A classification of generalized quantum statistics associated with classical Lie algebras

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Abstract

Generalized quantum statistics such as para-Fermi statistics is characterized by certain triple relations which, in the case of para-Fermi statistics, are related to the orthogonal Lie algebra $B_n = so(2n+1)$. In this paper, we give a quite general definition of "a generalized quantum statistics associated to a classical Lie algebra G". This definition is closely related to a certain $\mathbb{Z}$-grading of G. The generalized quantum statistics is then determined by a set of root vectors (the creation and annihilation operators of the statistics) and the set of algebraic relations for these operators. Then we give a complete classification of all generalized quantum statistics associated to the classical Lie algebras A_n, B_n, C_n and D_n. In the classification, several new classes of generalized quantum statistics are described.

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I Introduction

In classical quantum statistics one works exclusively with Bose and Fermi statistics (bosons and fermions). A historically important extension or generalization of these quantum statistics has been known for 50 years, namely the para-Bose and para-Fermi statistics as developed by Green [1]. Instead of the classical bilinear commutators or anti-commutators as for bosons and fermions, para-statistics is described by means of certain trilinear or triple relations. For example, for n pairs of para-Fermi creation and annihilation operators f_i^ξ ($\xi = \pm$ and $i = 1, \ldots, n$), the defining relations are:

$$[[f_j^\xi, f_k^\eta], f_l^\epsilon] = \frac{1}{2}(\epsilon - \eta)^2 \delta_{kl} f_j^\xi - \frac{1}{2}(\epsilon - \xi)^2 \delta_{jl} f_k^\eta, \qquad (1.1)$$
$$\xi, \eta, \epsilon = \pm \text{ or } \pm 1; \quad j, k, l = 1, \ldots, n.$$

About ten years after the introduction of para-Fermi relations by Green, it was proved that these relations are associated with the orthogonal Lie algebra $so(2n+1) = B_n$ [2]. More precisely, the Lie algebra generated by the $2n$ elements f_i^ξ, with $\xi = \pm$ and $i = 1, \ldots, n$, subject to the relations (1.1), is $so(2n+1)$ (as a Lie algebra defined by means of generators and relations). In fact, this can be considered as an alternative definition instead of the common definition by means of Chevalley generators and their known relations expressed by means of the Cartan matrix elements (inclusive the Serre relations). Moreover, there is a certain representation of $so(2n+1)$, the so-called Fermi representation $\mathcal{F}$, that yields the classical Fermi relations. In other words, the representatives $\mathcal{F}(f_i^\xi)$ satisfy the bilinear relations of classical Fermi statistics. Thus the usual Fermi statistics corresponds to a particular realization of para-Fermi statistics. For general para-Fermi statistics, a class of finite dimensional $so(2n+1)$ representations (of Fock type) needs to be investigated.

Twenty years after the connection between para-Fermi statistics and the Lie algebra $so(2n+1)$, a new connection, between para-Bose statistics and the orthosymplectic Lie superalgebra $osp(1|2n) = B(0, n)$ [3] was discovered [4]. The situation here is similar: the Lie superalgebra generated by $2n$ odd elements b_i^ξ, with $\xi = \pm$ and $i = 1, \ldots, n$, subject to the triple relations of para-Bose statistics, is $osp(1|2n)$ (as a Lie superalgebra defined by means of generators and relations). Also here there is a particular representation of $osp(1|2n)$, the so-called Bose representation $\mathcal{B}$, that yields the classical Bose relations, i.e. where the representatives $\mathcal{B}(b_i^\xi)$ satisfy the relations of classical Bose statistics. For more general para-Bose statistics, a class of infinite dimensional $osp(1|2n)$ representations needs to be investigated, and one of these representations corresponds with ordinary Bose statistics.

From these historical examples it is clear that para-statistics, as introduced by Green [1] and further developed by many other research teams (see [5] and the references therein), can be associated with representations of the Lie (super)algebras of class B (namely B_n and $B(0,n)$). The question that arises is whether alternative interesting types of generalized quantum statistics can be found in the framework of other classes of simple Lie algebras or superalgebras. In this paper we shall classify all the classes of generalized quantum statistics for the classical Lie algebras A_n, B_n, C_n and D_n, by means of their algebraic relations. In a forthcoming paper we hope to perform a similar classification for the classical Lie superalgebras.

We should mention that certain generalizations related to other Lie algebras have already been considered [6]-[10], although a complete classification was never made. For example, for the Lie algebra $sl(n+1) = A_n$ [7], a set of creation and annihilation operators has been described, and it was shown that n pairs of operators a_i^ξ, with $\xi = \pm$ and $i = 1, \ldots, n$, subject to the defining relations

$$\begin{aligned} &[[a_i^+, a_j^-], a_k^+] = \delta_{jk} a_i^+ + \delta_{ij} a_k^+, \\ &[[a_i^+, a_j^-], a_k^-] = -\delta_{ik} a_j^- - \delta_{ij} a_k^-, \\ &[a_i^+, a_j^+] = [a_i^-, a_j^-] = 0, \end{aligned} \tag{1.2}$$

$(i, j, k = 1, \ldots, n)$, generate the special linear Lie algebra $sl(n+1)$ (as a Lie algebra defined by means of generators and relations). Just as in the case of para-Fermi relations, (1.2) has two interpretations. On the one hand, (1.2) describes the algebraic relations of a new kind of generalized statistics, in this case A-statistics or statistics related to the Lie algebra A_n. On the other hand, (1.2) yields a set of defining relations for the Lie algebra A_n in terms of generators and relations. Observe that certain microscopic and macroscopic properties of A-statistics have already been studied [11]-[12].

The description (1.2) was given for the first time by N. Jacobson [13] in the context of "Lie triple systems". Therefore, this type of generators is often referred to as the "Jacobson generators" of $sl(n+1)$. In this context, we shall mainly use the terminology "creation and annihilation operators (CAOs) for $sl(n+1)$".

In the following section we shall give a precise definition of "generalized quantum statistics associated with a Lie algebra G" and the corresponding creation and annihilation operators. It will be clear that this notion is closely related to gradings of G, and to regular subalgebras of G. Following the definition, we go on to describe the actual classification method. In the remaining sections of this paper, the classification results are presented. The paper ends with some closing remarks and further outlook.

II Definition and classification method

Let G be a (classical) Lie algebra. A generalized quantum statistics associated with G is determined by a set of N creation operators x_i^+ and N annihilation operators x_i^-. Inspired by the para-Fermi case and the example of A-statistics, these $2N$ operators should satisfy certain conditions. First of all, these $2N$ operators should generate the Lie algebra G, subject to certain triple relations like (1.1) or (1.2). Let G_{+1} and G_{-1} be the subspaces of G spanned by these elements:

$$G_{+1} = \mathrm{span}\{x_i^+;\ i = 1\ldots, N\}, \qquad G_{-1} = \mathrm{span}\{x_i^-;\ i = 1\ldots, N\}. \tag{2.1}$$

Then $[G_{+1}, G_{+1}]$ can be zero (in which case the creation operators mutually commute, as in (1.2)) or non-zero (as in (1.1)). A similar statement holds for the annihilation operators and $[G_{-1}, G_{-1}]$. The fact that the defining relations should be triple relations, implies that it is natural to make the following requirements:

$$\begin{aligned}
&[[x_i^+, x_j^+], x_k^+] = 0,\\
&[[x_i^+, x_j^+], x_k^-] = \text{a lineair combination of } x_l^+,\\
&[[x_i^+, x_j^-], x_k^+] = \text{a lineair combination of } x_l^+,\\
&[[x_i^+, x_j^-], x_k^-] = \text{a lineair combination of } x_l^-,\\
&[[x_i^-, x_j^-], x_k^+] = \text{a lineair combination of } x_l^-,\\
&[[x_i^-, x_j^-], x_k^-] = 0.
\end{aligned}$$

So let $G_{\pm 2} = [G_{\pm 1, \pm 1}]$ and $G_0 = [G_{+1}, G_{-1}]$, then we may require $G_{-2} \oplus G_{-1} \oplus G_0 \oplus G_{+1} \oplus G_{+2}$ (direct sum as vector spaces) to be a $\mathbb{Z}$-grading of a subalgebra of G. Furthermore, since we want G to be generated by the $2N$ elements subject to the triple relations, one must have $G = G_{-2} \oplus G_{-1} \oplus G_0 \oplus G_{+1} \oplus G_{+2}$.

