A class of representations of the orthosymplectic Lie superalgebras $\mathcal{B}(n,n)$ and $\mathcal{B}(\infty,\infty)$

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Abstract In 1982 Palev showed that the algebraic structure generated by the creation and annihilation operators of a system of m parafermions and n parabosons, satisfying the mutual parafermion relations, is the Lie superalgebra $\mathfrak{osp}(2m+1|2n)$. The "parastatistics Fock spaces" of order p of such systems are then certain lowest weight representations of $\mathfrak{osp}(2m+1|2n)$. We investigate now the situation when the number of parafermions and parabosons becomes infinite, which is of interest not only in a physics context but also from the mathematical point of view. In this contribution, we will discuss the various steps that are needed to understand the infinite-rank case. First, we will introduce appropriate bases and Dynkin diagrams for $\mathfrak{B}(n,n)=\mathfrak{osp}(2n+1|2n)$ that allow us to extend $n\to\infty$. Then we will develop a new matrix form for $\mathfrak{B}(n,n)=\mathfrak{osp}(2n+1|2n)$, because the standard one is not appropriate for taking this limit. Following this, we construct a new Gelfand-Zetlin basis of the parastatistics Fock spaces in the finite rank case (in correspondence with this new matrix form). The new structures, related to a non-distinguished simple root system, allow the extension to $n\to\infty$. This leads to the definition of the algebra $\mathfrak{B}(\infty,\infty)$ as a Lie superalgebra generated by an infinite number of creation and annihilation operators (subject to certain relations), or as an algebra of certain infinite-dimensional matrices. We study the parastatistics Fock spaces, as certain lowest weight representations of $\mathfrak{B}(\infty,\infty)$. In particular, we construct a basis consisting of well-described row-stable Gelfand-Zetlin patterns.

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1 Introduction

Throughout this paper we will use square brackets for a commutator, $[A, B] = AB - BA$; curly brackets for an anti-commutator: $\{A, B\} = AB + BA$; and double brackets if we are dealing with operators from a $\mathbb{Z}_2$ graded algebra: $[\![A, B]\!] = AB - (-1)^{\langle A\rangle\langle B\rangle} BA$, where $\langle A\rangle = \deg(A) \in \{0, 1\}$ is the degree of A.

In this contribution we will consider Fock spaces for bosons, fermions, parabosons, parafermions, and combined systems of parabosons and parafermions. The emphasis is on algebraic structures behind these systems, on identifying Fock spaces with a class of representations of these algebras, and on constructing a basis for these representations.

For a system described by n pairs of boson (creation and annihilation) operators $B_i^\pm$ $(i = 1, \ldots, n)$, satisfying

$$[B_i^-, B_j^+] = \delta_{ij} \tag{1}$$

and all other commutators zero, the Fock space with vacuum vector $|0\rangle$ characterized by $(B_i^\pm)^\dagger = B_i^\mp$ and $B_i^-|0\rangle = 0$ has a very simple (orthonormal) basis:

$$|k_1, \ldots, k_n\rangle = \frac{(B_1^+)^{k_1} \cdots (B_n^+)^{k_n}}{\sqrt{k_1! \cdots k_n!}} |0\rangle \tag{2}$$

with $k_i \in \{0, 1, 2, \ldots\}$. Similarly, a system described by m pairs of fermion operators $F_i^\pm$ $(i = 1, \ldots, m)$, with

$$\{F_i^-, F_j^+\} = \delta_{ij} \tag{3}$$

and all other anti-commutators zero, the Fock space is characterized by $(F_i^\pm)^\dagger = F_i^\mp$ and $F_i^-|0\rangle = 0$, and has a basis similar to (2) but with all $k_i \in \{0, 1\}$.

More interesting structures are provided by parabosons and parafermions, especially from the algebraic point of view. These were first introduced by Green [1] and their Fock spaces were first studied by Greenberg and Messiah [2].

