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Lie Superalgebraic Approach to Quantum Statistics. osp(3|2) Wigner Quantum Oscillator*

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Abstract. The ordinary Bose and Fermi statistics and their generalizations para-Bose and para-Fermi statistics are considered from an algebraic point of view. It is indicated that they correspond to representations of Lie superalgebras $B(0|n)$ and B_n. Generalized statistics and nonstandard quantum systems associated to the other basic classical Lie superalgebras are discussed. The Wigner quantum oscillator corresponding to the Lie superalgebra $osp(3|2)$ is considered.

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1 Introduction

Every known particle in the Standard model is either a fermion or a boson. However the model does not describe the known universe, actually it corresponds to only a small percentage of the universe. A simple and natural generalization of bosons and fermions are so called parabosons and parafermions introduced in 1953 by Green [1]. The absence of empirical data for the existence of paraparticles could be because of their big masses, weak scale couplings, and lack of gauge couplings as indicated in [2]. Furthermore parabosons and parafermions, corresponding to order of statistics $p = 2$ are candidates for particles of dark matter and energy (see [2] for details).

A set of n pairs of parafermion operators F_i^ξ ($\xi = \pm$ and $i = 1, \ldots, n$), are defined by the following trilinear relations ($\xi, \eta, \epsilon = \pm$ or $\pm 1;\ j, k, l = 1, \ldots, n$):

$$[[F_j^\xi, F_k^\eta], F_l^\epsilon] = \frac{1}{2}(\epsilon - \eta)^2 \delta_{kl} F_j^\xi - \frac{1}{2}(\epsilon - \xi)^2 \delta_{jl} F_k^\eta, \tag{1}$$

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instead of the bilinear anticommutators for fermions $f_i^{\pm}$

$$\{f_i^-, f_j^+\} = \delta_{ij}, \qquad \{f_i^-, f_j^-\} = \{f_i^+, f_j^+\} = 0. \tag{2}$$

However, applying the formula

$$[AB, C] = A\{B, C\} - \{A, C\}B, \tag{3}$$

it is straightforward to see that fermions satisfy the parafermions relations (1). Therefore Fermi-Dirac statistics is a particular case of parafermion statistics. About ten years after the introduction of parafermion relations (1) by Green, it was proved that they are associated with the orthogonal Lie algebra $so(2n+1) \equiv B_n$ [3,4]. For general parafermion statistics, a class of finite dimensional $so(2n+1)$ representations (of Fock type) needs to be investigated (this is done in [5]), and a certain representation of $so(2n+1)$ corresponds to the classical Fermi relations (2).

Similarly, n pairs of paraboson operators B_i^ξ ($\xi = \pm$ and $i = 1, \ldots, n$), are defined by the trilinear relations ($\xi, \eta, \epsilon = \pm$ or ± 1; $\quad j, k, l = 1, \ldots, n$)

$$[\{B_j^\xi, B_k^\eta\}, B_l^\epsilon] = (\epsilon - \xi)\delta_{jl} B_k^\eta + (\epsilon - \eta)\delta_{kl} B_j^\xi, \tag{4}$$

instead of the bilinear commutators for bosons $b_i^{\pm}$

$$[b_i^-, b_j^+] = \delta_{ij}, \qquad [b_i^-, b_j^-] = [b_i^+, b_j^+] = 0. \tag{5}$$

Now applying the formula

$$[AB, C] = A[B, C] + [A, C]B, \tag{6}$$

it is easy to see that bosons satisfy the paraboson relations (4). Twenty years after the connection between parafermion statistics and the Lie algebra $so(2n+1)$, a new connection, between paraboson statistics and the orthosymplectic Lie superalgebra $B(0|n) \equiv osp(1|2n)$ [6] was discovered [7]. The Lie superalgebra generated by $2n$ odd elements B_i^ξ, with $\xi = \pm$ and $i = 1, \ldots, n$, subject to the triple relations (4), is $osp(1|2n)$. A certain representation of $osp(1|2n)$ corresponds to Bose-Einstein statistics. For more general paraboson statistics, a class of infinite dimensional $osp(1|2n)$ representations needs to be investigated [8].