There are two additional assumptions, again inspired by the known examples (1.1) and (1.2). One is related to the fact that creation and annihilation operators are usually considered to be each others conjugate. So, let ω be the standard antilinear anti-involutive mapping of the Lie algebra G (characterized by $\omega(x) = x^\dagger$ in the standard defining representation of G, where $x^\dagger$ denotes the transpose complex conjugate of the matrix x in this representation) then we should have $\omega(x_i^+) = x_i^-$. And finally, we shall assume that the generating elements $x_i^\pm$ are certain root vectors of the Lie algebra G.

Definition 1 *Let G be a classical Lie algebra, with antilinear anti-involutive mapping ω. A set of $2N$ root vectors $x_i^\pm$ ($i = 1, \ldots, N$) is called a set of creation and annihilation operators for G if:*

- $\omega(x_i^\pm) = x_i^\mp$,
- $G = G_{-2} \oplus G_{-1} \oplus G_0 \oplus G_{+1} \oplus G_{+2}$ *is a $\mathbb{Z}$-grading of G, with* $G_{\pm 1} = span\{x_i^\pm,\ i = 1\ldots, N\}$ *and* $G_{j+k} = [G_j, G_k]$.

The algebraic relations $\mathcal{R}$ satisfied by the operators $x_i^\pm$ are the relations of a generalized quantum statistics (GQS) associated with G.

So a GQS is characterized by a set $\{x_i^\pm\}$ of CAOs and the set of algebraic relations $\mathcal{R}$ they satisfy. A consequence of this definition is that G is generated by G_{-1} and G_{+1}, i.e. by the set of CAOs. Furthermore, since $G_{j+k} = [G_j, G_k]$, it follows that

$$G = \text{span}\{x_i^\xi,\ [x_i^\xi, x_j^\eta];\quad i,j = 1,\ldots,N,\ \xi,\eta = \pm\}. \tag{2.2}$$

This implies that it is necessary and sufficient to give all relations of the following type:

(R1) The set of all linear relations between the elements $[x_i^\xi, x_j^\eta]$ $(\xi,\eta = \pm,\ i,j = 1,\ldots,N)$.

(R2) The set of all triple relations of the form $[[x_i^\xi, x_j^\eta], x_k^\zeta] =$ linear combination of x_l^θ.

So in general $\mathcal{R}$ consists of a set of quadratic relations (linear combinations of elements of the type $[x_i^\xi, x_j^\eta]$) and a set of triple relations. This also implies that, as a Lie algebra defined by generators and relations, G is uniquely characterized by the set of generators $x_i^\pm$ subject to the relations $\mathcal{R}$.

Another consequence of this definition is that G_0 itself is a subalgebra of G spanned by root vectors of G, i.e. G_0 is a regular subalgebra of G. Even more: G_0 is a regular subalgebra containing the Cartan subalgebra H of G. And by the adjoint action, the remaining G_i's are G_0-modules. Thus the following technique can be used in order to obtain a complete classification of all GQS associated with G:

1. Determine all regular subalgebras G_0 of G. If not yet contained in G_0, replace G_0 by $G_0 + H$.

2. For each regular subalgebra G_0, determine the decomposition of G into simple G_0-modules g_k $(k = 1, 2, \ldots)$.

3. Investigate whether there exists a $\mathbb{Z}$-grading of G of the form

$$G = G_{-2} \oplus G_{-1} \oplus G_0 \oplus G_{+1} \oplus G_{+2}, \tag{2.3}$$

where each G_i is either directly a module g_k or else a sum of such modules $g_1 \oplus g_2 \oplus \cdots$, such that $\omega(G_{+i}) = G_{-i}$.

The first stage in this technique is a known one: to find regular subalgebras one can use the method of extended Dynkin diagrams [14]. The second stage is straightforward by means of Lie algebra representation techniques. The third stage requires most of the work: one must try out all possible combinations of the G_0-modules g_k, and see whether it is possible to obtain a grading of the type (2.3). In this process, if one of the simple G_0-modules g_k is such that $\omega(g_k) = g_k$, then it follows that this module should be part of G_0. In other words, such a case reduces essentially to another case with a larger regular subalgebra.

In general, when the rank of the semi-simple regular subalgebra is equal or close to the rank of G, the corresponding $\mathbb{Z}$-grading of G is "short" in the sense that $G_i = 0$ for $|i| > 1$ or $|i| > 2$. When the rank of the regular subalgebra becomes smaller, the corresponding $\mathbb{Z}$-grading of G is "long", and $G_i \neq 0$ for $|i| > 2$. Thus the analysis shows that it is usually sufficient to consider maximal regular subalgebras (same rank), or almost maximal regular subalgebras (rank of G minus 1 or 2).

Note that in [10] a definition of CAOs was already given. Our Definition 1 is inspired by the definition in [10], however it is different in the sense that the grading conditions $G_{j+k} = [G_j, G_k]$ are new. It is thanks to these new conditions that we are able to give a complete classification of CAOs and the corresponding GQS.

In the following sections we shall give a summary of the classification process for the classical Lie algebras A_n, B_n, C_n and D_n. Note that, in order to identify a GQS associated with G, it is sufficient to give only the set of CAOs, or alternatively, to give the subspace G_{-1} (then the x_i^- are the root vectors of G_{-1}, and $x_i^+ = \omega(x_i^-)$). The set $\mathcal{R}$ then consist of all quadratic relations (i.e. the linear relations between the elements $[x_i^\xi, x_j^\eta]$) and all triple relations, and all of these relations follow from the known commutation relations in G. Because, in principle, $\mathcal{R}$ can be determined from the set $\{x_i^\pm; i = 1, \ldots, N\}$, we will not always give it explicitly. In fact, when N is large, the corresponding relations can become rather numerous and long. Such examples of GQS would be too complicated for applications in physics. For this reason, we shall give $\mathcal{R}$ explicitly only when N is not too large, more precisely when N is either equal to the rank of G or at most double the rank of G.

Finally, observe that two different sets of CAOs $\{x_i^\pm; i = 1\ldots, N\}$ and $\{y_i^\pm; i = 1\ldots, N\}$ (same N) are said to be isomorphic if, for a certain permutation τ of $\{1, 2, \ldots, N\}$, the relations between the elements $x_{\tau(i)}^\pm$ and $y_i^\pm$ are the same. In that case, the regular subalgebra G_0 spanned by $\{[x_i^+, x_j^-]\}$ is isomorphic (as a Lie algebra) to the regular subalgebra spanned by $\{[y_i^+, y_j^-]\}$.

III The Lie algebra $A_n = sl(n+1)$

Let G be the special linear Lie algebra $sl(n+1)$, consisting of traceless $(n+1)\times(n+1)$ matrices. The Cartan subalgebra H of G is the subspace of diagonal matrices. The root vectors of G are known to be the elements e_{jk} $(j \neq k = 1, \ldots, n+1)$, where e_{jk} is a matrix with zeros everywhere except a 1 on the intersection of row j and column k. The corresponding root is $\epsilon_j - \epsilon_k$, in the usual basis. The anti-involution is such that $\omega(e_{jk}) = e_{kj}$. The simple roots and the Dynkin diagram of A_n are given in Table 1, and so is the extended Dynkin diagram.

In order to find regular subalgebras of $G = A_n$, one should delete nodes from the Dynkin diagram of G or from its extended Dynkin diagram. We shall start with the ordinary Dynkin diagram of A_n, and subsequently consider the extended diagram.

Step 1. Delete node i from the Dynkin diagram. The corresponding diagram is the

Dynkin diagram of $sl(i) \oplus sl(n-i+1)$, so $G_0 = H + sl(i) \oplus sl(n-i+1)$. In this case, there are only two G_0 modules and we can put

$$G_{-1} = \text{span}\{e_{kl};\ k=1,\ldots,i,\ l=i+1,\ldots,n+1\}, \quad G_{+1} = \omega(G_{-1}). \tag{3.1}$$

Therefore $sl(n+1)$ has the following grading:

$$sl(n+1) = G_{-1} \oplus G_0 \oplus G_{+1}, \tag{3.2}$$

and the number of creation and annihilation operators is $N = i(n-i+1)$. Note that the cases i and $n+1-i$ are isomorphic.

