The "odd" Gelfand-Zetlin basis for representations of general linear Lie superalgebras

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Abstract We introduce a new Gelfand-Zetlin (GZ) basis for covariant representations of $\mathfrak{gl}(n|n)$. The patterns in this basis are fixed according to a chain of subalgebras, all of which are Lie superalgebras themselves. The basic generators consist of odd elements only. This GZ basis is interesting because the limit when n goes to infinity becomes clear. This could be used in the description of systems with an infinite number of parabosons and parafermions.

1 Introduction and motivation

The generalization of bosons and fermions to so-called parabosons and parafermions was initiated by Green in 1953 [1]. In this process, the (anti-)commutation relations for the boson and fermion operators were replaced by certain triple relations [1, 2]. This allows more freedom when it comes to representations: where the standard bosons and fermions (with certain conditions such as a unique vacuum vector) allow only one irreducible unitary representation (namely the Fock space), parabosons and parafermions allow several such representations each characterized by a number p, the order of statistics. For the case $p=1$, the relations for parabosons and parafermions reduce to those for standard bosons and fermions.

The above generalization is especially interesting because of the underlying mathematical structure. A system consisting of k parafermions $f_j^{\pm}$ $(j=1,\ldots,k)$ is known to correspond to the defining relations of the Lie algebra $\mathfrak{so}(2k+1)$ [3, 4]. Similarly, a system consisting of n parabosons $b_j^{\pm}$ $(j=1,\ldots,n)$ corresponds to the

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defining relations of the Lie superalgebra $\mathfrak{osp}(1|2n)$ [5]. When it comes to a combined system of k parafermions and n parabosons (referred to as the parastatistics operators), there is some choice for the mixed triple relations [2]. The most natural choice implies that such a combined system corresponds to the defining relations of the Lie superalgebra $\mathfrak{osp}(2k+1|2n)$ [6].

The construction of the corresponding parastatistics Fock space of order p, which corresponds to an infinite-dimensional unitary representation of $\mathfrak{osp}(2k+1|2n)$, is far from trivial. This construction, including the explicit action of the parastatistics operators in an appropriate basis, was completed only recently [7].

For people working in quantum field theory, the main interest is in such systems with an infinite degree of freedom, i.e. where $k,n\to\infty$. In order to consider this, recall that the main ingredient in the construction of the parastatistics Fock space of order p is the branching $\mathfrak{osp}(2k+1|2n)\supset\mathfrak{gl}(k|n)$, and the use of Gel'fand-Zetlin (GZ) patterns of covariant representations of $\mathfrak{gl}(k|n)$ to label the states of this Fock space. The GZ-basis for covariant representations of $\mathfrak{gl}(k|n)$ was constructed in [8], and proceeds according to the subalgebra chain

$\mathfrak{gl}(k|n)\supset\mathfrak{gl}(k|n-1)\supset\cdots\supset\mathfrak{gl}(k|1)\supset\mathfrak{gl}(k)\supset\mathfrak{gl}(k-1)\supset\cdots\supset\mathfrak{gl}(2)\supset\mathfrak{gl}(1).$

The labels of the GZ-basis vectors for $\mathfrak{gl}(k|n)$ follow similar rules as those of the classical GZ-basis for the Lie algebra $\mathfrak{gl}(n)$ [9]. For example, a basis vector for a covariant representation of $\mathfrak{gl}(4|3)$ is given by

$$\left|\begin{array}{cccc:ccc}
\mu_{17} & \mu_{27} & \mu_{37} & \mu_{47} & \mu_{57} & \mu_{67} & \mu_{77}\\
\mu_{16} & \mu_{26} & \mu_{36} & \mu_{46} & \mu_{56} & \mu_{66} & \\
\mu_{15} & \mu_{25} & \mu_{35} & \mu_{45} & \mu_{55} & & \\
\hdashline
\mu_{14} & \mu_{24} & \mu_{34} & \mu_{44} & & & \\
\mu_{13} & \mu_{23} & \mu_{33} & & & & \\
\mu_{12} & \mu_{22} & & & & & \\
\mu_{11} & & & & & &
\end{array}\right).$$

In such a μ-triangle, all $\mu_{ij}\in\mathbb{Z}_+$, satisfying conditions such as