The above considerations show that it makes sense to investigate whether one can define generalized quantum statistics based on the other basic classical Lie superalgebras and this was done in [9]. Each such statistics is determined by M creation operators y_i^+ ($i = 1, \ldots, M$) and M annihilation operators y_i^- ($i = 1, \ldots, M$), which generate the corresponding superalgebra G subject to certain triple relations $\mathcal{R}$. This leads to a $\mathbb{Z}$-grading of the algebra G of the form

$$G = G_{-2} \oplus G_{-1} \oplus G_0 \oplus G_{+1} \oplus G_{+2}, \tag{7}$$

with $G_{\pm 1} = \text{span}\{y_i^{\pm},\ i = 1,\ldots,M\}$ and $G_{j+k} = [\![G_j, G_k]\!]$. One of the possible applications of the results in [9] is that generalized quantum statistics with only odd creation and annihilation operators are candidates for solutions of Wigner quantum systems [10]. The latter differ from a canonical quantum system only by the replacement of the postulate on canonical commutation relations by a new postulate, namely that Hamilton's equations and the Heisenberg equations hold and are identical (as operator equations) in the state space W of the system under consideration. In the present paper we will apply the results for a generalized quantum statistics with only odd creation and annihilation operators corresponding to the Lie superalgebra $B(m|n)$ for the simplest case $m = n = 1$ and we will investigate a 3D harmonic oscillator as a Wigner quantum system. In Section 2 we define the Lie superalgebra $B(1|1) \equiv osp(3|2)$ and construct a class of $osp(3|2)$ infinite dimensional irreducible representations. Section 3 is devoted to the $osp(3|2)$ Wigner quantum oscillator.

2 The Lie Superalgebra osp(3|2) and a Class of osp(3|2) Representations

The Lie superalgebra $B(1|1) \equiv osp(3|2)$ [6] consists of matrices of the form

$$\begin{pmatrix} a & 0 & b & x & u \\ 0 & -a & c & y & v \\ -c & -b & 0 & z & w \\ v & u & w & d & e \\ -y & -x & -z & f & -d \end{pmatrix}, \tag{8}$$

where the nonzero entries are arbitrary complex numbers. The even subalgebra $so(3) \oplus sp(2)$ consists of all matrices (8), for which $x = y = z = u = v = w = 0$, whereas the odd subspace is obtained taking $a = b = c = d = e = f = 0$. Let e_{ij} be a 5-by-5 matrix with 1 on the cross of the i^{th} row and the j^{th} column and zero elsewhere. The Cartan subalgebra H of $osp(3|2)$ is the subspace of diagonal matrices with basis $h_1 = e_{11} - e_{22},\ h_2 = e_{44} - e_{55}$. In terms of the dual basis ϵ, δ of H^*, the even root vectors and corresponding roots of $osp(3|2)$ are given by

$$\begin{aligned} &e_{13} - e_{32} \leftrightarrow \epsilon, \qquad && e_{45} \leftrightarrow 2\delta, \\ &e_{23} - e_{31} \leftrightarrow -\epsilon, \qquad && e_{54} \leftrightarrow -2\delta, \end{aligned}$$

and the odd ones by

$$\begin{aligned} &a_1^- = e_{14} - e_{52} \leftrightarrow \epsilon - \delta, \qquad && a_1^+ = e_{25} + e_{41} \leftrightarrow -\epsilon + \delta, \\ &a_2^- = e_{34} - e_{53} \leftrightarrow -\delta, \qquad && a_2^+ = e_{35} + e_{43} \leftrightarrow \delta, \\ &a_3^- = e_{24} - e_{51} \leftrightarrow -\epsilon - \delta, \qquad && a_3^+ = e_{15} + e_{42} \leftrightarrow \epsilon + \delta\,. \end{aligned} \tag{9}$$

It is straightforward to check that the matrices $a_i^\xi,\ \xi=\pm,\ i=1,2,3$ satisfy the following triple relations:

$$
\begin{aligned}
&[\{a_i^\xi, a_j^\xi\}, a_k^\xi] = 0,\\
&[\{a_i^\xi, a_j^\xi\}, a_k^{-\xi}] = -(-1)^\xi 2\delta_{4,i+j} a_{4-k}^\xi,\\
&[\{a_i^+, a_j^-\}, a_k^\xi] =\\
&(-1)^\xi \delta_{ij}(|i-k| + \delta_{6,i+j+k}) a_k^\xi + \delta_{i,j+1}(-1)^{i+k} a_{k+1}^\xi + \delta_{i+1,j}(-1)^{i+k} a_{k-1}^\xi .
\end{aligned}
\tag{10}
$$