A system of n pairs of parabosons $b_j^\pm$ $(j = 1, \ldots, n)$ is defined by means of triple relations:

$$[\{b_j^\xi, b_k^\eta\}, b_l^\epsilon] = (\epsilon - \xi)\delta_{jl} b_k^\eta + (\epsilon - \eta)\delta_{kl} b_j^\xi, \tag{4}$$

where $j, k, l \in \{1, 2, \ldots, n\}$ and $\eta, \epsilon, \xi \in \{+, -\}$ (to be interpreted as $+1$ and -1 in the algebraic expressions $\epsilon - \xi$ and $\epsilon - \eta$). In this case, there is not a unique Fock space, but for every positive integer p (referred to as the order of statistics) there is a Fock space $\mathcal{V}(p)$ characterized by $(b_j^\pm)^\dagger = b_j^\mp$, $b_j^-|0\rangle = 0$ and

$$\{b_j^-, b_k^+\}|0\rangle = p\, \delta_{jk}\, |0\rangle. \tag{5}$$

Similarly, a system of m pairs of parafermions $f_j^\pm$ $(j=1,\ldots,m)$ is defined by the triple relations

$$[[f_j^\xi, f_k^\eta], f_l^\epsilon] = |\epsilon-\eta|\delta_{kl} f_j^\xi - |\epsilon-\xi|\delta_{jl} f_k^\eta. \tag{6}$$

Their Fock spaces $\mathcal{W}(p)$, also labelled by a positive integer p, are characterized by $(f_j^\pm)^\dagger = f_j^\mp$, $f_j^-|0\rangle = 0$ and

$$[f_j^-, f_k^+]|0\rangle = p\,\delta_{jk}\,|0\rangle. \tag{7}$$

These cubic or triple relations involve nested (anti-)commutators, just like the Jacobi identity of Lie (super)algebras. It was indeed shown later [3, 4] that the parafermionic algebra generated by $2m$ elements $f_i^\pm$ subject to (6) is the orthogonal Lie algebra $\mathfrak{so}(2m+1)$. The Fock space $\mathcal{W}(p)$ is the unitary irreducible representation of $\mathfrak{so}(2m+1)$ with lowest weight $(-\frac{p}{2}, -\frac{p}{2}, \ldots, -\frac{p}{2})$ in the standard basis.

Many years later, it was shown that the parabosonic algebra generated by $2n$ *odd* elements $b_i^\pm$ subject to (4) is the orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2n)$ [5]. In this case the Fock space $\mathcal{V}(p)$ is the unitary irreducible $\mathfrak{osp}(1|2n)$ representation with lowest weight $(\frac{p}{2}, \frac{p}{2}, \ldots, \frac{p}{2})$ in the standard basis.

For $p=1$, $\mathcal{V}(p)$ becomes the ordinary boson Fock space and $\mathcal{W}(p)$ becomes the ordinary fermion Fock space.

Already in their first paper, Greenberg and Messiah [2] considered combined systems of parafermions and parabosons. In combined systems, it will be convenient to use negative indices for parafermions and positive indices for parabosons, and to use the common operator notation $c_i^\pm$:

$$c_j^\pm = f_j^\pm \quad (-m \le j \le -1); \qquad c_i^\pm = b_i^\pm \quad (1 \le i \le n). \tag{8}$$

Apart from two trivial combinations, there are two non-trivial relative commutation relations between parafermions and parabosons, also expressed by means of triple relations. The case considered here is the so-called "relative parafermion relation" and is determined by the parastatistics relations

$$[\![[\![c_j^+, c_k^-]\!], c_l^+]\!] = 2\delta_{kl} c_j^+, \qquad [\![[\![c_j^+, c_k^+]\!], c_l^+]\!] = 0, \tag{9}$$

$$[\![c_j^-, [\![c_k^+, c_l^-]\!]]\!] = 2\delta_{jk} c_l^-, \qquad [\![[\![c_j^-, c_k^-]\!], c_l^-]\!] = 0. \tag{10}$$

The complete set of relations can also be written in the somewhat complicated form

$$[\![[\![c_j^\xi, c_k^\eta]\!], c_l^\epsilon]\!] = -2\delta_{jl}\delta_{\epsilon,-\xi}\epsilon^{\langle l\rangle}(-1)^{\langle k\rangle\langle l\rangle} c_k^\eta + 2\epsilon^{\langle l\rangle}\delta_{kl}\delta_{\epsilon,-\eta} c_j^\xi, \tag{11}$$

where $\langle k\rangle$ refers to the grading of $c_k^\pm$, and thus is 0 for negative k and 1 for positive k, following (8).

