ON THE N-PARTICLE WIGNER QUANTUM OSCILLATOR: NON-COMMUTATIVE COORDINATES AND PARTICLE LOCALISATION

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After a general introduction to Wigner Quantum Systems, we define the three-dimensional n-particle Wigner Quantum Oscillator, and its relation to the Lie superalgebra $\mathfrak{sl}(1|3n)$. In this framework, coordinates, momentum and the angular momentum of the particles are defined and investigated. This investigation is done in state spaces, using representation theory of $\mathfrak{sl}(1|3n)$. For the case $n = 1$, the main properties are listed, with an emphasis on the non-commutative coordinates of the particle and its unconventional consequences. For general n, we study energy spectra, angular momentum, and in particular the particle configuration and its interpretation. Throughout, a comparison with the canonical oscillator solution is given.

1. Wigner Quantum Systems

In one of his famous papers entitled *Do the equations of motion determine the quantum mechanical commutation relations?* Wigner [1] introduced in 1950 the concept now referred to as a Wigner Quantum System. Wigner himself considered the example of a one-dimensional harmonic oscillator, with Hamiltonian $\hat{H} = \frac{1}{2}(\hat{p}^2 + \hat{q}^2)$ (using here $m = \omega = \hbar = 1$). Working in the Heisenberg picture, he abandoned the requirement $[\hat{q}, \hat{p}] = i$ (the canonical commutation relation - CCR), but instead searched for all operators $\hat{q}$ and $\hat{p}$ such that Hamilton's equations $\dot{\hat{q}} = \hat{p}$ and $\dot{\hat{p}} = -\hat{q}$ are identical with the Heisenberg equations $\dot{\hat{q}} = i[\hat{H}, \hat{q}]$ and $\dot{\hat{p}} = i[\hat{H}, \hat{p}]$. The conclusion of Wigner was that this system now has infinitely many solutions for the "unknown operators" $\hat{q}$ and $\hat{p}$, only one of which satisties the CCR $[\hat{q}, \hat{p}] = i$.

Nowadays, it is known [2] that this problem posed by Wigner also has a group-theoretical meaning. In fact, what Wigner did is equivalent to the classification of the unitary representations of the Lie superalgebra $\mathfrak{osp}(1|2)$.

Observe that at the time of Wigner it was already known that Hamilton's equations together with the CCRs imply the Heisenberg equations, and that the Heisenberg equations together with the CCRs imply Hamilton's equations (Ehrenfest theorem [3]). Wigner showed, on the example of the one-dimensional harmonic oscillator, that the compatibility of the Heisenberg equations and Hamilton's equations do not imply the validity of the CCRs.

The ideas of Wigner can be further generalized in those cases when the equations of motion admit more than one Hamiltonian [4]. For a recent development along this line we refer to [5] and the references therein. Here we are not concerned with this aspect.

For us, a Wigner Quantum System (WQS) [2, 6] is defined as an n-particle system in three dimensions with Hamiltonian

$$\hat{H} = \sum_{\alpha=1}^{n} \frac{\hat{\mathbf{P}}_\alpha^2}{2m_\alpha} + V(\hat{\mathbf{R}}_1, \hat{\mathbf{R}}_2, \ldots, \hat{\mathbf{R}}_n), \tag{1}$$

depending on $6n$ variables $\hat{\mathbf{R}}_\alpha = (\hat{R}_{\alpha 1}, \hat{R}_{\alpha 2}, \hat{R}_{\alpha 3})$ and $\hat{\mathbf{P}}_\alpha = (\hat{P}_{\alpha 1}, \hat{P}_{\alpha 2}, \hat{P}_{\alpha 3})$, with $\alpha = 1, 2, \ldots, n$, to be interpreted as (Cartesian) coordinates and momenta of the particles, respectively. All postulates of ordinary quantum mechanics should hold, i.e.

- **P1** The state space W is a Hilbert space. To every physical observable O there corresponds a Hermitian (self-adjoint) operator $\hat{O}$ acting in W.
- **P2** The observable O can take on only those values which are eigenvalues of $\hat{O}$. The expectation value of the observable O in a state ψ is given by $\langle \hat{O} \rangle_\psi = (\psi, \hat{O}\psi)/(\psi, \psi)$.

But the postulate on CCRs is replaced by:

P3 Hamilton's equations and the Heisenberg equations hold and are identical (as operator equations) in W.

In the following sections, we shall consider such systems for a simple Hamiltonian, i.e. the harmonic oscillator Hamiltonian.

2. The three-dimensional n-particle Wigner Quantum Oscillator (WQO)

As a special case of (1), we consider the harmonic oscillator Hamiltonian

$$\hat{H} = \sum_{\alpha=1}^{n} \Big(\frac{\hat{\mathbf{P}}_\alpha^2}{2m} + \frac{m\omega^2}{2} \hat{\mathbf{R}}_\alpha^2 \Big), \tag{2}$$

in terms of the $6n$ variables (operators) $\hat{R}_{\alpha i}$ and $\hat{P}_{\alpha i}$ ($i = 1, 2, 3$, $\alpha = 1, 2 \ldots, n$). By postulate **P3**: the 3D vector operators $\hat{\mathbf{R}}_1, \ldots, \hat{\mathbf{R}}_n$ and $\hat{\mathbf{P}}_1, \ldots, \hat{\mathbf{P}}_n$ must satisfy Hamilton's equations