Theorem 1. *([9], Section IV) As a Lie superalgebra defined by generators and relations, $osp(3|2)$ is generated by elements $a_j^\pm,\ j=1,2,3$ subject to the triple relations (10).*

Theorem 2. *An orthonormal basis for a class of unitary irreducible $osp(3|2)$ modules $V(p)$, $p=1,2,\ldots$, is given by the vectors ($\mu_{ij}\in\mathbb{Z}_+$)*

$$
\begin{aligned}
|\mu) = \left|\begin{matrix}\mu_{12},\mu_{22}\\ \mu_{11}\end{matrix}\right), &\ \text{for}\ \ \mu_{22}=0:\ \ \mu_{12}=0,1\ldots,p;\\
&\ \text{for}\ \ \mu_{22}=1,2,\ldots:\ \mu_{12}=1,2,\ldots,p;\\
&\ \mu_{12}-\mu_{11}=\theta\in\{0,1\}\ \ (\text{if}\ \ \mu_{12}=0,\theta=0).
\end{aligned}
\tag{11}
$$

The action of the Cartan algebra elements of $osp(3|2)$ is

$$
h_1|\mu) = \left(-\frac{p}{2}+\mu_{11}\right)|\mu), \quad h_2|\mu) = \left(\frac{p}{2}+\mu_{12}+\mu_{22}-\mu_{11}\right)|\mu). \tag{12}
$$

For the action of the operators $a_j^\pm,\ j=1,2,3$ we have

$$
a_1^+\left|\begin{matrix}\mu_{12},\mu_{22}\\ \mu_{11}\end{matrix}\right) = (1-\theta)\,(\mu_{12}+\mu_{22})^{\frac12}\left|\begin{matrix}\mu_{12},\mu_{22}\\ \mu_{11}-1\end{matrix}\right), \tag{13}
$$

$$
\begin{aligned}
&a_2^+\left|\begin{matrix}\mu_{12},\mu_{22}\\ \mu_{11}\end{matrix}\right) = (1-\theta)\left(\frac{\mu_{12}(p-\mu_{12})}{2(\mu_{12}+\mu_{22}+1-\mathcal{O}_{1+\mu_{22}}})\right)^{\frac12}\left|\begin{matrix}\mu_{12}+1,\mu_{22}\\ \mu_{11}\end{matrix}\right)\\
&+(-1)^\theta\left(\frac{(\mu_{12}+\mu_{22})(\mathcal{O}_{\mu_{22}}\mu_{22}+1)(\mathcal{E}_{1+\mu_{22}}(p+\mu_{22})+1)}{2(\mu_{11}+\mu_{22}+1)}\right)^{\frac12}\\
&\times\left(\frac{\mathcal{O}_{1+\mu_{22}}(\mu_{12}+\mu_{22})+1}{\mathcal{E}_{1+\mu_{22}}(\mu_{12}+\mu_{22}-1)+1}\right)^{\frac12}\left|\begin{matrix}\mu_{12},\mu_{22}+1\\ \mu_{11}\end{matrix}\right),
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
&a_3^+\left|\begin{matrix}\mu_{12},\mu_{22}\\ \mu_{11}\end{matrix}\right) = -(-1)^\theta\mathcal{E}_{\mu_{22}}\left(\frac{\mu_{12}(p-\mu_{12})}{(\mu_{12}+\mu_{22}+2\theta)}\right)^{\frac12}\left|\begin{matrix}\mu_{12}+1,\mu_{22}+1\\ \mu_{11}+1\end{matrix}\right)\\
&+\theta\left(\frac{(\mu_{22}+1+\mathcal{E}_{\mu_{22}})(p+\mu_{22}+1+\mathcal{E}_{\mu_{22}})}{(\mu_{12}+\mu_{22}+2\mathcal{E}_{\mu_{22}})}\right)^{\frac12}\left|\begin{matrix}\mu_{12},\mu_{22}+2\\ \mu_{11}+1\end{matrix}\right),
\end{aligned}
\tag{15}
$$

$$a_1^- \left| \begin{matrix} \mu_{12}, \mu_{22} \\ \mu_{11} \end{matrix} \right) = \theta \left(\mu_{12} + \mu_{22}\right)^{\frac{1}{2}} \left| \begin{matrix} \mu_{12}, \mu_{22} \\ \mu_{11} + 1 \end{matrix} \right), \tag{16}$$