The most interesting cases are those with $i=1$ and $i=2$, for which we shall explicitly give the relations $\mathcal{R}$ between the CAOs.

For $i=1$, $N=n$, the rank of A_n. Putting

$$a_j^- = e_{1,j+1}, \qquad a_j^+ = e_{j+1,1},\ j=1,\ldots,n, \tag{3.3}$$

(for A_n, the possible sets $\{x_i^\pm\}$ will be denoted $\{a_i^\pm\}$, for B_n, they will be denoted $\{b_i^\pm\}$, etc.) the corresponding relations $\mathcal{R}$ read $(j,k,l=1,\ldots,n)$:

$$\begin{aligned} &[a_j^+, a_k^+] = [a_j^-, a_k^-] = 0, \\ &[[a_j^+, a_k^-], a_l^+] = \delta_{jk} a_l^+ + \delta_{kl} a_j^+, \\ &[[a_j^+, a_k^-], a_l^-] = -\delta_{jk} a_l^- - \delta_{jl} a_k^-. \end{aligned} \tag{3.4}$$

These are the relations of A-statistics [6]-[7], [10]-[12] as considered in the Introduction.

For $i=2$, $N=2(n-1)$, let

$$\begin{aligned} &a_{-j}^- = e_{1,j+2}, \quad a_{+j}^- = e_{2,j+2}, \qquad j=1,\ldots,n-1, \\ &a_{-j}^+ = e_{j+2,1}, \quad a_{+j}^+ = e_{j+2,2}, \qquad j=1,\ldots,n-1. \end{aligned} \tag{3.5}$$

Now the corresponding relations are ($\xi, \eta, \epsilon = \pm;\ j, k, l = 1, \ldots, n-1$):

$$
\begin{aligned}
&[a^+_{\xi j}, a^+_{\eta k}] = [a^-_{\xi j}, a^-_{\eta k}] = 0, \\
&[a^+_{\xi j}, a^-_{-\xi k}] = 0, \qquad j \neq k, \\
&[a^+_{-j}, a^-_{-k}] = [a^+_{+j}, a^-_{+k}], \qquad j \neq k, \\
&[a^+_{+j}, a^-_{-j}] = [a^+_{+k}, a^-_{-k}], \\
&[a^+_{-j}, a^-_{+j}] = [a^+_{-k}, a^-_{+k}], \\
&[[a^+_{\xi j}, a^-_{\eta k}], a^+_{\epsilon l}] = \delta_{\eta\epsilon}\delta_{jk} a^+_{\xi l} + \delta_{\xi\eta}\delta_{kl} a^+_{\epsilon j}, \\
&[[a^+_{\xi j}, a^-_{\eta k}], a^-_{\epsilon l}] = -\delta_{\xi\epsilon}\delta_{jk} a^-_{\eta l} - \delta_{\xi\eta}\delta_{jl} a^-_{\epsilon k}.
\end{aligned}
\tag{3.6}
$$

These relations are already more complicated than (3.4). But they are still defining relations for the Lie algebra A_n.

Step 2. Delete node i and j from the Dynkin diagram. By the symmetry of the Dynkin diagram, it is sufficient to consider $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$ and $i < j < n+1-i$. We have $G_0 = H + sl(i) \oplus sl(j-i) \oplus sl(n+1-j)$. In this case, there are six simple G_0-modules. All the possible combinations of these modules give rise to gradings of the form

$$
sl(n+1) = G_{-2} \oplus G_{-1} \oplus G_0 \oplus G_{+1} \oplus G_{+2}.
$$

There are essentially three different ways in which these G_0-modules can be combined. To characterize these three cases, it is sufficient to give only G_{-1}:

$$
\begin{aligned}
G_{-1} &= \mathrm{span}\{e_{kl}, e_{lp};\ k = 1, \ldots, i,\ l = i+1, \ldots, j,\ p = j+1, \ldots, n+1\}, && (3.7) \\
&\quad \text{with } N = (j-i)(n+1-j+i); \\
G_{-1} &= \mathrm{span}\{e_{kl}, e_{pk};\ k = 1, \ldots, i,\ l = i+1, \ldots, j,\ p = j+1, \ldots, n+1\}, && (3.8) \\
&\quad \text{with } N = i(n+1-i); \\
G_{-1} &= \mathrm{span}\{e_{kl}, e_{lp};\ k = 1, \ldots, i,\ p = i+1, \ldots, j,\ l = j+1, \ldots, n+1\}, && (3.9) \\
&\quad \text{with } N = j(n+1-j).
\end{aligned}
$$

It turns out that the sets of CAOs corresponding to (3.8) and (3.9) are isomorphic to (3.7), so it is sufficient to consider only (3.7). Each case of (3.7) with $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$ and $i < j < n+1-i$ gives rise to a distinct GQS. For reasons explained earlier, we shall give the corresponding set of relations explicitly only for small N. In this case, it is interesting to give $\mathcal{R}$ for $j-i=1$, because then the number of creation or annihilation operators is $N=n$. One can label the CAOs as follows:

$$
\begin{aligned}
&a_k^- = e_{k,i+1}, \quad a_k^+ = e_{i+1,k}, \qquad k=1,\ldots,i; \\
&a_k^- = e_{i+1,k+1}, \quad a_k^+ = e_{k+1,i+1}, \qquad k=i+1,\ldots,n.
\end{aligned} \tag{3.10}
$$

Using

$$
\langle k \rangle = \begin{cases} 0 & \text{if} \quad k=1,\ldots,i \\ 1 & \text{if} \quad k=i+1,\ldots,n \end{cases} \tag{3.11}
$$

the quadratic and triple relations read:

$$
\begin{aligned}
&[a_k^+, a_l^+] = [a_k^-, a_l^-] = 0, \qquad k,l=1,\ldots,i \text{ or } k,l=i+1,\ldots,n, \\
&[a_k^-, a_l^+] = [a_k^+, a_l^-] = 0, \qquad k=1,\ldots,i,\ l=i+1,\ldots,n, \\
&[[a_k^+, a_l^-], a_m^+] = (-1)^{\langle l\rangle+\langle m\rangle}\delta_{kl}a_m^+ + (-1)^{\langle l\rangle+\langle m\rangle}\delta_{lm}a_k^+,\ k,l=1,\ldots,i \text{ or } k,l=i+1,\ldots,n, \\
&[[a_k^+, a_l^-], a_m^-] = -(-1)^{\langle l\rangle+\langle m\rangle}\delta_{kl}a_m^- - (-1)^{\langle l\rangle+\langle m\rangle}\delta_{km}a_l^-,\ k,l=1,\ldots,i \text{ or } k,l=i+1,\ldots,n, \\
&[[a_k^\xi, a_l^\xi], a_m^{-\xi}] = -\delta_{km}a_l^\xi + \delta_{lm}a_k^\xi,\ k=1,\ldots,i,\ l=i+1,\ldots,n, \\
&[[a_k^\xi, a_l^\xi], a_m^\xi] = 0, \qquad (\xi=\pm;\ k,l,m=1,\ldots,n).
\end{aligned} \tag{3.12}
$$

The existence of the set of CAOs (3.10) is pointed out in [6] as a possible example. The relations (3.12) with $n=2m$ and $i=m$ are the commutation relations of the so called causal A-statistics investigated in [9].

Step 3. If we delete 3 or more nodes from the Dynkin diagram, the resulting $\mathbb{Z}$-gradings of $sl(n+1)$ are no longer of the form $sl(n+1) = G_{-2} \oplus G_{-1} \oplus G_0 \oplus G_{+1} \oplus G_{+2}$, but there would be non-zero G_i with $|i|>2$, so these cases are not relevant for our classification.

Step 4. Next, we move on to the extended Dynkin diagram of G. If we delete node i from the extended Dynkin diagram, then remaining diagram is again of type A_n, so $G_0 = G$, and there are no CAOs.