It was shown by Palev [6] that the Lie superalgebra (LSA) generated by $2m$ even elements $f_j^\pm$ and $2n$ odd elements $b_j^\pm$ subject to the above relations (11) is $\mathfrak{B}(m,n) = \mathfrak{osp}(2m+1|2n)$. The Fock spaces, denoted by $V(p)$ and labelled by a positive integer p, are characterized by $(c_j^{\pm})^\dagger = c_j^{\mp}$, $c_j^-|0\rangle = 0$ and $[\![c_j^-, c_k^+]\!]|0\rangle = p\,\delta_{jk}\,|0\rangle$. $V(p)$ is the unitary irreducible representation of $\mathfrak{osp}(2m+1|2n)$ with lowest weight $(-\frac{p}{2},\ldots,-\frac{p}{2}|\frac{p}{2},\ldots,\frac{p}{2})$ in the standard basis. These are referred to as the parastatistics Fock spaces.

Understanding the algebraic structure behind such systems of parabosons / parafermions is one step. But understanding the structure of the corresponding Fock spaces is another important step. A major contribution here is the so-called Green ansatz, where one considers the p-fold tensor product of an ordinary boson/fermion Fock space and extracts an irreducible component herein. This is far from trivial, and computing matrix elements for generators remains a difficult problem in this approach [7,8]. For the case of parabosons ($\mathfrak{osp}(1|2n)$ representations $\mathcal{V}(p)$), a complete basis with all matrix elements was given for the first time in [9]. The same type of construction was given for parafermions ($\mathfrak{so}(2m+1)$ representations $\mathcal{W}(p)$) in [10]. Interesting character formulas for these representations were also given, and these could be extended to characters of the parastatistics representations $V(p)$ of $\mathfrak{osp}(2m+1|2n)$ [11]. An actual basis of the parastatistics Fock spaces was constructed in [12], where again all matrix elements of the generators could be computed.

All the above constructions of basis vectors rely on the development of an appropriate Gelfand-Zetlin (GZ) basis, which in turn depends on an appropriate chain of subalgebras under which the reduction of $V(p)$ is multiplicity free at every step of the chain. For the parastatistics case, this subalgebra chain is

$$\begin{aligned}
&\mathfrak{osp}(2m+1|2n) \supset \mathfrak{gl}(m|n) \supset \mathfrak{gl}(m|n-1) \supset \mathfrak{gl}(m|n-2) \supset \cdots \\
&\supset \mathfrak{gl}(m|1) \supset \mathfrak{gl}(m) \supset \mathfrak{gl}(m-1) \supset \cdots \supset \mathfrak{gl}(2) \supset \mathfrak{gl}(1).
\end{aligned} \tag{12}$$

Since it follows from the character formula [12] that the decomposition of $V(p)$ in the chain $\mathfrak{osp}(2m+1|2n) \supset \mathfrak{gl}(m|n)$ is easy and multiplicity free, the GZ-basis consists of a (triangular) pattern with $m+n$ rows, each row corresponding to a highest of a $\mathfrak{gl}$ algebra in the chain (12).

In the present contribution, we consider the case for which m and n become infinite. If one tries to extend the above mentioned GZ-patterns to infinite patterns, starting from the bottom row corresponding to $\mathfrak{gl}(1)$ and gradually increasing the rank of the algebra, it is obvious that one cannot let both m and n go to infinity.