$$\dot{\hat{\mathbf{P}}}_\alpha = -m\omega^2 \hat{\mathbf{R}}_\alpha, \quad \dot{\hat{\mathbf{R}}}_\alpha = \frac{1}{m} \hat{\mathbf{P}}_\alpha, \tag{3}$$

where, as usual, the dot refers to time derivation, and the Heisenberg equations

$$\dot{\hat{\mathbf{P}}}_\alpha = \frac{i}{\hbar} [\hat{H}, \hat{\mathbf{P}}_\alpha], \quad \dot{\hat{\mathbf{R}}}_\alpha = \frac{i}{\hbar} [\hat{H}, \hat{\mathbf{R}}_\alpha], \tag{4}$$

for $\alpha = 1, 2, \ldots, n$. Moreover (3) and (4) should be identical as operator equations. This leads to the following compatibility conditions (CCs):

$$[\hat{H}, \hat{\mathbf{P}}_\alpha] = i\hbar m\omega^2 \hat{\mathbf{R}}_\alpha, \quad [\hat{H}, \hat{\mathbf{R}}_\alpha] = -\frac{i\hbar}{m} \hat{\mathbf{P}}_\alpha. \tag{5}$$

The task is now to find operator solutions of (5). This turns out to be a difficult problem, for which not all solutions are known. On the other hand, one particular solution is well known, namely the canonical bose solution where $[\hat{R}_{\alpha j}, \hat{P}_{\beta k}] = i\hbar \delta_{\alpha\beta} \delta_{jk}$ and $[\hat{R}_{\alpha j}, \hat{R}_{\beta k}] = [\hat{P}_{\alpha j}, \hat{P}_{\beta k}] = 0$.

More than 20 years ago, other classes of solutions for (5) were discovered. One class of solutions can be formulated by means of the Lie superalgebra $\mathfrak{osp}(1|6n)$ [2], and is related to parabose operators [7]. In this paper we shall consider a second class of solutions, first observed by Palev [2], and related to the Lie superalgebra $\mathfrak{sl}(1|3n)$.

For this purpose, consider new (unkown) operators as linear combination of the old ones,

$$A_{\alpha k}^{\pm} = \sqrt{\frac{(3n-1)m\omega}{4\hbar}} \hat{R}_{\alpha k} \pm i \sqrt{\frac{(3n-1)}{4m\omega\hbar}} \hat{P}_{\alpha k}. \tag{6}$$

In terms of these operators, (2) becomes:

$$\hat{H} = \sum_{\alpha=1}^{n} \hat{H}_\alpha \quad \text{with} \quad \hat{H}_\alpha = \frac{\omega\hbar}{3n-1} \sum_{i=1}^{3} \{A_{\alpha i}^+, A_{\alpha i}^-\}, \tag{7}$$

and the compatibility conditions (5) read:

$$\sum_{\beta=1}^{n} \sum_{j=1}^{3} [\{A_{\beta j}^+, A_{\beta j}^-\}, A_{\alpha i}^\pm] = \mp(3n-1) A_{\alpha i}^\pm. \tag{8}$$

All solutions of (8) are not known, but the following yields an important family of solutions:

$$\begin{aligned} &[\{A_{\alpha i}^+, A_{\beta j}^-\}, A_{\gamma k}^+] = \delta_{jk}\delta_{\beta\gamma} A_{\alpha i}^+ - \delta_{ij}\delta_{\alpha\beta} A_{\gamma k}^+, \\ &[\{A_{\alpha i}^+, A_{\beta j}^-\}, A_{\gamma k}^-] = -\delta_{ik}\delta_{\alpha\gamma} A_{\beta j}^- + \delta_{ij}\delta_{\alpha\beta} A_{\gamma k}^-, \\ &\{A_{\alpha i}^+, A_{\beta j}^+\} = \{A_{\alpha i}^-, A_{\beta j}^-\} = 0. \end{aligned} \tag{9}$$

It is easy to verify that (9) implies indeed (8). The main observation of [2] is that the operators $A_{\alpha i}^\pm$ ($i = 1, 2, 3$ and $\alpha = 1, 2, \ldots, n$), subject to the above relations (9), are odd elements generating the Lie superalgebra $\mathfrak{sl}(1|3n)$. They are sometimes referred to as creation and annihilation operators (CAOs) of $\mathfrak{sl}(1|3n)$.

Observe that the operators $\hat{R}_{\alpha k}$ and $\hat{P}_{\alpha k}$ depend on time, and so do the CAOs. By Hamilton's equations

$$\dot{A}_{\alpha k}^\pm(t) = \mp i\omega A_{\alpha k}^\pm(t)$$

one has as solution $A_{\alpha k}^\pm(t) = A_{\alpha k}^\pm(0)\, e^{\mp i\omega t}$. Therefore, it is sufficient that the defining relations (9) hold at time $t = 0$ and we write $A_{\alpha k}^\pm \equiv A_{\alpha k}^\pm(0)$. Explicitly, one has:

$$\begin{aligned} \hat{R}_{\alpha k}(t) &= \sqrt{\frac{\hbar}{(3n-1)m\omega}} (A_{\alpha k}^+ e^{-i\omega t} + A_{\alpha k}^- e^{i\omega t}), \\ \hat{P}_{\alpha k}(t) &= -i\sqrt{\frac{m\omega\hbar}{(3n-1)}} (A_{\alpha k}^+ e^{-i\omega t} - A_{\alpha k}^- e^{i\omega t}). \end{aligned} \tag{10}$$

