijn.Lievens@rug.ac.be, rao@imsc.res.in, Joris.VanderJeugt@rug.ac.be

Abstract

In this short communication, which is self-contained, we show that the set of 24 Kummer solutions of the classical hypergeometric differential equation has an *elegant, simple group theoretic structure associated with the symmetries of a cube*; or, in other words, that the underlying symmetry group is the symmetric group S_4.

Keywords: Hypergeometric series, Kummer solutions, symmetries of the cube.

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Running head: On the Kummer solutions.

E.E. Kummer [5] showed that the second order ordinary differential equation, characterized by three regular singular points at $0, 1$ and ∞, i.e.

$$z(1-z)u''(z) + [c - (a+b+1)z]u'(z) - (ab)u(z) = 0, \tag{1}$$

with a, b and c real or complex parameters, has one solution as the hypergeometric series:

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) = F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, \tag{2}$$

where $(a)_n$ is the classical Pochhammer symbol:

$$\begin{aligned} (a)_n &= a(a+1)(a+2)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad n > 0, \\ &= 1, \quad n = 0, \end{aligned}$$

and that (2) belongs to a set of 24 functions. Kummer published a set of 6 distinct solutions of the hypergeometric equation. Each of these six solutions has four forms, related to one another by Euler's transformations, giving 24 forms in total [6, 2, 9]. Often, these 24 forms are referred to as the Kummer solutions of the hypergeometric equation.

In many classical textbooks these 24 solutions are given as a list. The fact that these 24 solutions are related to one another by a finite group of transformations was observed more recently; we know of two such references: in an article by Prosser [8], and in a book [4, pp. 36-40]. In none of these references, however, the finite group of order 24 (or, by a natural extension, of order 48) is characterized. In this short communication we show that the finite group relating the 24 Kummer solutions is the group of (rotational) symmetries of the cube, which is isomorphic to the symmetric group S_4. We shall not refer to any of the analytic properties of the solutions, since these have been discussed extensively in many books.

The series in (2) converges for $|z| < 1$, and the hypergeometric function $F(a, b; c; z)$ is the analytic continuation of this series in $\mathbb{C} \setminus [1, +\infty)$. The hypergeometric function has the obvious symmetry, that we shall refer to as the mirror symmetry:

$$F(a, b; c; z) = F(b, a; c; z). \tag{3}$$

Furthermore, the following transformations due to Euler and/or Pfaff [1, p. 68] are valid:

$$\begin{aligned} F(a, b; c; z) &= (1 - z)^{-a} F(a, c - b; c; z/(z - 1)), \\ F(a, b; c; z) &= (1 - z)^{-b} F(c - a, b; c; z/(z - 1)), \\ F(a, b; c; z) &= (1 - z)^{c-a-b} F(c - a, c - b; c; z). \end{aligned} \tag{4}$$

Thus, the solution $F(a, b; c; z)$ of (1) has four different forms, (2) and (4), or eight different forms if one includes the mirror symmetries (3). For this reason, Kummer's six solutions of (1) become a list of 24 solutions (or 48 solutions if one includes the mirror symmetries).

Kummer's 24 solutions are listed in Table 1 (column 3). The 24 solutions are grouped into six sets of four; the first element of each set refers to one of the six solutions given by Kummer, the remaining elements in the set follow by applying Euler's transformations (4).

It was already established in [8, 4] that these 24 solutions are related to one another by a finite group of order 24 (or, if one includes the mirror symmetries, by a finite group of order 48). Here we shall show that this group of order 24 is the group of (rotation)