In the next paragraph, we shall explain how the introduction of an "odd GZ-basis" can overcome this problem, however only in the case $m=n$. This will lead to a new basis for the Fock spaces of $\mathfrak{B}(n,n) = \mathfrak{osp}(2n+1|2n)$. This new basis was constructed in [13], to which we refer for further details. The current contribution summarizes some of the main results in [13] and it is inevitable to have some overlap with [13]. Here, we first give a justification for the necessity of a new GZ-basis. Then we will proceed to a new matrix realization of $\mathfrak{B}(n,n)$, and give the parastatistics generators in this new basis. The parastatistics Fock representations are then described in the new GZ-basis. We also include an example (given in the Appendix) to illustrate the various notions. Finally, it is shown how to extend this to the case when $n \to \infty$, where so-called row-stable GZ-patters are of importance. For some details and explicit formulas, the reader will be referred to [13].

2 Introducing an odd GZ-basis

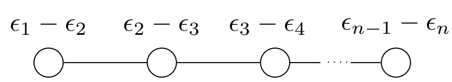
Looking back at the original idea of a Gelfand-Zeltin basis, for the case of the Lie algebra $\mathfrak{gl}(n)$, the construction of the basis is according to the chain of subalgebras

$$\mathfrak{gl}(n) \supset \mathfrak{gl}(n-1) \supset \cdots \supset \mathfrak{gl}(2) \supset \mathfrak{gl}(1). \tag{13}$$

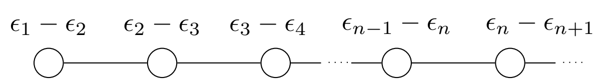
Every row of a GZ-basis vector consist of a highest weight of $\mathfrak{gl}(k)$, the top row ("row n") corresponding to $\mathfrak{gl}(n)$ and the bottom row ("row 1") to $\mathfrak{gl}(1)$. Such GZ-patterns can easily be extended to the infinite rank case by introducing infinitely large GZ-patterns according to

$$\mathfrak{gl}(1) \subset \mathfrak{gl}(2) \subset \cdots \subset \mathfrak{gl}(n-1) \subset \mathfrak{gl}(n) \subset \cdots . \tag{14}$$

In order to label basis vectors of an irreducible $\mathfrak{gl}(\infty)$ representation, with locally finite action of $\mathfrak{gl}(\infty)$ generators, one should require certain stability properties of the infinite GZ-patterns. The main idea is however that one can reverse the chain (13) to (14) allowing the limit $n \to \infty$. Also in terms of Dynkin diagrams, this process of letting n increase to infinity is somehow clear from the Dynkin diagram of $\mathfrak{gl}(n)$,



and its extension as n increases:



For the Lie superalgebra $\mathfrak{gl}(m|n)$, one can also construct (at least for a class of representations) a GZ-basis [14] according to the chain

$$\begin{aligned} &\mathfrak{gl}(m|n) \supset \mathfrak{gl}(m|n-1) \supset \mathfrak{gl}(m|n-2) \supset \cdots \\ &\supset \mathfrak{gl}(m|1) \supset \mathfrak{gl}(m) \supset \mathfrak{gl}(m-1) \supset \cdots \supset \mathfrak{gl}(2) \supset \mathfrak{gl}(1). \end{aligned} \tag{15}$$

In an attempt to let m and n increase to infinity, the GZ-patterns corresponding to the above chain are no longer appropriate. Indeed, if one reverses the chain (15) in which m grows to infinity,

$$\mathfrak{gl}(1) \subset \mathfrak{gl}(2) \subset \cdots \subset \mathfrak{gl}(m-1) \subset \mathfrak{gl}(m) \subset \mathfrak{gl}(m+1) \subset \cdots \tag{16}$$

one somehow never reaches the point where a Lie superalgebra can be included, and there is no way of having also $n \to \infty$.