For later comparison, we recall here also the canonical solution for (2):

$$\begin{aligned} \overline{R}_{\alpha k}(t) &= \sqrt{\frac{\hbar}{(3n-1)m\omega}} (B_{\alpha k}^+ e^{i\omega t} + B_{\alpha k}^- e^{-i\omega t}), \\ \overline{P}_{\alpha k}(t) &= i\sqrt{\frac{m\omega\hbar}{(3n-1)}} (B_{\alpha k}^+ e^{i\omega t} - B_{\alpha k}^- e^{-i\omega t}), \end{aligned} \tag{11}$$

where $B^{\pm}_{\alpha k}$ are ordinary Bose creation and annihilation operators, satisfying $[B^{-}_{\alpha j}, B^{+}_{\beta k}] = \delta_{jk}\delta_{\alpha\beta}$ and $[B^{\pm}_{\alpha j}, B^{\pm}_{\beta k}] = 0$. It is known that the elements $B^{\pm}_{\alpha k}$ (as odd generators) and their anti-commutators $\{B^{+}_{\alpha j}, B^{-}_{\beta k}\}$ generate the Lie superalgebra $\mathfrak{osp}(1|6n)$ [8].

3. The angular momentum operators

The operators $\hat{\mathbf{R}}_\alpha = (\hat{R}_{\alpha 1}, \hat{R}_{\alpha 2}, \hat{R}_{\alpha 3})$ and $\hat{\mathbf{P}}_\alpha = (\hat{P}_{\alpha 1}, \hat{P}_{\alpha 2}, \hat{P}_{\alpha 3})$ of the WQO are interpreted and referred to as the position and momentum operators of the particle α. We introduce one more physical notion, namely the angular momentum of this particle. Since the operator $\hat{\mathbf{R}}_\alpha \times \hat{\mathbf{P}}_\alpha$ no longer satisfies the usual requirements, we follow an alternative definition for the angular momentum. For the particle α, its angular momentum operator $\hat{\mathbf{M}}_\alpha = (\hat{M}_{\alpha 1}, \hat{M}_{\alpha 2}, \hat{M}_{\alpha 3})$ should be such that

- its components are in the enveloping algebra of $\hat{\mathbf{R}}_\alpha = (\hat{R}_{\alpha 1}, \hat{R}_{\alpha 2}, \hat{R}_{\alpha 3})$ and $\hat{\mathbf{P}}_\alpha = (\hat{P}_{\alpha 1}, \hat{P}_{\alpha 2}, \hat{P}_{\alpha 3})$ (and linear in these components);
- $\hat{M}_{\alpha 1}$, $\hat{M}_{\alpha 2}$ and $\hat{M}_{\alpha 3}$ commute with the Hamiltonian;
- they span a basis of the Lie algebra $\mathfrak{so}(3)$.

This leads to a unique solution [6], given by:

$$\hat{M}_{\alpha j} = -\frac{3n-1}{2\hbar} \sum_{k,l=1}^{3} \epsilon_{jkl}\{\hat{R}_{\alpha k}, \hat{P}_{\alpha l}\} = -i \sum_{k,l=1}^{3} \epsilon_{jkl}\{A^{+}_{\alpha k}, A^{-}_{\alpha l}\}, \qquad (12)$$

where ϵ_{jkl} is the familiar antisymmetric tensor. Using (12) and (9), one can verify that the components of $\hat{\mathbf{M}}_\alpha$ and $\hat{\mathbf{M}}_\beta$ commute for $\alpha \neq \beta$, and that indeed the following relations hold:

$$[\hat{M}_{\alpha j}, \hat{M}_{\alpha k}] = i\epsilon_{jkl}\hat{M}_{\alpha l}, \qquad [\hat{M}_{\alpha j}, \hat{H}] = 0. \qquad (13)$$

Apart from the single particle angular momentum, we define as components of the total angular momentum of the n-particle system:

$$\hat{M}_j = \sum_{\alpha=1}^{n} \hat{M}_{\alpha j}, \qquad j = 1, 2, 3. \qquad (14)$$

It is easy to verify that these operators generate an $\mathfrak{so}(3)$ subalgebra of $\mathfrak{sl}(1|3n)$.

4. The state spaces (representations)

In order to investigate physical properties of the WQO, we need to consider the state spaces in which the operators act. According to the postulates, the state space W is a Hilbert space, and $(A^{\pm}_{\alpha j})^{\dagger} = A^{\mp}_{\alpha j}$. The last condition implies that the operators $A^{\pm}_{\alpha j}$ act in "unitary" (or star) representations of $\mathfrak{sl}(1|3n)$.

All the unitary irreducible representations (irreps) of $\mathfrak{sl}(M|N)$ have been classified [9]: essentially they are the typical, the covariant and the contravariant representations. Here, we make one further choice: instead of considering all unitary representations, we shall study only one class of them, namely the so-called Fock type representations [10] of $\mathfrak{sl}(1|3n)$. The advantage of these representations is the fact that all operators have a simple action in a particular basis.