Step 5. If we delete node i and j from the extended Dynkin diagram ($0 \leq i < j \leq n+1$), then $sl(n+1) = G_{-1} \oplus G_0 \oplus G_{+1}$ with $G_0 = H + sl(j-i) \oplus sl(n-j+i+1)$, and

$$G_{-1} = \text{span}\{e_{kl};\ k = i+1 \ldots, j,\ l \neq i+1, \ldots, j\}.$$

The number of annihilation operators is $N = (j-i)(n+1-j+i)$. It is not difficult to see that all these cases are isomorphic to those of Step 1. This can also be deduced from the symmetry of the Dynkin diagram.

Step 6. If we delete nodes i, j and k from the extended Dynkin diagram ($i < j < k$), then the corresponding $\mathbb{Z}$-gradings are of the form

$$sl(n+1) = G_{-2} \oplus G_{-1} \oplus G_0 \oplus G_{+1} \oplus G_{+2}.$$

All the corresponding CAOs, however, are isomorphic to those of Step 2 (which can again be seen from the remaining Dynkin diagram).

Step 7. If we delete 4 or more nodes from the extended Dynkin diagram, the corresponding $\mathbb{Z}$-grading of $sl(n+1)$ has no longer the required properties (i.e. there are non-zero subspaces G_i with $|i| > 2$).

IV The Lie algebra $B_n = so(2n+1)$

$G = so(2n+1)$ is the subalgebra of $sl(2n+1)$ consisting of matrices of the form:

$$\begin{pmatrix} a & b & c \\ d & -a^t & e \\ -e^t & -c^t & 0 \end{pmatrix}, \tag{4.1}$$

where a is any $(n \times n)$-matrix, b and d are antisymmetric $(n \times n)$-matrices, and c and e are $(n \times 1)$-matrices. The Cartan subalgebra H of G is again the subspace of diagonal matrices. The root vectors and corresponding roots of G are given by:

$$\begin{aligned}
e_{jk} - e_{k+n,j+n} &\leftrightarrow \epsilon_j - \epsilon_k, \qquad j \neq k = 1, \ldots, n, \\
e_{j,k+n} - e_{k,j+n} &\leftrightarrow \epsilon_j + \epsilon_k, \qquad j < k = 1, \ldots, n, \\
e_{j+n,k} - e_{k+n,j} &\leftrightarrow -\epsilon_j - \epsilon_k, \qquad j < k = 1, \ldots, n, \\
e_{j,2n+1} - e_{2n+1,j+n} &\leftrightarrow \epsilon_j, \qquad j = 1, \ldots, n, \\
e_{n+j,2n+1} - e_{2n+1,j} &\leftrightarrow -\epsilon_j, \qquad j = 1, \ldots, n.
\end{aligned}$$

The anti-involution is such that $\omega(e_{jk}) = e_{kj}$. The simple roots, the Dynkin diagram and the extended Dynkin diagram of B_n are given in Table 1. Just as for A_n, we now start the process of deleting nodes from the Dynkin diagram or from the extended Dynkin diagram.

Step 1. Delete node 1 from the Dynkin diagram. The remaining diagram is that of B_{n-1}, so $G_0 = H + so(2n-1) \equiv H + B_{n-1}$. There are two G_0-modules:

$$G_{-1} = \text{span}\{e_{1,2n+1} - e_{2n+1,n+1},\ e_{1,k+n} - e_{k,n+1},\ e_{1k} - e_{k+n,n+1};\ k = 2, \ldots, n\}, \tag{4.2}$$

and $G_{+1} = \omega(G_{-1})$. Thus $so(2n+1)$ has the following grading:

$$so(2n+1) = G_{-1} \oplus G_0 \oplus G_{+1}$$

and the number of (mutually commuting) creation and annihilation operators is $N = 2n-1$. Let us denote the CAOs by:

$$\begin{aligned}
&b_{00}^- = e_{1,2n+1} - e_{2n+1,n+1},\ b_{00}^+ = e_{2n+1,1} - e_{n+1,2n+1}, \\
&b_{-k}^- = e_{1,n+k+1} - e_{k+1,n+1},\ b_{-k}^+ = e_{n+k+1,1} - e_{n+1,k+1}, \qquad k = 1, \ldots, n-1, \\
&b_{+k}^- = e_{1,k+1} - e_{n+k+1,n+1},\ b_{+k}^+ = e_{k+1,1} - e_{n+1,n+k+1}, \qquad k = 1, \ldots, n-1.
\end{aligned} \tag{4.3}$$

The corresponding relations $\mathcal{R}$ are given by ($\xi, \eta, \epsilon = 0, \pm;\ i, j, k = 1, \ldots, n-1$):

$$[b_{\xi i}^+, b_{\eta j}^+] = [b_{\xi i}^-, b_{\eta j}^-] = 0,$$

$$
\begin{aligned}
&[b^+_{-i}, b^-_{-j}] = [b^-_{+i}, b^+_{+j}], \qquad i \neq j, \\
&[b^+_{00}, b^-_{-j}] = [b^-_{00}, b^+_{+j}], \\
&[b^+_{00}, b^-_{+j}] = [b^-_{00}, b^+_{-j}], \\
&[[b^+_{\xi i}, b^-_{\eta j}], b^+_{\epsilon k}] = \delta_{ij}\delta_{\xi\eta} b^+_{\epsilon k} + \delta_{jk}\delta_{\eta\epsilon} b^+_{\xi i} - \delta_{ik}\delta_{\xi,-\epsilon} b^+_{-\eta j}, \\
&[[b^+_{\xi i}, b^-_{\eta j}], b^-_{\epsilon k}] = -\delta_{ij}\delta_{\xi\eta} b^-_{\epsilon k} - \delta_{ik}\delta_{\xi\epsilon} b^-_{\eta j} + \delta_{jk}\delta_{\eta,-\epsilon} b^-_{-\xi i}.
\end{aligned} \tag{4.4}
$$

Step 2. Delete node i ($i = 2, \ldots, n$) from the Dynkin diagram; then the corresponding subalgebra is $G_0 = H + sl(i) \oplus so(2(n-i)+1)$. Now there are four G_0-modules, with the following grading for G:

$$
so(2n+1) = G_{-2} \oplus G_{-1} \oplus G_0 \oplus G_{+1} \oplus G_{+2}
$$

with

$$
\begin{aligned}
G_{-1} &= \mathrm{span}\{e_{j,2n+1} - e_{2n+1,n+j},\ e_{j,k+n} - e_{k,n+j},\ e_{jk} - e_{k+n,n+j}; \\
&\qquad j = 1, \ldots, i,\ k = i+1, \ldots, n\}, \\
G_{-2} &= \mathrm{span}\{e_{j,k+n} - e_{k,j+n};\ 1 \leq j < k \leq i\}.
\end{aligned} \tag{4.5}
$$

The number of the annihilation operators is $N = 2i(n-i) + i$. The most interesting case is that with $i = n$: this is the para-Fermi case presented in the Introduction. Indeed, let

$$
f^-_j = \sqrt{2}(e_{j,2n+1} - e_{2n+1,n+j}), \quad f^+_j = \sqrt{2}(e_{2n+1,j} - e_{n+j,2n+1}), \qquad j = 1, \ldots, n. \tag{4.6}
$$

Then there are no quadratic relations, and $\mathcal{R}$ consists of triple relations only:

$$
\begin{aligned}
&[[f^\xi_j, f^\eta_k], f^\epsilon_l] = \frac{1}{2}(\epsilon - \eta)^2 \delta_{kl} f^\xi_j - \frac{1}{2}(\epsilon - \xi)^2 \delta_{jl} f^\eta_k, \\
&\qquad \xi, \eta, \epsilon = \pm \text{ or } \pm 1; \quad j, k, l = 1, \ldots, n.
\end{aligned} \tag{4.7}
$$

Step 3. Delete two or more nodes from the Dynkin diagram. Then the corresponding $\mathbb{Z}$-grading of $so(2n+1)$ has no longer the required properties (i.e. there are non-zero G_i with $|i| > 2$).

Step 4. Now we turn to the extended Dynkin diagram. Deleting node i from this diagram, leaves the Dynkin diagram of $so(2n+1)$ for $i = 0, 1$, of $so(2n)$ for $i = n$, of $sl(2)\oplus sl(2)\oplus so(2n-3)$ for $i = 2$, of $sl(4)\oplus so(2n-5)$ for $i = 3$, and of $so(2i)\oplus so(2n-2i+1)$ for $i \geq 4$. In all these cases there is only one G_0-module, so there are no contributions to our classification.