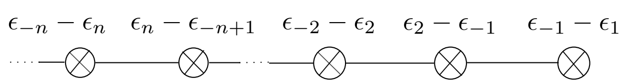
In a previous paper [15], this was solved by introducing the so-called odd GZ-basis for $\mathfrak{gl}(n|n)$ (m and n must be equal). This arises from the chain of superalgebras

$$\begin{aligned} &\mathfrak{gl}(n|n) \supset \mathfrak{gl}(n|n-1) \supset \mathfrak{gl}(n-1|n-1) \supset \cdots \\ &\cdots \supset \mathfrak{gl}(2|2) \supset \mathfrak{gl}(2|1) \supset \mathfrak{gl}(1|1) \supset \mathfrak{gl}(1). \end{aligned} \tag{17}$$

This chain can easily be reversed and continued to infinity,

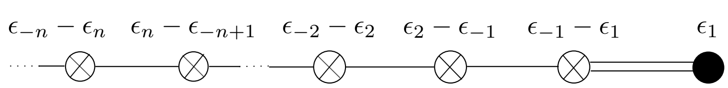
$$\begin{aligned} &\mathfrak{gl}(1) \subset \mathfrak{gl}(1|1) \subset \mathfrak{gl}(2|1) \subset \mathfrak{gl}(2|2) \subset \cdots \\ &\subset \mathfrak{gl}(n-1|n-1) \subset \mathfrak{gl}(n|n-1) \subset \mathfrak{gl}(n|n) \subset \cdots \end{aligned} \tag{18}$$

leading to an appropriate GZ-basis for $\mathfrak{gl}(\infty|\infty)$ representations [15], in which each row of the infinite GZ-pattern corresponds to a highest weight in the chain (18) (with certain stability requirements). Note that such a chain corresponds to a consecutive inclusion of Dynkin diagrams of Lie superalgebras of type $\mathfrak{gl}$ with odd simple roots only. In a convenient basis $(\ldots, \epsilon_{-3}, \epsilon_{-2}, \epsilon_{-1}; \epsilon_1, \epsilon_2, \epsilon_3, \ldots)$, the Dynkin diagram is



Hence, starting from the right and extending each time by one node to the left, one finds consecutively the Dynkin diagrams of $\mathfrak{gl}(1|1)$, $\mathfrak{gl}(1|2)$, $\mathfrak{gl}(2|2)$, etc. This process can continue to the left basically up to infinity.

It is in this context that the convenient GZ-basis and Dynkin diagrams for $\mathfrak{B}(n,n) = \mathfrak{osp}(2n+1|2n)$ are introduced. Adding the extra odd root ϵ_1 to the right, one finds by extending to the left consecutive Dynkin diagrams of $\mathfrak{B}(n,n)$ or $\mathfrak{B}(n,n+1)$.



3 New matrix realization of $\mathfrak{B}(n,n)$

Following the previous remarks, it is convenient to work in a new matrix realization of $\mathfrak{B}(n,n)$. Rows and columns, and indices of other objects, will be labelled by both negative and positive numbers. For non-negative integers m and n we will use the following notation for ordered sets:

$$\begin{aligned}[-m,n] &= \{-m,\ldots,-2,-1,0,1,2,\ldots,n\},\\ [-m,n]^* &= \{-m,\ldots,-2,-1,1,2,\ldots,n\}.\end{aligned} \tag{19}$$

When more convenient, we write the minus sign of an index as an overlined number, e.g. $[\bar{2},3]^* = \{\bar{2},\bar{1},1,2,3\}$. We will also use $\mathbb{Z}^* = \mathbb{Z}\setminus\{0\}$, $\mathbb{Z}_+ = \{0,1,2,\ldots\}$, $\mathbb{Z}_+^* = \{1,2,3,\ldots\}$.

Let I and J be the (2×2)-matrices

$$I := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{20}$$

and let B be the $(4n+1)\times(4n+1)$-matrix, with indices in $[-2n,2n]$, given by $B = I\oplus\cdots\oplus I\oplus 1\oplus J\oplus\cdots\oplus J$, or, written in block form:

$$B := \left(\begin{array}{ccc|c|ccc} I & 0 & & & & & \\ 0 & \ddots & 0 & & & & \\ & 0 & I & & & & \\ \hline & & & 1 & & & \\ \hline & & & & J & 0 & \\ & & & & 0 & \ddots & 0 \\ & & & & & 0 & J \end{array}\right). \tag{21}$$

Herein, 0 stands for the zero (2×2)-matrix, the entry 1 is at position $(0,0)$, and the empty parts of the matrix consist of zeros.