The Fock representations are characterized by a positive integer p: $W = W(p)$ is a finite dimensional covariant irrep with highest weight $(p, 0, \ldots, 0)$. It is determined by the relations:

$$A^{-}_{\alpha j}\,|\,0\rangle = 0, \quad A^{-}_{\alpha j}A^{+}_{\beta k}\,|\,0\rangle = p\delta_{\alpha\beta}\delta_{jk}\,|\,0\rangle, \tag{15}$$

a formulation reminiscent of the definition of "parastatistics of order p". Observe that $W(p)$ is a *typical* irrep when $p \geq 3n$ and *atypical* when $p < 3n$.

It is easy to construct a basis for $W(p)$. Let Θ be a string of 0's and 1's:

$$\Theta \equiv (\theta_{11}, \theta_{12}, \theta_{13}, \ldots \theta_{n1}, \theta_{n2}, \theta_{n3}), \qquad \theta_{\alpha i} \in \{0, 1\}. \tag{16}$$

Then, an orthonormal basis of $W(p)$ is given by the vectors

$$\begin{aligned} |\,p;\Theta\rangle &\equiv |p; \theta_{11}, \theta_{12}, \theta_{13}, \ldots, \theta_{n1}, \theta_{n2}, \theta_{n3}\rangle \\ &= \sqrt{\frac{(p-q)!}{p!}}(A^{+}_{11})^{\theta_{11}}(A^{+}_{12})^{\theta_{12}}(A^{+}_{13})^{\theta_{13}}\cdots \\ &\qquad \cdots(A^{+}_{n1})^{\theta_{n1}}(A^{+}_{n2})^{\theta_{n2}}(A^{+}_{n3})^{\theta_{n3}}\,|\,0\rangle, \end{aligned} \tag{17}$$

where

$$q \equiv \sum_{\alpha=1}^{n}\sum_{i=1}^{3}\theta_{\alpha i} \tag{18}$$

must satisfy:

$$0 \leq q \leq \min(p, 3n). \tag{19}$$

The action of the CAOs reads:

$$A_{\alpha i}^{-}\,|\,p;\Theta\rangle = \theta_{\alpha i}(-1)^{\psi_{\alpha i}}\sqrt{p-q+1}\,|\,p;\Theta\rangle_{\overline{\alpha i}}, \tag{20}$$

$$A_{\alpha i}^{+}\,|\,p;\Theta\rangle = (1-\theta_{\alpha i})(-1)^{\psi_{\alpha i}}\sqrt{p-q}\,|\,p;\Theta\rangle_{\overline{\alpha i}}, \tag{21}$$

where

$$\psi_{\alpha i} = \sum_{(\beta j)<(\alpha i)} \theta_{\beta j},$$

with lexicographical ordering on the pairs (αi). In (20)-(21), $|\,p;\Theta\rangle_{\overline{\alpha i}}$ stands for the state obtained from $|\,p;\Theta\rangle$ after the replacement of $\theta_{\alpha i}$ by $\overline{\theta}_{\alpha i} = (1-\theta_{\alpha i})$.

The explicit Fock space representation of the WQO defined by (16)-(21) is reminiscent of the familiar Fock space representation of the canonical quantum oscillator which has as basis vectors

$$\begin{aligned} |\,\Phi\rangle &\equiv |\phi_{11}, \phi_{12}, \phi_{13}, \ldots, \phi_{n1}, \phi_{n2}, \phi_{n3}\rangle \\ &= \prod_{\alpha=1}^{n}\prod_{i=1}^{3} \frac{1}{\sqrt{\phi_{\alpha i}!}} (B_{11}^{+})^{\phi_{11}}(B_{12}^{+})^{\phi_{12}}(B_{13}^{+})^{\phi_{13}}\cdots \\ &\quad \cdots (B_{n1}^{+})^{\phi_{n1}}(B_{n2}^{+})^{\phi_{n2}}(B_{n3}^{+})^{\phi_{n3}}\,|\,0\rangle \end{aligned} \tag{22}$$

with $\phi_{\alpha i} \in \{0, 1, 2, \ldots\}$. Clearly, it is infinite-dimensional, and the action of the Bose creation and annihilation operators is as usual:

$$B_{\alpha i}^{-}\,|\,\Phi\rangle = \sqrt{\phi_{\alpha i}}\,|\,\Phi\rangle_{-\alpha i}, \qquad B_{\alpha i}^{+}\,|\,\Phi\rangle = \sqrt{\phi_{\alpha i}+1}\,|\,\Phi\rangle_{+\alpha i}, \tag{23}$$

where $|\,\Phi\rangle_{\pm\alpha i}$ is obtained from $|\,\Phi\rangle$ through replacing $\phi_{\alpha i}$ by $\phi_{\alpha i} \pm 1$. It is known that this Fock space is a particular (infinite-dimensional) irreducible representation of the Lie superalgebra $\mathfrak{osp}(1|6n)$ [8]

5. Properties of the single particle 3D WQO [11]

Let us first consider the case $n = 1$; since the subscript α is now superfluous, it will be dropped from the notation in this section.