Step 5. Delete the adjacent nodes $(i-1)$ and i $(i = 3, \ldots, n)$ from the extended Dynkin diagram. The remaining diagram is that of $\tilde{G}_0 = sl(2) \oplus sl(2) \oplus so(2(n-i)+1)$ for $i = 3$, of $\tilde{G}_0 = sl(4) \oplus so(2(n-i)+1)$ for $i = 4$, and of $\tilde{G}_0 = so(2(i-1)) \oplus so(2(n-i)+1)$ for $i > 4$. In each case, there are five $\tilde{G}_0$-modules g_k, one of which is invariant under ω (say g_1). Then one has to put $G_0 = H + \tilde{G}_0 + g_1$, and in each case one finds $G_0 \equiv H + B_{n-1}$.

Now, there are only two G_0-modules and

$$so(2n+1) = G_{-1} \oplus G_0 \oplus G_{+1}$$

with

$$G_{-1} = \mathrm{span}\{e_{i,2n+1} - e_{2n+1,n+i};\ e_{ik} - e_{k+n,n+i};\ e_{i,k+n} - e_{k,n+i};\ k \neq i = 1, \ldots, n\}. \quad (4.8)$$

The number of the anihilation operators is $N = 2n-1$, and all these cases are isomorphic to those of Step 1.

Step 6. Delete two nonadjacent nodes from the extended Dynkin diagram, say i and j, $i < j$, $i, j \neq 0, 1$. The remaining diagram is that of $\tilde{G}_0 = so(2i)\oplus sl(j-i)\oplus so(2(n-j)+1)$ (if $i = 2$ we have $sl(2) \oplus sl(2)$ instead of $so(2i)$). There are seven $\tilde{G}_0$-modules g_k, one of which (say g_1) with $\omega(g_1) = g_1$. Thus one has to take $G_0 = H + \tilde{G}_0 + g_1$, and this is in fact $G_0 \equiv H + so(2(n-j+i)+1) \oplus sl(j-i)$

The corresponding grading is:

$$so(2n+1) = G_{-2} \oplus G_{-1} \oplus G_0 \oplus G_{+1} \oplus G_{+2}$$

with

$$\begin{aligned} G_{-1} &= \text{span}\{e_{k,2n+1} - e_{2n+1,n+k},\ e_{kl} - e_{l+n,n+k},\ e_{k,n+l} - e_{l,n+k}; \\ &\qquad k = i+1,\ldots,j,\ l = 1,\ldots,i,j+1,\ldots,n\}, \\ G_{-2} &= \text{span}\{e_{k,n+l} - e_{l,n+k};\ i+1 \leq k < l \leq j\}. \end{aligned} \tag{4.9}$$

The number of the annihilation operators is $N = 2(j-i)(n-j+i) + j - i$, and all these cases turn out to be isomorphic to those of Step 2.

Step 7. If we delete 3 or more nodes from the extended Dynkin diagram, the corresponding $\mathbb{Z}$-grading of $so(2n+1)$ has no longer the required properties (i.e. there are non-zero subspaces G_i with $|i| > 2$).

V The Lie algebra $C_n = sp(2n)$

$G = sp(2n)$ is the subalgebra of $sl(2n)$ consisting of matrices of the form:

$$\begin{pmatrix} a & b \\ c & -a^t \end{pmatrix}, \tag{5.1}$$

where a is any $(n \times n)$-matrix, and b and c are symmetric $(n \times n)$-matrices. The Cartan subalgebra H consist of the diagonal matrices, and the root vectors and corresponding roots of G are:

$$\begin{aligned} e_{jk} - e_{k+n,j+n} &\leftrightarrow \epsilon_j - \epsilon_k, \qquad j \neq k = 1,\ldots,n, \\ e_{j,k+n} + e_{k,j+n} &\leftrightarrow \epsilon_j + \epsilon_k, \qquad j \leq k = 1,\ldots,n, \\ e_{j+n,k} + e_{k+n,j} &\leftrightarrow -\epsilon_j - \epsilon_k, \qquad j \leq k = 1,\ldots,n. \end{aligned}$$

The simple roots, Dynkin diagram and extended Dynkin diagram are given in Table 1. Again, the anti-involution is such that $\omega(e_{jk}) = e_{kj}$. Next, we describe the process of deleting nodes and its consequences for the classification of GQS.

Step 1. Delete node i $(i=1,\ldots,n-1)$ from the Dynkin diagram. The remaining diagram is that of $sl(i)\oplus sp(2(n-i))$, so $G_0 = H + sl(i)\oplus sp(2(n-i))$. There are four G_0-modules, leading to the following grading:

$$sp(2n) = G_{-2}\oplus G_{-1}\oplus G_0\oplus G_{+1}\oplus G_{+2}$$

with

$$\begin{aligned} G_{-1} &= \mathrm{span}\{e_{k,n+l}+e_{l,n+k},\ e_{kl}-e_{n+l,n+k};\ k=1,\ldots,i,\ l=i+1,\ldots,n\}, \\ G_{-2} &= \mathrm{span}\{e_{k,n+l}+e_{l,n+k};\ 1\leq k\leq l\leq i\}. \end{aligned} \tag{5.2}$$

The number of the annihilation operators is $N=2i(n-i)$. The most interesting cases are $i=1$ and $i=n-1$, which we shall describe in more detail.

For $i=1$, let us denote the CAOs by

$$\begin{aligned} &c^-_{-j} = e_{1,n+j+1}+e_{j+1,n+1},\ c^-_{+j} = e_{1,j+1}-e_{n+j+1,n+1}, \quad j=1,\ldots,n-1, \\ &c^+_{-j} = e_{n+j+1,1}+e_{n+1,j+1},\ c^+_{+j} = e_{j+1,1}-e_{n+1,n+j+1}, \quad j=1,\ldots,n-1. \end{aligned} \tag{5.3}$$

Then the corresponding relations $\mathcal{R}$ read, with $\xi,\eta,\epsilon,\gamma=\pm$ or ±1, and $j,k,l=1,\ldots,n-1$:

$$\begin{aligned} &[c^\eta_{\xi j}, c^\eta_{\xi k}] = 0, \\ &[c^+_{-j}, c^-_{-k}] = [c^-_{+j}, c^+_{+k}], \qquad j\neq k, \\ &[c^-_{-j}, c^-_{+k}] = [c^+_{-j}, c^+_{+k}] = 0, \qquad j\neq k, \\ &[[c^+_{\xi j}, c^-_{\eta k}], c^+_{\epsilon l}] = \delta_{\xi\eta}\delta_{jk}c^+_{\epsilon l} + \delta_{\eta\epsilon}\delta_{kl}c^+_{\xi j} + (-1)^{\eta\epsilon}\delta_{\xi,-\epsilon}\delta_{jl}c^+_{-\eta k}, \\ &[[c^+_{\xi j}, c^-_{\eta k}], c^-_{\epsilon l}] = -\delta_{\xi\eta}\delta_{jk}c^-_{\epsilon l} - \delta_{\xi\epsilon}\delta_{jl}c^-_{\eta k} + (-1)^{\xi\eta}\delta_{\eta,-\epsilon}\delta_{kl}c^-_{-\xi j}, \\ &[[c^\xi_{-j}, c^\xi_{+k}], c^{-\xi}_{\eta l}] = 2\eta\delta_{jk}c^\xi_{-\eta l}, \\ &[[c^\gamma_{\xi j}, c^\gamma_{\eta k}], c^\gamma_{\epsilon l}] = 0. \end{aligned} \tag{5.4}$$

For $i=n-1$, let us also denote the CAOs by $c^\pm_j$:

$$c^-_{-j} = e_{j,2n}+e_{n,n+j},\ c^-_{+j} = e_{jn}-e_{2n,n+j}, \quad j=1,\ldots,n-1,$$

$$c^+_{-j} = e_{2n,j} + e_{n+j,n},\ c^+_{+j} = e_{nj} - e_{n+j,2n}, \quad j = 1, \ldots, n-1. \tag{5.5}$$