- betweenness conditions ($1\leq i\leq j\leq k-1$ or $k+1\leq i\leq j\leq k+n-1$)
$$\mu_{i,j+1}\geq\mu_{ij}\geq\mu_{i+1,j+1}$$
- θ-conditions or 0-1-conditions ($1\leq i\leq k$, $k+1\leq s\leq k+n$)
$$\mu_{is}-\mu_{i,s-1}\equiv\theta_{i,s-1}\in\{0,1\}.$$

For a complete description of the conditions, see [8]. Note also that the top row of the above μ-triangle corresponds to the highest weight of the covariant representation. A GZ-basis also includes the explicit action of a set of generators on the basis vectors: for the standard GZ-basis given above, this set consists of the Chevalley generators of $\mathfrak{gl}(k|n)$ [8, Theorem 7] (corresponding to the distinguished set of simple roots).

Although this GZ-basis is perfectly well suited for the finite rank case of $\mathfrak{gl}(k|n)$, the problem is that it cannot be extended to a class of irreducible representations (irreps) of the infinite rank Lie superalgebra $\mathfrak{gl}(\infty|\infty)$. In order to solve this, one needs to use a different GZ-basis according to a different chain of subalgebras:

$\mathfrak{gl}(n|n) \supset \mathfrak{gl}(n|n-1) \supset \mathfrak{gl}(n-1|n-1) \supset \mathfrak{gl}(n-1|n-2) \supset \cdots \supset \mathfrak{gl}(1|1) \supset \mathfrak{gl}(1)$. This can then be "reversed" in order to give a GZ-basis for $\mathfrak{gl}(\infty|\infty)$:
$\mathfrak{gl}(1) = \mathfrak{gl}(1|0) \subset \mathfrak{gl}(1|1) \subset \mathfrak{gl}(2|1) \subset \mathfrak{gl}(2|2) \subset \mathfrak{gl}(3|2) \subset \mathfrak{gl}(3|3) \subset \cdots \subset \mathfrak{gl}(\infty|\infty)$. Thus, first we need to construct a new GZ-basis (the "odd" GZ-basis) for $\mathfrak{gl}(n|n)$ according to the above subalgebra chain. A striking property is that the generators for which the action takes its simplest form is now different: they are the (positive and negative) root vectors corresponding to a non-distinguished simple root system of $\mathfrak{gl}(n|n)$ consisting of odd roots only (justifying the name "odd" GZ-basis).

All results of the current proceedings contribution have been given in [10]. Here we shortly review the problem and list some additional properties and remarks.

2 Overview of the main results

The Lie superalgebra $\mathfrak{g} = \mathfrak{gl}(k|n)$ is defined by [11]:

$$\mathfrak{gl}(k|n) = \{x = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\},$$

with $A \in M_{k\times k}$, $B \in M_{k\times n}$, $C \in M_{n\times k}$ and $D \in M_{n\times n}$. The even subalgebra $\mathfrak{gl}(k|n)_{\bar{0}}$ has $B = 0$ and $C = 0$; the odd subspace $\mathfrak{gl}(k|n)_{\bar{1}}$ has $A = 0$ and $D = 0$. It is convenient to use the ordered set $\{-k,\ldots,-2,-1;1,2,\ldots n\}$ as index set for the rows and columns of the above matrices. The Weyl basis is given by elements E_{ij} ($i,j = -k,\ldots,-2,-1;1,2,\ldots,n$), with Lie superalgebra bracket

$$[\![E_{ab},E_{cd}]\!] = \delta_{bc}E_{ad} - (-1)^{\deg(E_{ab})\deg(E_{cd})}\delta_{ad}E_{cb}.$$

The Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is span(E_{jj}) with $j = -k,\ldots,-2,-1;\ 1,2,\ldots,n$. The dual space $\mathfrak{h}^*$ (or weight space) is spanned by the forms ε_i ($i = -k,\ldots,-2,-1;$ $1,2,\ldots,n$). For $\Lambda \in \mathfrak{h}^*$,

$$\Lambda = \textstyle\sum_{i=-k\ (i\neq 0)}^{n} m_{ir}\varepsilon_i,$$

the components are written as ($r = k+n$)

$$[m]^r = [m_{-k,r},\ldots,m_{-2,r},m_{-1,r};m_{1r},m_{2r},\ldots,m_{nr}].$$

The roots of $\mathfrak{gl}(k|n)$ are the elements $\varepsilon_i - \varepsilon_j$ ($i \neq j$); the positive roots consist of $\varepsilon_i - \varepsilon_j$ ($i < j$), and the positive odd roots of $\varepsilon_i - \varepsilon_j$ with $i < 0$ and $j > 0$. The distinguished set of simple roots [11] is:

$$\varepsilon_{-k} - \varepsilon_{-k+1},\ \varepsilon_{-k+1} - \varepsilon_{-k+2},\ldots,\varepsilon_{-1} - \varepsilon_1, \varepsilon_1 - \varepsilon_2,\ldots,\varepsilon_{n-1} - \varepsilon_n.$$

In general, an integral dominant weight Λ corresponds to a finite-dimensional irrep $V(\Lambda)$ and vice versa. Here, we are only dealing with covariant representations: these are labelled by a partition $\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_\ell)$ such that λ is inside the (k,n)-hook: $\lambda_{k+1} \leq n$ [12]. The corresponding highest weight: $\Lambda^\lambda \equiv [m]^r$ is determined by [13]

$$\begin{aligned} m_{ir} &= \lambda_{k+i+1}, \quad -k \leq i \leq -1, \\ m_{ir} &= \max\{0, \lambda_i' - k\}, \quad 1 \leq i \leq n, \end{aligned}$$

where λ' is the partition conjugate to λ. Conversely, if $[m]^r$ is integral dominant and $m_{-1,r} \geq \#\{i : m_{ir} > 0,\ 1 \leq i \leq n\}$ then this corresponds to covariant module with

$$\begin{aligned}
&\lambda_i = m_{i-k-1,r}, \quad 1 \le i \le k,\\
&\lambda_{k+i} = \#\{j : m_{jr} \le i,\ 1 \le j \le n\}, \quad 1 \le i \le n.
\end{aligned}$$

The main property of covariant representations is that their character is known to be a supersymmetric Schur function [12]. With $x_i = e^{\varepsilon_i}$ ($i \le -1$) and $y_i = e^{\varepsilon_i}$ ($1 \le i$),

$$\mathrm{char}\, V([\Lambda^{\lambda}]) = s_{\lambda}(x_{\bar{k}}, \ldots, x_{\bar{2}}, x_{\bar{1}} | y_1, y_2, \ldots, y_n).$$

(For convenience, we sometimes write $\bar{j}$ instead of $-j$, as in the indices of the x's).

Using properties of these supersymmetric Schur functions [14], one can "peel off" a variable y_n or a variable $x_{\bar{k}}$. This allows the decomposition of a covariant representation of $\mathfrak{gl}(n|n)$ according to the subalgebra chain $\mathfrak{gl}(n|n) \supset \mathfrak{gl}(n|n-1) \supset \mathfrak{gl}(n-1|n-1)$. Labelling the highest weights of the respective covariant representations as follows:

$$\begin{aligned}
&\mathfrak{gl}(n|n) \ \leftrightarrow\ [m]^r = [m_{-n,r}, \ldots, m_{-2,r}, m_{-1,r}; m_{1r}, m_{2r}, \ldots, m_{nr}]\\
&\mathfrak{gl}(n|n-1) \ \leftrightarrow\ [m]^{r-1} = [m_{-n,r-1}, \ldots, m_{-1,r-1}; m_{1,r-1}, \ldots, m_{n-1,r-1}]\\
&\mathfrak{gl}(n-1|n-1) \ \leftrightarrow\ [m]^{r-2} = [m_{-n+1,r-2}, \ldots, m_{-1,r-2}; m_{1,r-2}, \ldots, m_{n-1,r-2}],
\end{aligned}$$

the decompositions $\mathfrak{gl}(n|n) \to \mathfrak{gl}(n|n-1)$ and $\mathfrak{gl}(n|n-1) \to \mathfrak{gl}(n-1|n-1)$ are given by, respectively,