The matrices X of the Lie superalgebra $\mathfrak{B}(n,n)$ will have the following block form:

$$X := \left(\begin{array}{ccc|c|ccc} X_{\bar{n},\bar{n}} & \cdots & X_{\bar{n},\bar{1}} & X_{\bar{n},0} & X_{\bar{n},1} & \cdots & X_{\bar{n},n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ X_{\bar{1},\bar{n}} & \cdots & X_{\bar{1},\bar{1}} & X_{\bar{1},0} & X_{\bar{1},1} & \cdots & X_{\bar{1},n} \\ \hline X_{0,\bar{n}} & \cdots & X_{0,\bar{1}} & 0 & X_{0,1} & \cdots & X_{0,n} \\ \hline X_{1,\bar{n}} & \cdots & X_{1,\bar{1}} & X_{1,0} & X_{1,1} & \cdots & X_{1,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ X_{n,\bar{n}} & \cdots & X_{n,\bar{1}} & X_{n,0} & X_{n,1} & \cdots & X_{n,n} \end{array}\right). \tag{22}$$

Herein, any matrix of the form X_{ij} with $i,j\in[\bar{n},n]^*$ is a (2×2)-matrix, $X_{0,i}$ is a (1×2)-matrix and $X_{i,0}$ a (2×1)-matrix.

The Lie superalgebra $\mathfrak{B}(n,n) = \mathfrak{osp}(2n+1|2n)$ is $\mathbb{Z}_2$-graded and its homogeneous elements are referred to as even and odd elements, with the degree denoted by $\deg(X)$. The even matrices X will have zeros in the upper right and bottom left blocks, i.e. $X_{ij} = 0$ for all $(i,j) \in [\bar{n}, 0] \times [1,n]$ and $(i,j) \in [1,n] \times [\bar{n}, 0]$. The odd matrices X will have zeros in the upper left and bottom right blocks, i.e. $X_{ij} = 0$ for all $(i,j) \in [\bar{n}, 0] \times [\bar{n}, 0]$ and $(i,j) \in [1,n] \times [1,n]$.

The actual definition, derived from [16], is then as follows: $\mathfrak{B}(n,n)_0$ consists of all even matrices X of the form (22) such that

$$X^T B + BX = 0;$$

$\mathfrak{B}(n,n)_1$ consists of all odd matrices X of the form (22) such that

$$X^{ST} B - BX = 0.$$

Herein X^T is the ordinary transpose of X and X^{ST} is the supertranspose of X [13, 16]. For homogeneous elements of type (22), the Lie superalgebra bracket is

$$[\![X, Y]\!] = XY - (-1)^{\deg(X)\deg(Y)} YX,$$

with ordinary matrix multiplication in the right hand side.

Denote, as usual, by e_{ij} the matrix with zeros everywhere except a 1 on position (i,j), where the row and column indices run from $-2n$ to $2n$. A basis of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{B}(n,n)$ consists of the elements $h_i = e_{2i-1,2i-1} - e_{2i,2i}$ $(i \in [1,n])$ and $h_i = e_{2i,2i} - e_{2i+1,2i+1}$ $(i \in [\bar{n}, \bar{1}])$. The corresponding dual basis of $\mathfrak{h}^*$ will be denoted by ϵ_i $(i \in [\bar{n}, n]^*)$. The following elements are even root vectors with roots ϵ_{-i} and $-\epsilon_{-i}$ respectively $(i \in [1,n])$:

$$\begin{aligned} c_{-i}^{+} &\equiv f_{-i}^{+} = \sqrt{2}(e_{-2i,0} - e_{0,-2i+1}), \\ c_{-i}^{-} &\equiv f_{-i}^{-} = \sqrt{2}(e_{0,-2i} - e_{-2i+1,0}), \end{aligned} \tag{23}$$

and odd root vectors with roots ϵ_i and $-\epsilon_i$ respectively $(i \in [1,n])$ are given by:

$$\begin{aligned} c_{i}^{+} &\equiv b_{i}^{+} = \sqrt{2}(e_{0,2i} + e_{2i-1,0}), \\ c_{i}^{-} &\equiv b_{i}^{-} = \sqrt{2}(e_{0,2i-1} - e_{2i,0}). \end{aligned} \tag{24}$$