Now, the corresponding relations read, with $\xi, \eta, \epsilon, \gamma = \pm$ or ± 1, $j, k, l = 1, \ldots, n-1$:

$$\begin{aligned}
&[c^\eta_{\xi j}, c^\eta_{\xi k}] = 0,\\
&[c^+_{+j}, c^-_{-k}] = [c^-_{+j}, c^+_{-k}] = 0, \qquad j \neq k,\\
&[[c^\epsilon_{\xi j}, c^{-\epsilon}_{\xi k}], c^\epsilon_{\eta l}] = \xi\eta\delta_{jk}c^\epsilon_{\eta l} + \delta_{kl}c^\epsilon_{\eta j},\\
&[[c^\epsilon_{+j}, c^{-\epsilon}_{-k}], c^\xi_{\eta l}] = (\epsilon\xi - \eta)\delta_{jk}c^\xi_{-\eta l},\\
&[[c^\epsilon_{+j}, c^\epsilon_{-k}], c^{-\epsilon}_{\xi l}] = -\xi\delta_{jl}c^\epsilon_{-\xi k} - \xi\delta_{kl}c^\epsilon_{-\xi j},\\
&[[c^\gamma_{\xi j}, c^\gamma_{\eta k}], c^\gamma_{\epsilon l}] = 0.
\end{aligned} \tag{5.6}$$

This set of CAOs, together with their relations (5.6), was constructed earlier in [6]. Also the CAOs (5.3) were already mentioned in [6] as a possible example, without giving the actual relations (5.4).

Step 2. When node n is deleted from the Dynkin diagram of C_n, the corresponding diagram is that of $sl(n)$, and $G_0 = H + sl(n)$. In this case, there are two G_0-modules, and $sp(2n)$ has the grading $sp(2n) = G_{-1} \oplus G_0 \oplus G_{+1}$ with

$$G_{-1} = \{e_{j,n+k} + e_{k,n+j};\ 1 \leq j \leq k \leq n\}. \tag{5.7}$$

There are $N = \frac{n(n+1)}{2}$ commuting annihilation operators, and the relations $\mathcal{R}$ will not be given explicitly.

Step 3. Upon deleting two or more nodes from the Dynkin diagram of C_n, the corresponding $\mathbb{Z}$-gradings have no longer the required property (there are non-zero G_i with $|i| > 2$).

Step 4. Now we turn to the extended Dynkin diagram. Deleting one node from this diagram leads to a situation with only one G_0-module, irrelevant for our classification.

Step 5. Delete the adjacent nodes $(i-1)$ and i $(i = 2, \ldots, n)$ from the extended Dynkin diagram. The remaining diagram is that of $\tilde{G}_0 = sp(2(i-1)) \oplus sp(2(n-i))$. There are seven $\tilde{G}_0$-modules g_k, one of which satisfies $\omega(g_1) = g_1$. Putting $G_0 = H + \tilde{G}_0 + g_1$, it turns out that $G_0 \equiv H + C_{n-1}$. In that case, there are only four G_0-modules and G has the grading $sp(2n) = G_{-2} \oplus G_{-1} \oplus G_0 \oplus G_{+1} \oplus G_{+2}$ with

$$G_{-1} = \text{span}\{e_{i,n+j} + e_{j,n+i},\ e_{ij} - e_{n+j,n+i};\ j \neq i = 1, \ldots, n\}. \tag{5.8}$$

The number of the annihilation operators is $N = 2(n-1)$, and all these cases are isomorphic to the $i = 1$ case of Step 1.

Step 6. Delete two nonadjacent nodes $i < j$ (excluding the case $i = 1$ and $j = n$) from the extended Dynkin diagram. The remaining diagram is that of $\tilde{G}_0 = sp(2i) \oplus sl(j-i) \oplus sp(2(n-j))$. There are again seven $\tilde{G}_0$-modules g_k, among which one with $\omega(g_1) = g_1$. Then $G_0 = H + \tilde{G}_0 + g_1 \equiv H + sl(j-i) \oplus sp(2(n-j+i))$. There are only four G_0-modules and the grading is $sp(2n) = G_{-2} \oplus G_{-1} \oplus G_0 \oplus G_{+1} \oplus G_{+2}$ with

$$G_{-1} = \text{span}\{e_{k,n+l} + e_{l,n+k},\ e_{kl} - e_{n+l,n+k};\ k = i+1, \ldots, j,\ l \neq i+1, \ldots, j\}. \tag{5.9}$$

The number of annihilation operators is $N = 2(j-i)(n-j+i)$, and all these cases are isomorphic to those of Step 1 with $i \neq 1$.

Step 7. Delete node 1 and n from the extended Dynkin diagram. The remaining diagram is that of $sl(2) \oplus sl(n-1)$. With $G_0 = sl(2) \oplus sl(n-1)$, there are four G_0-modules and the corresponding grading is $sp(2n) = G_{-2} \oplus G_{-1} \oplus G_0 \oplus G_{+1} \oplus G_{+2}$ with

$$G_{-1} = \text{span}\{e_{1,n+k} + e_{k,n+1},\ e_{k1} - e_{n+1,n+k};\ k = 2, \ldots, n\}. \tag{5.10}$$

This case is isomorphic to the $i = n-1$ case of Step 1.

Step 8. If we delete 3 or more nodes from the extended Dynkin diagram, the corresponding $\mathbb{Z}$-grading of $sp(2n)$ has no longer the required properties (i.e. there are non-zero subspaces G_i with $|i| > 2$).

VI The Lie algebra $D_n = so(2n)$

$G = so(2n)$ is the subalgebra of $sl(2n)$ consisting of matrices of the form:

$$\begin{pmatrix} a & b \\ c & -a^t \end{pmatrix}, \tag{6.1}$$

where a is any $(n\times n)$-matrix, and b and c are antisymmetric $(n\times n)$-matrices. The Cartan subalgebra H consist of the diagonal matrices, and the root vectors and corresponding roots of G are:

$$\begin{aligned}
e_{jk} - e_{k+n,j+n} \quad &\leftrightarrow \quad \epsilon_j - \epsilon_k, \qquad j \neq k = 1, \ldots, n, \\
e_{j,k+n} - e_{k,j+n} \quad &\leftrightarrow \quad \epsilon_j + \epsilon_k, \qquad j < k = 1, \ldots, n, \\
e_{j+n,k} - e_{k+n,j} \quad &\leftrightarrow \quad -\epsilon_j - \epsilon_k, \qquad j < k = 1, \ldots, n.
\end{aligned}$$

The simple roots, Dynkin diagram and extended Dynkin diagram are given in Table 1. Again, the anti-involution is such that $\omega(e_{jk}) = e_{kj}$. Next, we describe the process of deleting nodes and its consequences for the classification of GQS.

Step 1. When node 1 is deleted from the Dynkin diagram of D_n, the remaining diagram is that of D_{n-1}, so $G_0 = H + D_{n-1} = H + so(2(n-1))$. There are two G_0-modules,

$$G_{-1} = \mathrm{span}\{e_{1i} - e_{n+i,n+1},\ e_{1,n+i} - e_{i,n+1};\ i = 2, \ldots, n\}, \tag{6.2}$$

and $G_{+1} = \omega(G_{-1})$. G has the corresponding grading $so(2n) = G_{-1} \oplus G_0 \oplus G_{+1}$, and there are $N = 2(n-1)$ commuting annihilation operators. Denoting the CAOs by

$$\begin{aligned}
&d^-_{-i} = e_{1,n+i+1} - e_{i+1,n+1},\ d^-_{+i} = e_{1,i+1} - e_{n+i+1,n+1}, \qquad i = 1, \ldots, n-1, \\
&d^+_{-i} = e_{n+i+1,1} - e_{n+1,i+1},\ d^+_{+i} = e_{i+1,1} - e_{n+1,n+i+1}, \qquad i = 1, \ldots, n-1,
\end{aligned} \tag{6.3}$$

then, for $\xi, \eta, \epsilon = \pm$ and $i, j, k = 1, \ldots, n-1$, the relations $\mathcal{R}$ are given by:

$$[d^\epsilon_{\xi i}, d^\epsilon_{\eta j}] = 0,$$

$$
\begin{aligned}
&[d^+_{-i}, d^-_{+i}] = [d^+_{+i}, d^-_{-i}] = 0, \\
&[[d^+_{\xi i}, d^-_{\eta j}], d^-_{\epsilon k}] = -\delta_{\xi\eta}\delta_{ij}d^-_{\epsilon k} - \delta_{\xi\epsilon}\delta_{ik}d^-_{\eta j} + \delta_{\eta,-\epsilon}\delta_{jk}d^-_{-\xi,i}, \\
&[[d^+_{\xi i}, d^-_{\eta j}], d^+_{\epsilon k}] = \delta_{\xi\eta}\delta_{ij}d^+_{\epsilon k} + \delta_{\eta\epsilon}\delta_{jk}d^+_{\xi i} - \delta_{\xi,-\epsilon}\delta_{ik}d^+_{-\eta,j}.
\end{aligned} \tag{6.4}
$$

Although the relations (6.4) are new, the existence of the set of CAOs (6.3) was pointed out in [6].