$$V([m]^r) = \bigoplus_k V_k([m]^{r-1}), \qquad V([m]^{r-1}) = \bigoplus_k V_k([m]^{r-2})$$

according to the rules

$$\begin{aligned}
&(1)\ m_{ir} - m_{i,r-1} = \theta_{i,r-1} \in \{0,1\} \quad (-n \le i \le -1)\\
&(2)\ m_{ir} - m_{i,r-1} \text{ and } m_{i,r-1} - m_{i+1,r} \in \mathbb{Z}_+ \quad (1 \le i \le n-1)\\
&(3)\ m_{i,r-2} - m_{i,r-1} = \theta_{i,r-2} \in \{0,1\} \quad (1 \le i \le n-1)\\
&(4)\ m_{i,r-1} - m_{i+1,r-2} \text{ and } m_{i+1,r-2} - m_{i+1,r-1} \in \mathbb{Z}_+ \quad (-n \le i \le -2).
\end{aligned}$$

This process can now be repeated, and thus one obtains a new GZ-basis for covariant representations $V([m]^r)$ of $\mathfrak{gl}(n|n)$. The m-patterns of these vectors take the form

$$|m)^r = \left|\begin{array}{ccccc|cccccc}
m_{\bar{n}r} & m_{\overline{n-1},r} & \cdots & m_{\bar{2}r} & m_{\bar{1}r} & m_{1r} & m_{2r} & \cdots & m_{n-2,r} & m_{n-1,r} & m_{nr}\\
m_{\bar{n},r-1} & m_{\overline{n-1},r-1} & \cdots & m_{\bar{2},r-1} & m_{\bar{1},r-1} & m_{1,r-1} & m_{2,r-1} & \cdots & m_{n-2,r-1} & m_{n-1,r-1} & \\
 & m_{\overline{n-1},r-2} & \cdots & m_{\bar{2},r-2} & m_{\bar{1},r-2} & m_{1,r-2} & m_{2,r-2} & \cdots & m_{n-2,r-2} & m_{n-1,r-2} & \\
 & m_{\overline{n-1},r-3} & \cdots & m_{\bar{2},r-3} & m_{\bar{1},r-3} & m_{1,r-3} & m_{2,r-3} & \cdots & m_{n-2,r-3} & & \\
 & & \ddots & \vdots & \vdots & \vdots & \vdots & \iddots & & & \\
 & & & m_{\bar{2}4} & m_{\bar{1}4} & m_{14} & m_{24} & & & & \\
 & & & m_{\bar{2}3} & m_{\bar{1}3} & m_{13} & & & & & \\
 & & & & m_{\bar{1}2} & m_{12} & & & & & \\
 & & & & m_{\bar{1}1} & & & & & &
\end{array}\right),$$

where the inbetweenness conditions and θ-conditions to be satisfied for the integers m_{ij} follow from the above rules (1)-(4), and we have followed the same notational convention as before: $\bar{j}$ stands for $-j$. The set of all vectors $|m)^r$ satisfying these conditions constitute a basis in $V([m]^r)$ [10].

Recall that in the standard GZ-basis the action of the Lie superalgebra is determined by the (diagonal) action of the Cartan subalgebra elements E_{ii} and the explicit action of the Chevalley generators, i.e. the root vectors

$$E_{-n,-n+1},\ldots,E_{-2,-1},E_{-1,1},E_{1,2},\ldots,E_{n-1,n},$$

corresponding to the simple roots (in the distinguished basis) and those corresponding to the negatives of the simple roots. In this distinguished choice for the simple roots, there is only one odd simple root, depicted by a cross in the Dynkin diagram:

$$\bigcirc\!\!-\!\!\!-\!\!\bigcirc\!\!-\!\!\!-\cdots-\!\!\!-\!\!\bigcirc\!\!-\!\!\!-\!\!\bigotimes\!\!-\!\!\!-\!\!\bigcirc\!\!-\!\!\!-\cdots-\!\!\!-\!\!\bigcirc\!\!-\!\!\!-\!\!\bigcirc.$$