Table 1: Kummer's 24 solutions

i	Permutation	Function	α_i
1	$()$	$F(a,b;c;z)$	0
	$(2,4,5,3)$	$(1-z)^{-b}F(b,c-a;c;z/(z-1))$	
	$(2,5)(3,4)$	$(1-z)^{c-a-b}F(c-a,c-b;c;z)$	
	$(2,3,5,4)$	$(1-z)^{-a}F(c-b,a;c;z/(z-1))$	
2	$(1,2,6,5)$	$z^{1-c}F(1+a-c,1+b-c;1+a+b-c;1-z)$	$(a+b)/4-1/12$
	$(1,2)(3,4)(5,6)$	$F(b,a;1+a+b-c;1-z)$	
	$(1,2,3)(4,6,5)$	$z^{-a}F(a,1+a-c;1+a+b-c;1-1/z)$	
	$(1,2,4)(3,6,5)$	$z^{-b}F(1+b-c,b;1+a+b-c;1-1/z)$	
3	$(1,3,6,4)$	$z^{-a}F(1+a-c,a;1+a-b;1/z)$	$(3a+b-c)/4+1/12$
	$(1,3)(2,5)(4,6)$	$z^{b-c}(1-z)^{c-a-b}F(c-b,1-b;1+a-b;1/z)$	
	$(1,3,2)(4,5,6)$	$(1-z)^{-a}F(a,c-b;1+a-b;1/(1-z))$	
	$(1,3,5)(2,6,4)$	$z^{1-c}(1-z)^{c-a-1}F(1-b,1+a-c;1+a-b;1/(1-z))$	
4	$(1,4,6,3)$	$z^{-b}F(b,1+b-c;1+b-a;1/z)$	$(a+3b-c)/4+1/12$
	$(1,4)(2,5)(3,6)$	$z^{a-c}(1-z)^{c-a-b}F(1-a,c-a;1+b-a;1/z)$	
	$(1,4,2)(3,5,6)$	$(1-z)^{-b}F(c-a,b;1+b-a;1/(1-z))$	
	$(1,4,5)(2,6,3)$	$z^{1-c}(1-z)^{c-b-1}F(1+b-c,1-a;1+b-a;1/(1-z))$	
5	$(1,5,6,2)$	$(1-z)^{c-a-b}F(c-b,c-a;1+c-a-b;1-z)$	$-(a+b-2c)/4-1/12$
	$(1,5)(2,6)(3,4)$	$z^{1-c}(1-z)^{c-a-b}F(1-a,1-b;1+c-a-b;1-z)$	
	$(1,5,3)(2,4,6)$	$z^{a-c}(1-z)^{c-a-b}F(c-a,1-a;1+c-a-b;1-1/z)$	
	$(1,5,4)(2,3,6)$	$z^{b-c}(1-z)^{c-a-b}F(1-b,c-b;1+c-a-b;1-1/z)$	
6	$(1,6)(2,5)$	$z^{1-c}(1-z)^{c-a-b}F(1-b,1-a;2-c;z)$	$c/2-1/2$
	$(1,6)(3,4)$	$z^{1-c}F(1+b-c,1+a-c;2-c;z)$	
	$(1,6)(2,4)(3,5)$	$z^{1-c}(1-z)^{c-b-1}F(1-a,1+b-c;2-c;z/(z-1))$	
	$(1,6)(2,3)(4,5)$	$z^{1-c}(1-z)^{c-a-1}F(1+a-c,1-b;2-c;z/(z-1))$	

symmetries of the cube, also known as the "direct symmetry group of the cube", or as the octahedral group $\mathcal{O}$ [7]. The group of order 48 is the complete group of (rotation and reflection) symmetries of the cube, known also as the complete symmetry group of the cube $\mathcal{O}_h$ [7].

In order to describe the group $\mathcal{O}$, consider a cube of which the faces are labelled by 1, 2, 3, 4, 5 and 6, in such a way that the sum of the labels of opposite faces is 7 (like the markings on a dice), see Figure 1. Consider the original configuration of the cube, given in Figure 2(a). Each rotation symmetry of the cube, i.e. each element of $\mathcal{O}$, is then described by a particular permutation of the faces of the cube. For example, a rotation over $-\pi/2$ among the axis passing through the midpoints of faces 1 and 6 yields the configuration in Figure 2(b). Clearly, the transformation from (a) to (b) is described by the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 2 & 5 & 3 & 6 \end{pmatrix},$$

or, in cycle notation, $(2,4,5,3)$. Similarly, Figure 2(c) describes a rotation through the axis passing through the midpoints of faces 3 and 4; the corresponding permutation of the faces is given by $(1,2,6,5)$. The group $\mathcal{O}$ is thus a subgroup of the symmetric group S_6. Its 24 elements are given in the second column of Table 1. It is easy to see that $\mathcal{O}$ is the subgroup of S_6 generated by the two elements $g_1 = (2,4,5,3)$ and $g_2 = (1,2,6,5)$. Observe also that $\mathcal{O}$ is isomorphic to the symmetric group S_4. This can be seen by considering the four main diagonals of the cube: every symmetry is then uniquely described by a permutation of these four diagonals (see, e.g. [3]).