Step 2. When node i ($i = 2, \ldots, n-2$) is deleted from the Dynkin diagram of D_n, the remaining diagram is that of $sl(i) \oplus so(2(n-i))$ (or $sl(n-2) \oplus sl(2) \oplus sl(2)$ in the case $i = n-2$). With $G_0 = sl(i) \oplus so(2(n-i))$, there are four G_0-modules, and $so(2n)$ has the following grading $so(2n) = G_{-2} \oplus G_{-1} \oplus G_0 \oplus G_{+1} \oplus G_{+2}$ with

$$
G_{-1} = \text{span}\{e_{kl} - e_{n+l,n+k},\ e_{k,n+l} - e_{l,n+k};\ k = 1, \ldots, i,\ l = i+1, \ldots, n\}. \tag{6.5}
$$

The number of annihilation operators is $N = 2i(n-i)$.

Step 3. Delete node $n-1$ or n from the Dynkin diagram; the remaining diagram is that of $sl(n)$, and $G_0 = H + sl(n)$. There are only two G_0-modules and G has the grading $so(2n) = G_{-1} \oplus G_0 \oplus G_{+1}$, with

$$
\begin{aligned}
G_{-1} &= \text{span}\{e_{j,n+k} - e_{k,n+j};\ 1 \leq j < k \leq n-1\} \cup \\
&\quad\ \text{span}\{e_{jn} - e_{2n,n+j};\ j = 1, \ldots, n-1\},\ \text{for } i = n-1, \qquad (6.6)\\
G_{-1} &= \text{span}\{e_{j,k+n} - e_{k,j+n};\ 1 \leq j < k \leq n\},\ \text{for } i = n.
\end{aligned}
$$

There are $N = \frac{n(n-1)}{2}$ commuting annihilation operators, and these two cases are isomorphic. The relations are not given explicitly.

Step 4. Upon deleting two nodes i and j ($i < j = 1, \ldots, n-2$) or more from the Dynkin diagram of D_n, the corresponding $\mathbb{Z}$-gradings have no longer the required property (there are non-zero G_i with $|i| > 2$).

Step 5. Delete nodes $n-1$ and n from the Dynkin diagram. The remaining diagram is that of $sl(n-1)$. For $G_0 = H + sl(n-1)$, there are six G_0-modules. There are three different ways in which these G_0-modules can be combined, each of them yielding a $\mathbb{Z}$-grading of the form $so(2n) = G_{-2} \oplus G_{-1} \oplus G_0 \oplus G_{+1} \oplus G_{+2}$, namely:

$$G_{-1} = \mathrm{span}\{e_{jn} - e_{2n,n+j},\ e_{j,2n} - e_{n,n+j};\ j = 1, \ldots, n-1\}, \tag{6.7}$$

$$\begin{aligned} G_{-1} &= \mathrm{span}\{e_{jn} - e_{2n,n+j},\ j = 1, \ldots, n-1; \\ &\quad e_{n+j,k} - e_{n+k,j},\ 1 \leq j < k \leq n-1\}, \end{aligned} \tag{6.8}$$

$$\begin{aligned} G_{-1} &= \mathrm{span}\{e_{j+n,n} - e_{2n,j},\ j = 1, \ldots, n-1; \\ &\quad e_{j,k+n} - e_{k,j+n},\ 1 \leq j < k \leq n-1\}. \end{aligned} \tag{6.9}$$

For (6.7), we have $N = 2(n-1)$; for (6.8) and (6.9), we have $N = \frac{n(n-1)}{2}$. It turns out that (6.8) and (6.9) are isomorphic to each other. Here, we shall give the relations only for (6.7). Denote the CAOs of (6.7) by

$$\begin{aligned} &d_{-i}^{-} = e_{i,2n} - e_{n,n+i},\ d_{+i}^{-} = e_{in} - e_{2n,n+i}, \qquad i = 1, \ldots, n-1, \\ &d_{-i}^{+} = e_{2n,i} - e_{n+i,n},\ d_{+i}^{+} = e_{ni} - e_{n+i,2n}, \qquad i = 1, \ldots, n-1. \end{aligned} \tag{6.10}$$

Then, with $\xi, \eta, \epsilon, \gamma = \pm$ or ± 1 and $i, j, k = 1, \ldots, n-1$, the relations are explicitly given by:

$$\begin{aligned} &[d_{\xi i}^{\eta}, d_{\xi j}^{\eta}] = 0, \\ &[d_{-i}^{+}, d_{+j}^{-}] = [d_{+i}^{+}, d_{-j}^{-}] = 0, \\ &[d_{+i}^{-}, d_{-i}^{-}] = [d_{+i}^{+}, d_{-i}^{+}] = 0, \\ &[[d_{\xi i}^{\gamma}, d_{\eta j}^{\gamma}], d_{\epsilon k}^{\gamma}] = 0, \\ &[[d_{+i}^{\xi}, d_{-j}^{\xi}], d_{\epsilon k}^{-\xi}] = -\delta_{ik} d_{-\epsilon j}^{\xi} + \delta_{jk} d_{-\epsilon i}^{\xi}, \\ &[[d_{\xi i}^{\eta}, d_{\xi j}^{-\eta}], d_{\epsilon k}^{\eta}] = \xi\epsilon\delta_{ij} d_{\epsilon k}^{\eta} + \delta_{jk} d_{\epsilon i}^{\eta}. \end{aligned} \tag{6.11}$$

The set of CAOs (6.10) with relations (6.11) is the example that was considered earlier in [6] and [8].

Step 6. Now we move to the extended Dynkin diagram. Deleting node i leaves the Dynkin diagram of $so(2n)$ for $i = 0, 1, n-1, n$, of $sl(2) \oplus sl(2) \oplus so(2(n-2))$ for $i = 2$, of

$sl(3) \oplus so(2(n-3))$ for $i = 3$, and of $= so(2i) \oplus so(2(n-i))$ for $i \geq 4$. In all these cases there is only one G_0-module, so there are no contributions to our classification.

Note that deleting nodes i and j ($1 < i < j < \lfloor \frac{n+1}{2} \rfloor$) from the extended Dynkin diagram is equivalent to delete nodes $(n-j)$ and $(n-i)$.

Step 7. Delete the adjacent nodes $(j-1)$ and j. For $j = 1$ we are back to Step 1, and for $j = 2$ to Step 2 with $i = 2$. For $j \geq 3$ the remaining diagram is that of $\tilde{G}_0 = so(2(j-1)) \oplus so(2(n-j))$ (for $j = 3$ this is $sl(2) \oplus sl(2) \oplus so(2(n-j))$ and for $j = 4$ this is $sl(4) \oplus so(2(n-j)))$. There are five $\tilde{G}_0$-modules g_k, one with $\omega(g_5) = g_5$, so one has to put $G_0 = H + \tilde{G}_0 + g_5 \equiv H + so(2(n-1))$. Now, there are only two G_0-modules, G has the grading $so(2n) = G_{-1} \oplus G_0 \oplus G_{+1}$, and all these cases are isomorphic to those of Step 1.

Step 8. Delete the nonadjacent nodes i and j ($i < j-1$) from the extended Dynkin diagram. The remaining diagram is that of $\tilde{G}_0 = so(2i) \oplus sl(j-i) \oplus so(2(n-j))$ (for $i = 2$ this is $sl(2) \oplus sl(2) \oplus sl(j-i) \oplus so(2(n-j))$; for $i = 3$ this is $sl(3) \oplus sl(j-i) \oplus so(2(n-j)))$. There are nine $\tilde{G}_0$-modules g_k, one with $\omega(g_9) = g_9$. Putting $G_0 = H + \tilde{G}_0 + g_9 \equiv H + sl(j-i) \oplus so(2(n-j+i))$, there are only four G_0-modules. All these cases are isomorphic to those of Step 2.