The remaining root vectors of $\mathfrak{B}(n,n)$ are given by elements of the form $[\![c_i^{\xi}, c_j^{\eta}]\!]$. The matrices (23)-(24) satisfy the triple relations (11), hence they realize the parastatistics operators.

In our development, it is also important to note that the $4n^2$ elements

$$[\![c_i^{+}, c_j^{-}]\!] \qquad (i,j \in [\bar{n}, n]^*) \tag{25}$$

are a basis of the subalgebra $\mathfrak{gl}(n|n)$. Observe also that

$$[c_i^+, c_i^-] = 2h_i \quad (i \in [\bar{n}, \bar{1}]), \qquad \{c_i^+, c_i^-\} = 2h_i \quad (i \in [1, n]). \tag{26}$$

Hence $\mathfrak{h} = \text{span}\{h_i,\ i \in [\bar{n}, n]^*\}$, the Cartan subalgebra of $\mathfrak{B}(n,n)$, is also the Cartan subalgebra of $\mathfrak{gl}(n|n)$.

4 The Fock representations $V(p)$ of $\mathfrak{B}(n,n)$

The Fock representation $V(p)$ of $\mathfrak{B}(n,n)$ was already introduced in the first section. Note that the condition $[\![c_j^-, c_k^+]\!]|0\rangle = p\delta_{jk}\,|0\rangle$ implies that we are dealing with a lowest weight representation of $\mathfrak{B}(n,n)$, with lowest weight $(-\frac{p}{2},\ldots,-\frac{p}{2}|\frac{p}{2},\ldots,\frac{p}{2})$ in the basis $\{\epsilon_{-n},\ldots,\epsilon_{-1};\epsilon_1,\ldots,\epsilon_n\}$. These representations have been analyzed in [12]. The main result is the decomposition with respect to the subalgebra chain $\mathfrak{B}(n,n) \supset \mathfrak{gl}(n|n)$, because then the Gelfand-Zetlin basis of the $\mathfrak{gl}(n|n)$ representations can be used to label the vectors of $V(p)$. In the decomposition of $V(p)$ with respect to $\mathfrak{B}(n,n) \supset \mathfrak{gl}(n|n)$, all covariant representations of $\mathfrak{gl}(n|n)$ labelled by a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ appear with multiplicity 1, subject to $\lambda_1 \le p$ and $\lambda_{n+1} \le n$. For each $\mathfrak{gl}(n|n)$ covariant representation labelled by λ, the highest weight can be determined [17], and is given by an array of $2n$ integers denoted by

$$[m]^{2n} = [m_{\bar{n},2n}, \ldots, m_{\bar{2},2n}, m_{\bar{1},2n}; m_{1,2n}, m_{2,2n}, \ldots, m_{n,2n}] \tag{27}$$

satisfying certain conditions. Next, one can follow the chain (17), leading in each step to the highest weight of the subalgebra, and thus yielding a labelling with $2n$ rows for the corresponding vectors. This is the actual odd GZ-basis for the Fock representation $V(p)$ of $\mathfrak{B}(n,n)$.