$$a_2^- \left| \begin{matrix} \mu_{12}, \mu_{22} \\ \mu_{11} \end{matrix} \right) = \theta \left(\frac{(\mu_{12} - 1)(p - \mu_{12} + 1)}{2(\mu_{12} + \mu_{22} - \mathcal{O}_{1+\mu_{22}}}) \right)^{\frac{1}{2}} \left| \begin{matrix} \mu_{12} - 1, \mu_{22} \\ \mu_{11} \end{matrix} \right)$$
$$+(-1)^{\theta} \left(\frac{(\mathcal{O}_{1+\mu_{22}}(\mu_{22} - 1) + 1)(\mathcal{E}_{\mu_{22}}(p + \mu_{22} - 1) + 1)}{2(\mu_{11} + \mu_{22})} \right)^{\frac{1}{2}}$$
$$\times \left(\frac{(\mu_{12} + \mu_{22} - 1)(\mathcal{O}_{\mu_{22}}(\mu_{12} + \mu_{22} - 1) + 1)}{\mathcal{E}_{\mu_{22}}(\mu_{12} + \mu_{22} - 2) + 1} \right)^{\frac{1}{2}} \left| \begin{matrix} \mu_{12}, \mu_{22} - 1 \\ \mu_{11} \end{matrix} \right), \tag{17}$$

$$a_3^- \left| \begin{matrix} \mu_{12}, \mu_{22} \\ \mu_{11} \end{matrix} \right)$$
$$= -(-1)^{\theta} \mathcal{E}_{1+\mu_{22}} \left(\frac{(\mu_{12} - 1)(p - \mu_{12} + 1)}{(\mu_{12} + \mu_{22} - 2 + 2\theta)} \right)^{\frac{1}{2}} \left| \begin{matrix} \mu_{12} - 1, \mu_{22} - 1 \\ \mu_{11} - 1 \end{matrix} \right)$$
$$+(1 - \theta) \left(\frac{(\mu_{22} - 1 + \mathcal{E}_{\mu_{22}})(p + \mu_{22} - 1 + \mathcal{E}_{\mu_{22}})}{(\mu_{12} + \mu_{22} - 2 + 2\mathcal{E}_{\mu_{22}})} \right)^{\frac{1}{2}} \left| \begin{matrix} \mu_{12}, \mu_{22} - 2 \\ \mu_{11} - 1 \end{matrix} \right), \tag{18}$$

$$\mathcal{E}_j = 1 \textit{ if } j \textit{ is even and 0 otherwise}, \quad \mathcal{O}_j = 1 \textit{ if } j \textit{ is odd and 0 otherwise}. \tag{19}$$

In order to prove that the explicit actions (13)-(18) give a representation of $osp(3|2)$ we have checked that (13)-(18) satisfy the defining $osp(3|2)$ relations (10). The irreducibility follows from the fact that for any nonzero vector $y \in V(p)$ there exists a polynomial $\mathcal{P}$ of $osp(3|2)$ generators such that $\mathcal{P}y = V(p)$.

Note that another way to get the actions of $a_j^{\pm}, j = 1, 2, 3$ is to express them in terms of a pair of para-Fermi and a pair of para-Bose operators and apply the results given in [11].

3 osp(3|2) Wigner Quantum Oscillator

Consider a three-dimensional harmonic oscillator

$$H = \frac{\mathbf{p}^2}{2m} + \frac{m\omega^2}{2}\mathbf{r}^2 \tag{20}$$

as a Wigner quantum system. Therefore we consider

$$\mathbf{r} = (r_1, r_2, r_3) \quad \text{and} \quad \mathbf{p} = (p_1, p_2, p_3),$$

as unknown operators and must find them such that Hamilton's equations

$$\dot{\mathbf{p}} = -m\omega^2\mathbf{r}, \quad \dot{\mathbf{r}} = \frac{\mathbf{p}}{m} \tag{21}$$

and the Heisenberg equations

$$\dot{\mathbf{p}} = -\frac{i}{\hbar}[\mathbf{p}, H], \quad \dot{\mathbf{r}} = -\frac{i}{\hbar}[\mathbf{r}, H], \tag{22}$$

are identical (as operator equations) in the state space W, which is a Hilbert space, of the oscillator. Moreover, we must define the projections of the angular momentum **M=**(M_1, M_2, M_3), so that **M**, **r** and **p** transform as vectors

$$[M_j, c_k] = i\sum_{l=1}^{3} \varepsilon_{jkl} c_l, \; c_k = M_k, r_k, p_k, \; j,k = 1,2,3. \tag{23}$$