The Hamiltonian $\hat{H}$ is diagonal in the basis $|p;\Theta\rangle$, and one finds:

$$\hat{H}\,|\,p;\Theta\rangle = E_q\,|\,p;\Theta\rangle \quad \text{with} \quad E_q = \frac{\hbar\omega}{2}\,(3p-2q)\,, \tag{24}$$

so the energy levels are equidistant in steps of $\hbar\omega/2$. Also the operator $\hat{\mathbf{M}}^2$ is diagonal, with

$$\hat{\mathbf{M}}^2|p;\Theta\rangle = \begin{cases} 0 & \text{if } \theta_1 = \theta_2 = \theta_3; \\ 2|p;\Theta\rangle & \text{otherwise.} \end{cases} \tag{25}$$

So the angular momentum of the single particle is 0 or 1. This also follows from a decomposition of $W(p)$ with respect to the subalgebra $\mathfrak{so}(3)$.

The most striking properties of the single particle WQO are related to the fact that its position or momentum operators are non-commutative:

$$[\hat{R}_i, \hat{R}_j] \neq 0 \quad \text{and} \quad [\hat{P}_i, \hat{P}_j] \neq 0 \text{ for} \quad i \neq j = 1, 2, 3. \tag{26}$$

On the other hand, the square operators $\hat{\mathbf{R}}^2$ and $\hat{\mathbf{P}}^2$ are integrals of motion; in fact,

$$\hat{\epsilon} \equiv \frac{2}{\omega\hbar}\hat{H} = \frac{2m\omega}{\hbar}\hat{\mathbf{R}}^2 = \frac{2}{m\omega\hbar}\hat{\mathbf{P}}^2 = \sum_{i=1}^{3}\{A_i^+, A_i^-\}. \tag{27}$$

This implies that in "stationary states" (a superposition of states with the same q) the particle is at a fixed distance $\varrho_q = \sqrt{\frac{\hbar}{2m\omega}(3p-2q)}$ from the origin.

However, one can even be more specific about possible outcomes of a position measurement. Indeed, it is not difficult to verify that the set of operators

$$\hat{H},\ \hat{\mathbf{R}}^2, \hat{\mathbf{P}}^2,\ \hat{R}_1^2,\ \hat{R}_2^2,\ \hat{R}_3^2,\ \hat{P}_1^2,\ \hat{P}_2^2,\ \hat{P}_3^2$$

mutually commute, and are all diagonal in the $|p;\Theta\rangle$-basis. Before deducing physical consequences from this, let us for convenience introduce dimensionless notation for energy, position and momentum:

$$\hat{\varepsilon} = \frac{2}{\omega\hbar}\hat{H}, \quad \hat{r}_i(t) = \sqrt{\frac{2m\omega}{\hbar}}\hat{R}_i(t), \quad \hat{p}_i(t) = \sqrt{\frac{2}{m\omega\hbar}}\hat{P}_i(t). \tag{28}$$

From (20)-(21) it now follows that $(k = 1, 2, 3)$:

$$\hat{r}_k^2|p;\Theta\rangle = \hat{p}_k^2|p;\Theta\rangle = (p - q + \theta_k)|p;\Theta\rangle, \tag{29}$$

$$\hat{r}^2|p;\Theta\rangle = (3p - 2q)|p;\Theta\rangle, \qquad (q = \theta_1 + \theta_2 + \theta_3). \tag{30}$$

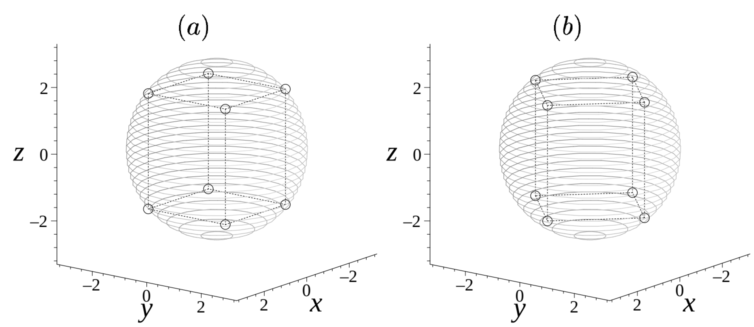
Furthermore, since $\hat{r}_1^2$, $\hat{r}_2^2$ and $\hat{r}_3^2$ mutually commute, their eigenvalues can be measured simultaneously. So, if the system is in a fixed state $|p;\Theta\rangle$, measurements of position imply that the particle will be detected in one of the eight "nests" with coordinates

$$r_1 = \pm\sqrt{p - q + \theta_1}, \quad r_2 = \pm\sqrt{p - q + \theta_2}, \quad r_3 = \pm\sqrt{p - q + \theta_3}, \tag{31}$$

on a sphere with radius $\rho_q = \sqrt{3p - 2q}$.

Consider, as an example, the state $|p;0,0,1\rangle$ (with $p > 2$). The possible outcomes of measurements of the position operators imply that the particle is in one of eight nests indicated in Figure 1(a), even though its precise position cannot be determined due to the non-commutativity of the operators $\hat{r}_k$ $(k = 1, 2, 3)$. For each of the eight basis states $|p; \theta\rangle$, such a configuration holds, except when $p \leq 2$. For $p \leq 2$, the representation is atypical and of smaller dimension; then also the coordinate configurations collapse.