Figure 1: The cube, with top face 1, bottom face 6, front face 2, back face 5, left face 3 and right face 4.

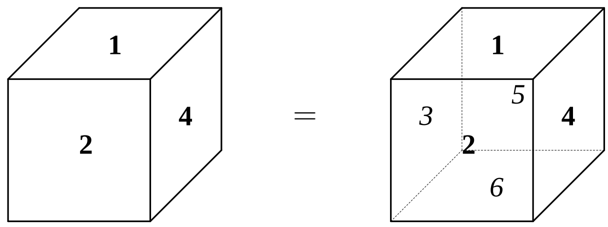
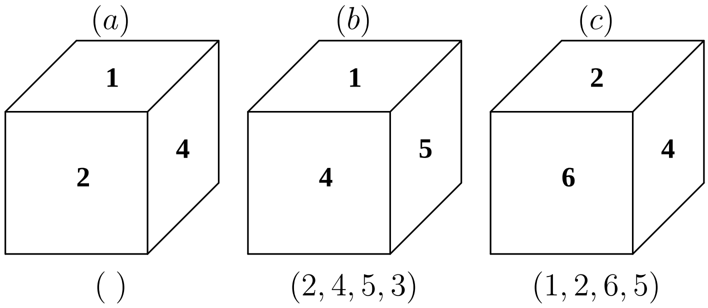


Figure 2: The cube, and two of its rotation symmetries



Let x_i $(i = 1, \ldots, 6)$ be six variables satisfying the constraint

$$\sum_{i=1}^{6} x_i = 0 \tag{5}$$

and consider the following function:

$$f(x) = F(\frac{1}{2} + x_1 + x_2 + x_3, \frac{1}{2} + x_1 + x_2 + x_4; 1 + x_1 - x_6; -\frac{x_1 + x_6}{x_3 + x_4}). \tag{6}$$

Identify the four arguments of F with a, b, c and z; solving this system with respect to the x_i leaves one degree of freedom (since there are 5 independent x_i's). Consider any element g of $\mathcal{O}$; the action of g on x is determined by permuting the indices of the x_i. So, acting with $g_1 = (2,4,5,3)$ on $f(x)$ gives

$$f(g_1 \cdot x) = F(\frac{1}{2} + x_1 + x_4 + x_2, \frac{1}{2} + x_1 + x_4 + x_5; 1 + x_1 - x_6; -\frac{x_1 + x_6}{x_2 + x_5}), \tag{7}$$

and this is equal to $F(b, c-a; c; z/(z-1))$ when the original $f(x)$ is identified with $F(a,b;c;z)$. Similarly, one finds with $g_2 = (1,2,6,5)$ that $f(g_2 \cdot x) = F(1+a-c, 1+b-c; 1+a+b-c; 1-z)$. For each element g of $\mathcal{O}$, the corresponding function $f(g \cdot x)$ is given in the third column of Table 1 (only the hypergeometric function; not yet the powers of z and $1-z$). Thus with every element of $\mathcal{O}$ one of the 24 Kummer solutions is associated. One can do even better and reproduce the complete solution, including the power functions in front of the hypergeometric functions (even though this is slightly more technical). For this purpose, consider

$$C(x) = (-1)^{\frac{x_1 - x_6}{4} + \frac{1}{12}} (x_1 + x_6)^{\frac{x_1 - x_6}{2} + \frac{1}{6}} (x_2 + x_5)^{\frac{x_2 - x_5}{2} + \frac{1}{6}} (-x_3 - x_4)^{\frac{x_1 - x_6}{2} + \frac{x_2 - x_5}{2} + \frac{1}{3}}. \tag{8}$$

The action of any element $g \in \mathcal{O}$ is again by permutation of the indices, as follows:

$$g : f(x) \to \frac{C(g \cdot x)}{C(x)} f(g \cdot x). \tag{9}$$

It is now easy to verify that the 24 elements of $\mathcal{O}$ yield the 24 solutions given in Table 1. For completeness, we should mention that (9) gives in fact a constant times the Kummer solution, but this constant is irrelevant. Here, this constant is $(-1)^{\alpha_i}$, where α_i is also given in Table 1.