Consider a new set of unknown operators

$$A_k^{\pm} = \sqrt{\frac{3m\omega}{4\hbar}} r_k \mp i \frac{3}{\sqrt{4m\omega\hbar}} p_k, \; k = 1,2,3. \tag{24}$$

In terms of $A_k^{\pm}$ the Hamiltonian reads

$$H = \frac{\omega\hbar}{3} \sum_{k=1}^{3} \{A_k^+, A_k^-\}$$

and the compatibility conditions to be satisfied by the Wigner quantum system yields ($k = 1,2,3$)

$$\sum_{i=1}^{3} [\{A_i^+, A_i^-\}, A_k^{\pm}] = \pm 3A_k^{\pm}. \tag{25}$$

As a solution to (25) one can choose the operators $A_k^{\pm}$ as follows:

$$A_1^{\xi} = -\frac{1}{\sqrt{2}}(a_1^{\xi} + a_3^{\xi}), \;\; A_2^{\xi} = -\xi\frac{i}{\sqrt{2}}(a_1^{\xi} - a_3^{\xi}), \;\; A_3^{\xi} = a_2^{\xi}, \;\; \xi = \pm, \tag{26}$$

where $a_i^{\pm}$ satisfy the defining triple $osp(3|2)$ relations (10). The projections (M_1, M_2, M_3) of the angular momentum can be defined as

$$\begin{aligned} M_1 &= \frac{1}{\sqrt{2}}(\{a_1^+, a_2^-\} + \{a_2^+, a_1^-\}), \\ M_2 &= \frac{i}{\sqrt{2}}(\{a_1^+, a_2^-\} - \{a_2^+, a_1^-\}), \\ M_3 &= \{a_1^+, a_1^-\} - \{a_2^+, a_2^-\}. \end{aligned} \tag{27}$$

Let W be the $osp(3|2)$ module $V(p)$. In each such module $(a_i^-)^{\dagger} = a_i^+, i = 1,2,3$. Therefore the position and the momentum operators are Hermitian operators (hence also the Hamiltonian H, the square of the angular momentum

$\mathbf{M}^2$ and its projections M_1, M_2, M_3 are Hermitian operators). From (26) and Theorem 2 one immediately derives that

$$H|\mu) = \frac{\hbar\omega}{3}\sum_{i=1}^{3}\{a_i^+, a_i^-\}|\mu) = \hbar\omega(\frac{p}{2} + \mu_{22} + \theta)|\mu) \tag{28}$$

Thus, the energy spectrum $E_n, n = 0, 1, 2, \ldots$ of the Wigner quantum oscillator is given by

$$E_n = \hbar\omega(n + \frac{p}{2}),\ n = 0, 1, 2, \ldots. \tag{29}$$

The case $p = 1$ corresponds to anticommuting pairs of Bose and Fermi operators and the energy spectrum is the same as for the one-dimensional canonical harmonic oscillator.

The coordinates, the momenta and the angular momenta operators are on the same footing: the different components do not commute with each other

$$[r_i, r_j] \neq 0,\ [p_i, p_j] \neq 0,\ [M_i, M_j] \neq 0,\ i \neq j = 1, 2, 3. \tag{30}$$

The geometry of the oscillator is noncommutative.

Each state $|\mu)$ is an eigenvector of M_3

$$M_3|\mu) = (-\frac{p}{2} + \mu_{11})|\mu), \quad \mu_{11} = 0, 1, \ldots, p. \tag{31}$$

Consequently the Wigner oscillator has an angular momentum

$$M = \frac{p}{2}, \quad p = 1, 2, \ldots.$$

One of the outcomes of the present model is the possibility to have an oscillator with a half-integer (for p odd) angular momentum M. Following the ideas of Ref. [12] one can consider the oscillator as describing the internal motion of two point particles interacting via a harmonic potential. To this end one has to assume (something that always holds in canonical quantum mechanics) that the operators of the coordinates and momenta of the center of mass commute with the internal variables $\mathbf{r}$ and $\mathbf{p}$. In this picture the $osp(3|2)$ oscillator describes the internal motion of two constituents. The angular momentum M is the spin of the composite system. Restated in this way our result could be viewed as a classical model of spin: two (nonrelativistic noncanonical) point particles are curling around each other in such a way that the resulting angular momentum of the composite system can take also half-integer values.

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