Figure 1. (a) Possible outcomes of the measurements of the coordinates $\hat{r}_k$ of the particle in the stationary state $|p; 0, 0, 1\rangle$ are illustrated by means of eight nests. (b) Possible outcomes of the measurements of rotated coordinates $\hat{s}_k$ of the particle in this eigenstate.



By examining the eigenvectors of $\hat{r}_k$, one can draw conclusions about the occupation probabilities of the 8 nests, even though the 8 probabilities themselves cannot be determined.

Note that similar conclusions can be drawn with respect to the momentum of the particle.

It should be emphasized that due to the non-commutativity, the position of the particle is not absolute but relative to which coordinates are being measured. For example, one can consider measurements of the coordinates $s_k(t)$ with respect to an alternative frame of reference. In case of an orientation obtained by rotating the frame of reference through an angle ϕ about the third axis, one has

$$\begin{aligned} \hat{s}_1(t) &= \cos\phi\, \hat{r}_1(t) + \sin\phi\, \hat{r}_2(t); \\ \hat{s}_2(t) &= -\sin\phi\, \hat{r}_1(t) + \cos\phi\, \hat{r}_2(t); \\ \hat{s}_3(t) &= \hat{r}_3(t). \end{aligned} \tag{32}$$

Once again, the squares of these operators $\hat{s}_k^2$ mutually commute (but not with $\hat{r}_k^2$), and their eigenvalues are of the same type $p - q + \theta$. The eigenvectors of $\hat{s}_k^2$ are in general linear combination of the states $|p;\theta\rangle$, although our example state $|p;0,0,1\rangle$ happens to be directly an eigenvector of the $\hat{s}_k^2$. The actions indicate that the sites corresponding to possible values of measurements of the coordinates $\hat{s}_k$ are again nests on a sphere of radius ρ_q, but the nests define a rectangular parallelepiped obtain by rotating the original one about the third axis through an angle ϕ, see Figure 1(b).

The explanation for these different outcomes lies in the fact that the particle itself cannot be localised, since measurements of its coordinates do not mutually commute. It is the choice of coordinate to be measured that leads to the observed value corresponding to the associated eigenvalue.

6. The n-particle WQO [12]

We now return to the n-particle WQO, with Hamiltonian given by (7). Its energy spectrum can be computed using the basis (17), with actions (20)-(21):

$$\hat{H}\,|\,p;\Theta\rangle = E_q\,|\,p;\Theta\rangle \quad \text{with} \quad E_q = \hbar\omega\left(\frac{3np}{3n-1} - q\right) \tag{33}$$

for $q = 0,1,2,\ldots,\min(3n,p)$ and determined by (18).

This is to be compared with the spectrum of the canonical solution in the Fock space with basis vectors (22):

$$\hat{H}\,|\,\Phi\rangle = \bar{E}_q\,|\,\Phi\rangle \quad \text{where} \quad \bar{E}_q = \hbar\omega\left(\frac{3}{2}n + q\right) \tag{34}$$

with $q = \sum_{\alpha=1}^{n}\sum_{i=1}^{3}\phi_{\alpha i}$.

These energy spectra, together with their multiplicities, can also be found from branching rules with respect to a $\mathfrak{gl}(1)$ subalgebra of the Lie superalgebra under consideration. For the WQO, this branching rule is described by (see e.g. [13])

$$\begin{aligned}
\mathfrak{sl}(1|3n) &\longrightarrow \mathfrak{gl}(1)\oplus\mathfrak{sl}(3n)\\
W(p) &\longrightarrow \sum_{q=0}^{\min(p,3n)} V_{\mathfrak{gl}(1)}^{q+3n(p-q)}\otimes V_{\mathfrak{sl}(3n)}^{1^q}
\end{aligned}$$

where the superscripts refer to the highest weights of the representation in partition notation. The $\mathfrak{gl}(1)$ generator is $(3n-1)/(\hbar\omega)$ times the Hamiltonian. The branching rule also implies that the multiplicity of E_q is equal to the dimension of the $\mathfrak{sl}(3n)$ irrep with partition labels (1^q), thus $\text{mult}(E_q) = \binom{3n}{q}$.

For the canonical solution, the Fock space described by the basis vectors (22) is the infinite-dimensional irrep of $\mathfrak{osp}(1|6n)$ which decomposes into the sum of the two infinite-dimensional irreducible metaplectic or oscillator representations of $\mathfrak{sp}(6n)$ [14, 15], and then into finite-dimensional irreducible representations of $\mathfrak{gl}(1) \oplus \mathfrak{sl}(3n)$, according to the following branching rules:

$$
\begin{aligned}
&\mathfrak{osp}(1|6n) \longrightarrow \mathfrak{sp}(6n) \longrightarrow \mathfrak{gl}(1) \oplus \mathfrak{sl}(3n) \\
&\qquad \text{Fock} \longrightarrow V^{\varepsilon+}_{\mathfrak{sp}(6n)} \oplus V^{\varepsilon-}_{\mathfrak{sp}(6n)} \longrightarrow \sum_{q=0}^{\infty} V^{\frac{3}{2}n+q}_{\mathfrak{gl}(1)} \otimes V^{q}_{\mathfrak{sl}(3n)}.
\end{aligned}
$$

So now the energy multiplicity is equal to the dimension of the $\mathfrak{sl}(3n)$ irrep with partition labels (q), i.e. $\mathrm{mult}(\bar{E}_q) = \binom{3n-1+q}{q}$.