Step 9. If we delete 3 or more nodes from the extended Dynkin diagram, the corresponding $\mathbb{Z}$-grading of $so(2n)$ has no longer the required properties (i.e. there are non-zero subspaces G_i with $|i| > 2$).

VII Summary and conclusions

We have obtained a complete classification of all GQS associated with the classical Lie algebras. The familiar cases (para-Fermi statistics and A-statistics) appear as simple examples in our classification. It is worth observing that some other examples in this classification are also rather simple. The GQS given in (3.6) and (3.12), e.g., seem to be closely related to A-statistics, except that there are two kind of 'particles' corresponding to the CAOs (see (3.5) and (3.11)). The GQS of type D given in (6.4) has also particularly simple defining relations. For convenience, a comprehensive summary of the classification of all GQS is given in Table 2.

As we have already mentioned in the main text, several cases in our classification appear as examples in Ref. [7]-[12] and in Palev's thesis [6]. In these papers or in the thesis, however, no classification is given: only a number of examples inspired by the para-Fermi case are considered. Furthermore, for some of these examples Fock type representations are constructed.

In order to study the physical properties of a GQS, one should determine the action of the CAOs in a Fock space. Thus one is automatically led to representation theory. Here, the Lie algebraic framework is useful, since a lot is known about Lie algebra representations. Apart from other properties to be satisfied, these Fock spaces should be 'unitary' (with respect to the given anti-involution ω). Whether the class of finite dimensional representations of G plays a role, or whether it is a class of infinite dimensional representations, depends on the choice of ω. With the standard choice considered in this paper, the unitary representations are finite dimensional. For another choice of ω (still with $\omega(G_{-1}) = G_{+1}$, but no longer all +-signs in $\omega(x_i^-) = \pm x_i^+$), our classification of GQS remains valid, but the unitary representations will be infinite dimensional.

It is only after a classification of the Fock spaces for a particular GQS that one can study its macroscopic and microscopic properties. Such a program is feasable, and can give rise to interesting quantum statistical properties. For example, for A-statistics, the microscopic properties (i.e. the properties of the CAOs and their action on the Fock spaces) have been described in [7]-[11], whereas the macroscopic properties (i.e. the statistical properties of ensembles of 'particles' satisfying this GQS) have been studied in [12]. We hope that some other cases of this classification will yield similar interesting GQS.

From the mathematical point of view, a set of CAOs together with a complete set of relations $\mathcal{R}$ unambiguously describes the Lie algebra. So each case of our classification also gives the description of a classical Lie algebra in terms of a number of generators subject to certain relations. This can also be reformulated in terms of the notion of Lie triple systems [13]. According to the definition, a Lie triple system L of an associative algebra A is a subspace of A that is closed under the ternary composition $[[a, b], c]$, where $[a, b] = ab - ba$. It is easy to see that in our case the subspace $G_{-1} \oplus G_{+1}$ (i.e. the subspace spanned by all CAOs) is a Lie triple system for the universal enveloping algebra $U(G)$.

This paper was devoted to classical Lie algebras only. The exceptional Lie algebras are not considered here. Although it would be possible to perform a mathematical classification of the GQS associated with G_2, F_4, E_6, E_7 and E_8, it is obvious that in such a case the number of CAOs is a fixed integer. For physical applications, it is of importance that the number of CAOs is not a fixed number but an integer parameter N. In fact, in quantum field theoretical applications, one is mainly interested in the case $N \to \infty$.

As mentioned in the introduction, para-Bose statistics is connected with a Lie superalgebra, the orthosymplectic superalgebra $osp(1|2n)$. In a future paper, we hope to classify all GQS associated with the classical Lie superalgebras.

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Table 1. Classical Lie algebras, their (extended) Dynkin diagrams with a labelling of the nodes and the corresponding simple roots.

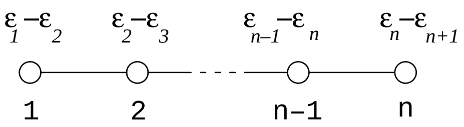
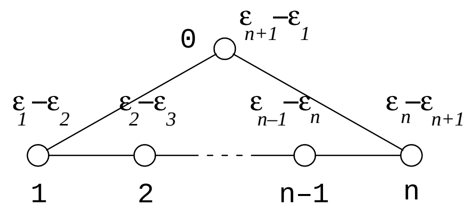
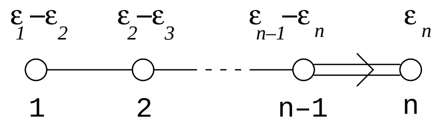
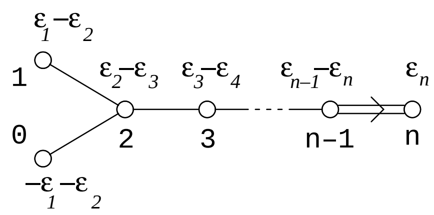
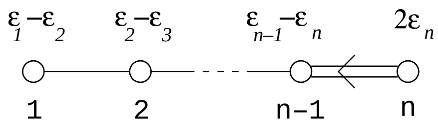
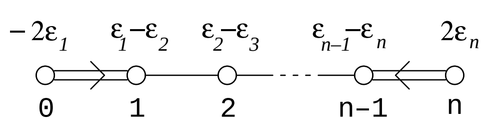
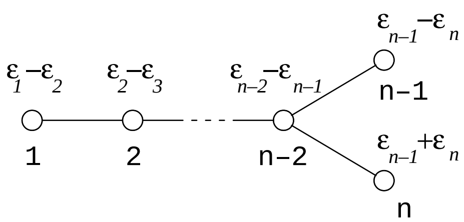
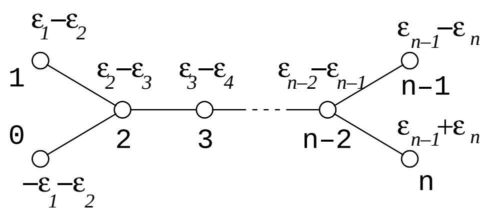
Lie algebra	Dynkin diagram	extended Dynkin diagram
A_n $(n > 0)$		
B_n $(n > 1)$		
C_n $(n > 2)$		
D_n $(n > 3)$		

Table 2. Summary of the classification: all non-isomorphic GQS associated with a classical Lie algebra are given. For each GQS, we list: the Dynkin diagram of G_0 (described in terms of the Dynkin diagram D of G), the subspace G_{-1} (as a reference to the main text), the number of annihilation operators (N), and the relations $\mathcal{R}$ (if given in the text).

Lie algebra	Dynkin diagram of G_0	G_{-1}	N	$\mathcal{R}$
A_n	$D-\{i\}$ ($i \leq \lfloor \frac{n+1}{2} \rfloor$)	(3.1)	$i(n+1-i)$	$i=1:$ (3.4) $i=2:$ (3.6)
	$D-\{i,j\}$ ($i \leq \lfloor \frac{n}{2} \rfloor$ $i<j<n+1-i$)	(3.7)	$(j-i)(n+1-j+i)$	$j-i=1:$ (3.12)
B_n	$D-\{1\}$	(4.2)	$2n-1$	(4.4)
	$D-\{i\}$ ($2 \leq i \leq n$)	(4.5)	$2i(n-i)+i$	$i=n$: (4.7)
C_n	$D-\{i\}$ ($1 \leq i \leq n-1$)	(5.2)	$2i(n-i)$	$i=1:$ (5.4) $i=n-1:$ (5.6)
	$D-\{n\}$	(5.7)	$\frac{n(n+1)}{2}$	-
D_n	$D-\{1\}$	(6.2)	$2(n-1)$	(6.4)
	$D-\{i\}$ ($2 \leq i \leq n-2$)	(6.5)	$2i(n-i)$	-
	$D-\{n\}$	(6.6)	$\frac{n(n-1)}{2}$	-
	$D-\{n-1,n\}$	(6.7) (6.8)	$2(n-1)$ $\frac{n(n-1)}{2}$	(6.11) -