In the “odd” GZ-basis, the situation is different. For the GZ-patterns $|m)^r$, one can again give the (diagonal) action of the Cartan subalgebra elements E_{ii}. The set of positive root vectors for which an explicit action can be computed is now different and given by

$$E_{-1,1},E_{-2,1},E_{-2,2},E_{-3,2},E_{-3,3},\ldots,E_{-n,n-1},E_{-n,n},$$

consisting of odd roots only. (Similarly, there is the action of the corresponding set of negative root vectors.) Thus the root vectors $E_{\pm\alpha}$ correspond to the following choice of simple roots (with only odd roots):

$$\varepsilon_{-1}-\varepsilon_1,\varepsilon_1-\varepsilon_{-2},\varepsilon_{-2}-\varepsilon_2,\varepsilon_2-\varepsilon_{-3},\ldots,\varepsilon_{-n+1}-\varepsilon_{n-1},\varepsilon_{n-1}-\varepsilon_{-n},\varepsilon_{-n}-\varepsilon_n$$

with a Dynkin diagram of the form

$$\bigotimes\!\!-\!\!\!-\!\!\bigotimes\!\!-\!\!\!-\!\!\bigotimes\!\!-\!\!\!-\!\!\bigotimes\!\!-\!\!\!-\!\!\bigotimes\!\!-\!\!\!-\cdots-\!\!\!-\!\!\bigotimes\!\!-\!\!\!-\!\!\bigotimes\!\!-\!\!\!-\!\!\bigotimes.$$

The main result of [10] is the determination of the explicit action of the above generators on the new GZ basis vectors. We shall not repeat these formulae here, but just note that the action of the E_{ii} is a simple diagonal action on $|m)^r$, whereas the action of the remaining generators takes the form

$$\begin{aligned}
E_{-i,i}|m)^r &= \sum_{k=-i}^{-1} A_{ik}|m)^r_{+(k,2i-1)} + \sum_{k=1}^{i-1} A_{ik}|m)^r_{+(k,2i-1)} \\
E_{-i-1,i}|m)^r &= \sum_{k=-i}^{-1} B_{ik}|m)^r_{-(k,2i)} + \sum_{k=1}^{i} B_{ik}|m)^r_{-(k,2i)} \\
E_{i,-i}|m)^r &= \sum_{k=-i}^{-1} C_{ik}|m)^r_{-(k,2i-1)} + \sum_{k=1}^{i-1} C_{ik}|m)^r_{-(k,2i-1)} \\
E_{i,-i-1}|m)^r &= \sum_{k=-i}^{-1} D_{ik}|m)^r_{+(k,2i)} + \sum_{k=1}^{i} D_{ik}|m)^r_{+(k,2i)}.
\end{aligned}$$

Herein, $|m)^r_{\pm(ij)}$ is the pattern obtained from $|m)^r$ by replacing the entry m_{ij} by $m_{ij} \pm 1$; the actual expressions for the matrix elements $A_{ik}, B_{ik}, C_{ik}, D_{ik}$ can be found in [10, Theorem 4]. Observe that the action of a generator makes changes in only one row of the pattern of $|m)^r$.

The major advantage is that the "odd" GZ basis for $\mathfrak{gl}(n|n)$ can easily be extended to the infinite rank Lie superalgebra $\mathfrak{gl}(\infty|\infty)$, defined as the set of matrices with index set $\{\ldots,-3,-2,-1;1,2,3,\ldots\} = \mathbb{Z}^* \equiv \mathbb{Z}\setminus\{0\}$ with only a finite number of nonzero elements, and with the appropriate bracket. A highest weight is an infinite sequence $[m] \equiv [\ldots, m_{-k}, \ldots, m_{-2}, m_{-1}; m_1, m_2, \ldots, m_k, \ldots]$, and provided these numbers satisfy certain conditions, the corresponding highest weight representation $V([m])$ is a covariant representation. The basis vectors of $V([m])$ consist of "infinite GZ-patterns": similar to those of $\mathfrak{gl}(n|n)$, but with the above sequence as top row and consisting of an infinite set of rows in a triangular pattern. These GZ-patterns should – apart from inbetweenness conditions and θ-conditions – also satisfy a *stability condition*. The set of infinite stable GZ-patterns $|m)$ form a basis of the irreducible representation $V([m])$, and the transformation of the basis under the action of the $\mathfrak{gl}(\infty|\infty)$ generators is easily obtained from the finite rank case [10].

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