Next, we turn our attention to the angular momentum of the n-particle WQO. Similarly to the one particle case, one obtains

$$
\hat{\mathbf{M}}^2_\alpha |p;\Theta\rangle = \delta_\alpha\, 2\, |p;\Theta\rangle, \tag{35}
$$

where

$$
\delta_\alpha = \begin{cases} 0 & \text{if } \theta_{\alpha 1} = \theta_{\alpha 2} = \theta_{\alpha 3}; \\ 1 & \text{otherwise.} \end{cases}
$$

In other words, the WQO behaves like a collection of spin zero and spin one particles.

Although the stationary states $|p;\Theta\rangle$ are eigenvectors of $\hat{\mathbf{M}}^2_\alpha$, they are not eigenstates of either $\hat{M}_3$ or $\hat{\mathbf{M}}^2$. This makes it more difficult to compute the possible values of the total angular momentum of the system. One classical way to determine these values, is to compute the decomposition of the representation space according to a branching rule with respect to the $\mathfrak{so}(3)$ subalgebra generated by the total angular momentum components. For the WQO, this branching rule is described by

$$
\begin{aligned}
&\mathfrak{sl}(1|3n) \to \mathfrak{gl}(1) \oplus \mathfrak{sl}(3n) \to \mathfrak{gl}(1) \oplus \mathfrak{sl}(3) \oplus \mathfrak{sl}(n) \\
&\to \mathfrak{gl}(1) \oplus \mathfrak{so}(3) \oplus \mathfrak{sl}(n) \to \mathfrak{gl}(1) \oplus \mathfrak{so}(3)
\end{aligned}
$$

and the decomposition of the irrep $W(p)$ can be computed for each n (although no closed formula can be given for all n). The total angular momentum ranges from 0 to n (with multiplicities).

Observe that for the canonical case, one should examine the branching rule

$$\begin{aligned}&\mathfrak{osp}(1|6n) \to \mathfrak{sp}(6n) \to \mathfrak{gl}(1) \oplus \mathfrak{sl}(3n) \to \mathfrak{gl}(1) \oplus \mathfrak{sl}(3) \oplus \mathfrak{sl}(n)\\ &\to \mathfrak{gl}(1) \oplus \mathfrak{so}(3) \oplus \mathfrak{sl}(n) \to \mathfrak{gl}(1) \oplus \mathfrak{so}(3)\end{aligned}$$

for the Fock space spanned by (22). In this case, the total angular momentum can take infinitely many values.

Finally, we shall make some observations on the possible oscillator configurations for the n-particle WQO. As before, it is convenient to use dimensionless operators:

$$\hat{r}_{\alpha k}(t) = \sqrt{\frac{(3n-1)m\omega}{\hbar}}\hat{R}_{\alpha k}(t) = A^+_{\alpha k}\, e^{-i\omega t} + A^-_{\alpha k}\, e^{i\omega t}. \qquad (36)$$

Then

$$\hat{r}^2_{\alpha k} = \{A^+_{\alpha k}, A^-_{\alpha k}\} \qquad (37)$$

and the basis vectors $|p;\Theta\rangle$ are directly eigenvectors of these coordinate operators:

$$\hat{r}^2_{\alpha k}\,|p;\Theta\rangle = r^2_{\alpha k}\,|p;\Theta\rangle, \qquad \text{with } r^2_{\alpha k} = p - q + \theta_{\alpha k}. \qquad (38)$$

Since the squares of the coordinate operators all commute, $[\hat{r}^2_{\alpha i}, \hat{r}^2_{\beta j}] = 0$ for all α, β, i, j, the conclusions about particle coordinates are similar as in the $n = 1$ case: as far as coordinates is concerned, one simply gets a superposition of n single particle systems. In other words, if the system is in the state $|p;\Theta\rangle$, measurements of the coordinates $r_{\alpha k}$ of the αth particle can yield only

$$r_{\alpha k} = \pm\sqrt{p - q + \theta_{\alpha k}} \quad \text{for } k = 1, 2, 3.$$

This corresponds to measuring the particle α in one of 8 nests on a sphere with radius $\rho_\alpha = \sqrt{3p - 3q + q_\alpha}$, where $q_\alpha = \theta_{\alpha 1} + \theta_{\alpha 2} + \theta_{\alpha 3}$.

As in the single particle case, however, one has to be careful about conclusions related to the position of particles, since the actual position operators do not commute (only their squares commute), and so the particles cannot be absolutely localised. The non-commutativity of the underlying geometry is also reflected by the action of the following commutator upon a basis state (for $(\alpha i) < (\beta j)$):

$$\begin{aligned}
&[\hat{r}_{\alpha i}(t), \hat{r}_{\beta j}(t)]\,|\,p;\ldots,\theta_{ai},\ldots,\theta_{\beta j},\ldots\rangle \\
&= (-1)^{\psi_{\beta j}-\psi_{\alpha i}}\Bigg(2e^{i2\omega t}\theta_{\alpha i}\theta_{\beta j}\sqrt{(p-q+1)(p-q+2)} \\
&\quad +(\theta_{\alpha i}-\theta_{\beta j})^2(2p-2q+1) \\
&\quad +2e^{-i2\omega t}\overline{\theta}_{\alpha i}\overline{\theta}_{\beta j}\sqrt{(p-q-1)(p-q)}\Bigg)\,|\,p;\ldots,\overline{\theta}_{\alpha i},\ldots,\overline{\theta}_{\beta j},\ldots\rangle,
\end{aligned}$$

where $\overline{\theta}_{\gamma k} = 1-\theta_{\gamma k}$. The right hand side of this expression is nonzero for all $p \geq q+2$.