As a first remark, note that $\mathcal{O}$ does not include the mirror symmetries. If one extends $\mathcal{O}$ to $\mathcal{O}_h$ by including reflection symmetries of the cube, one gets a group of order 48. This can be done by adding the additional generator $g_3 = (3,4)$ to g_1 and g_2. In Figure 1 one can see that g_3 corresponds to a reflection about a plane. In (6), the permutation $(3,4)$ corresponds to the mirror symmetry (3). One can verify that the 48 elements of $\mathcal{O}_h$ yield the 24 Kummer solutions given in Table 1 plus their 24 mirror symmetries.

Secondly, note that the subgroup of $\mathcal{O}$ consisting of those symmetries that leave the top and bottom face (with labels 1 and 6) invariant is a cyclic group of order 4. This group describes the four solutions related to one another by Euler's transformations.

The fact that $\mathcal{O}$ is the symmetry group of the Kummer solutions can be best understood from the hypergeometric equation. It is well known that this equation is related to the Riemann equation [10, Chapter XIV]:

$$u''(z) + \left(\frac{1-\alpha-\alpha'}{z-z_a} + \frac{1-\beta-\beta'}{z-z_b} + \frac{1-\gamma-\gamma'}{z-z_c}\right) u'(z) +$$
$$\left(\frac{\alpha\alpha'(z_a-z_b)(z_a-z_c)}{z-z_a} + \frac{\beta\beta'(z_b-z_c)(z_b-z_a)}{z-z_b} + \frac{\gamma\gamma'(z_c-z_a)(z_c-z_b)}{z-z_c}\right)$$
$$\times \frac{u(z)}{(z-z_a)(z-z_b)(z-z_c)} = 0, \tag{10}$$

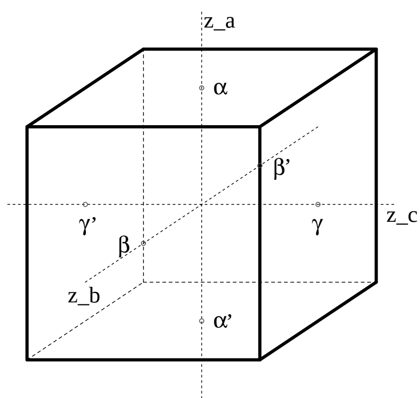
where $\alpha+\alpha'+\beta+\beta'+\gamma+\gamma'=1$. Putting the regular singularities $(z_a, z_b, z_c) = (0, \infty, 1)$ and (α, α'), (β, β'), (γ, γ') to $(0, 1-c)$, (a, b), $(0, c-a-b)$, one obtains (1).

A solution of (10) is given by:

$$\left(\frac{z-z_a}{z-z_b}\right)^{\alpha} \left(\frac{z-z_c}{z-z_b}\right)^{\gamma} F\left(\alpha+\beta+\gamma, \alpha+\beta'+\gamma; 1+\alpha-\alpha'; \frac{(z_c-z_b)(z-z_a)}{(z_c-z_a)(z-z_b)}\right). \tag{11}$$

In total, one can list 24 such solutions, see e.g. [10, p. 284], or – including the trivial mirror symmetries – 48 solutions. The 48 solutions arise from the $3! = 6$ permutations of the singularities (z_a, z_b, z_c) and the $2^3 = 8$ transpositions of the primed and unprimed parameters (α, α'), (β, β'), (γ, γ'), in the following way. Consider again the cube, with opposite faces now being labelled by α and α', β and β', γ and γ'. The axis passing through the midpoints of faces α and α' is labelled by z_a, and similarly for z_b and z_c, see Figure 3.

Figure 3: The cube, with labels of faces and axes referring to the Riemann equation.



The 24 (or, including mirror symmetries, 48) solutions of (10), as given e.g. in [10], are now in an obvious way related to the symmetries of the cube. Here, this follows immediately from the fact that each symmetry of the cube in Figure 3 leaves the equation (10) invariant.

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