In these circumstances, we must expect some difficulties over the interpretation of the measurement of the distance between two particles α and β. Let us, in line with classical notions of distance, define $\hat{d}^2_{\alpha\beta}(t)$ as the square distance operator for particles α and β, where

$$\hat{d}^2_{\alpha\beta}(t) = \sum_{i=1}^{3}\left(\hat{r}_{\alpha i}(t) - \hat{r}_{\beta i}(t)\right)^2. \tag{39}$$

In particular, consider the case $n = 2$, with $\alpha = 1$ and $\beta = 2$. A computation of the eigenvalues of $\hat{d}^2_{12}(t)$ in a typical irrep $W(p)$ leads to the following list:

$$\begin{array}{l|l|l}
(6p)_1 & (6p-12)_4 & (6p-22)_3 \\
(6p-4)_3 & (6p-14)_9 & (6p-24)_3 \\
(6p-6)_3 & (6p-16)_9 & (6p-26)_3 \\
(6p-8)_3 & (6p-18)_4 & (6p-30)_1 \\
(6p-10)_9 & (6p-20)_9 &
\end{array} \tag{40}$$

The multiplicity of each eigenvalue is given here as a subscript.

Suppose that the system is in the state $|p;\dot{0}\rangle \equiv |p;0,0,0,0,0,0\rangle$. The nest coordinates for the two particles are $r_{1i} = \pm\sqrt{p}$ and $r_{2i} = \pm\sqrt{p}$ for $i = 1,2,3$. Thus, the square distances between these nests are determined by

$$\sum_{i=1}^{3}(r_{1i}-r_{2i})^2 \in \{0, 4p, 8p, 12p\}. \tag{41}$$

So the spectrum of eigenvalues of $\hat{d}^2_{12}(t)$ given in (40) does not contain for general p the values of the squares of the distances between nests as given in (41). This is only an apparent contradiction. After all, the states $|p;\Theta\rangle$ are not eigenstates of $\hat{d}^2_{12}(t)$, and $\hat{d}^2_{12}(t)$ does not commute with $\hat{r}^2_{1i}(t)$ and $\hat{r}^2_{2j}(t)$.

There is, however, a way in which even these considerations about square distances between particles are consistent with the nest interpretation. For this purpose, let us return to the general case (arbitrary $n \geq 2$) with particles α and β. As usual in quantum mechanics, the expectation value of the square distance operator is given by:

$$\overline{d^2}_{\alpha\beta}(t) = \langle p;\Theta|\ \hat{d}^2_{\alpha\beta}(t)\ |p;\Theta\rangle, \tag{42}$$

and it yields the average value of the square distance of the two particles. A careful computation, expressing $\hat{d}^2_{\alpha\beta}(t)$ in terms of the operators $A^{\pm}_{\alpha k}$ and using their action, leads to the following result:

$$\overline{d^2}_{\alpha\beta}(t) = 6p - 6q + q_\alpha + q_\beta = \rho_\alpha^2 + \rho_\beta^2, \tag{43}$$

where $q_\alpha = \theta_{\alpha 1} + \theta_{\alpha 2} + \theta_{\alpha 3}$ and $\rho_\alpha = \sqrt{3p - 3q + q_\alpha}$ (similarly for q_β and ρ_β). Note that ρ_α and ρ_β are the radii of the spheres on which particles α resp. β are located.

As a consistency check, we will compare this to the classical average, i.e. the value $d^2_{\alpha\beta}$ being the average of the square of the distance between the nests available to particles α and β. In this case, another careful computation (including relations satisfied by occupancy probabilities) leads indeed to the same value:

$$d^2_{\alpha\beta} = 6p - 6q + q_\alpha + q_\beta = \rho_\alpha^2 + \rho_\beta^2. \tag{44}$$

Quite generally, the expectation values of all operators related to localisation of particles are in agreement with the computation of a classical average using the notion of nests as being the actual positions of the particles.

7. Conclusions

We have considered a major class of non-canonical solutions for the oscillator Hamiltonian of n particles in three-dimensional space, i.e. the n-particle WQO in 3D. Our class of solutions is formulated in the framework of the Lie superalgebra $\mathfrak{sl}(1|3n)$.

We have constructed Fock type representations $W(p)$ as state spaces for the WQO. These representations are finite-dimensional. The energy of the system is quantized, with equally spaced finite spectrum. The angular momentum is also quantized, with single particle angular momenta 0 or 1. But also the coordinates and momenta are quantized, having a finite spectrum and leading to a finite number of "nests". The coordinate operators do not commute. There is a sense in which WQSs provide a natural framework for considering non-commutative coordinates, since the non-commutativity comes automatically (and does not need to be introduced ad hoc as in many other approaches).

Acknowledgments

NIS was supported by a Marie Curie Individual Fellowship of the European Community Programme 'Improving the Human Research Potential and the Socio-Economic Knowledge Base' under contract number HPMF-CT-2002-01571.

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