Explicitly, it is described as follows. For any positive integer p, a basis of the Fock representation $V(p)$ of $\mathfrak{B}(n,n)$ is given by the set of vectors of the following form:

$$|p;m)^{2n} \equiv |m)^{2n} = \left| \begin{matrix} [m]^{2n} \\ |m)^{2n-1} \end{matrix} \right) = \tag{28}$$

$$\left|\begin{array}{lllll|llllll}
m_{\bar{n},2n} & m_{\overline{n-1},2n} & \cdots & m_{\bar{2},2n} & m_{\bar{1},2n} & m_{1,2n} & m_{2,2n} & \cdots & m_{n-2,2n} & m_{n-1,2n} & m_{n,2n} \\
\uparrow & \uparrow & \cdots & \uparrow & \uparrow & & & & & & \\
m_{\bar{n},2n-1} & m_{\overline{n-1},2n-1} & \cdots & m_{\bar{2},2n-1} & m_{\bar{1},2n-1} & m_{1,2n-1} & m_{2,2n-1} & \cdots & m_{n-2,2n-1} & m_{n-1,2n-1} & \\
 & & & & & \downarrow & \downarrow & \cdots & \downarrow & \downarrow & \\
 & m_{\overline{n-1},2n-2} & \cdots & m_{\bar{2},2n-2} & m_{\bar{1},2n-2} & m_{1,2n-2} & m_{2,2n-2} & \cdots & m_{n-2,2n-2} & m_{n-1,2n-2} & \\
 & \uparrow & \cdots & \uparrow & \uparrow & & & & & & \\
 & m_{\overline{n-1},2n-3} & \cdots & m_{\bar{2},2n-3} & m_{\bar{1},2n-3} & m_{1,2n-3} & m_{2,2n-3} & \cdots & m_{n-2,2n-3} & & \\
 & & \ddots & \vdots & \vdots & \vdots & \vdots & \cdot^{\cdot^{\cdot}} & & & \\
 & & & m_{\bar{2}4} & m_{\bar{1}4} & m_{14} & m_{24} & & & & \\
 & & & \uparrow & \uparrow & & & & & & \\
 & & & m_{\bar{2}3} & m_{\bar{1}3} & m_{13} & & & & & \\
 & & & & & \downarrow & & & & & \\
 & & & & m_{\bar{1}2} & m_{12} & & & & & \\
 & & & & \uparrow & & & & & & \\
 & & & & m_{\bar{1}1} & & & & & &
\end{array}\right)$$

where all $m_{ij} \in \mathbb{Z}_+$, satisfying $m_{\bar{n},2n} \le p$ and the GZ-conditions

$$\begin{array}{l}
1.\ m_{j,2n} - m_{j+1,2n} \in \mathbb{Z}_+,\ j \in [\bar{n},\bar{2}] \cup [1,n] \text{ and} \\
\quad m_{-1,2n} \ge \#\{i : m_{i,2n} > 0,\ i \in [1,n]\}; \\
2.\ m_{-i,2s} - m_{-i,2s-1} \equiv \theta_{-i,2s-1} \in \{0,1\}, \quad 1 \le i \le s \le n; \\
3.\ m_{i,2s} - m_{i,2s+1} \equiv \theta_{i,2s} \in \{0,1\}, \quad 1 \le i \le s \le n-1; \\
4.\ m_{-1,2s} \ge \#\{i : m_{i,2s} > 0,\ i \in [1,s]\},\ s \in [1,n]; \\
5.\ m_{-1,2s-1} \ge \#\{i : m_{i,2s-1} > 0,\ i \in [1,s-1]\},\ s \in [2,n]; \\
6.\ m_{i,2s} - m_{i,2s-1} \in \mathbb{Z}_+ \text{ and } m_{i,2s-1} - m_{i+1,2s} \in \mathbb{Z}_+, \\
\qquad 1 \le i \le s-1 \le n-1; \\
7.\ m_{-i-1,2s+1} - m_{-i,2s} \in \mathbb{Z}_+ \text{ and } m_{-i,2s} - m_{-i,2s+1} \in \mathbb{Z}_+, \\
\qquad 1 \le i \le s \le n-1.
\end{array} \tag{29}$$

Conditions 2 and 3 are referred to as “θ-conditions”. Conditions 6 and 7 are often referred to as “betweenness conditions.” Conditions 1, 4 and 5 assure that each row of (28) corresponds to the highest weight of a covariant representation of $\mathfrak{gl}(t|t)$ or $\mathfrak{gl}(t|t-1)$ in the chain (17). Note that the arrows in this pattern have no real function, and can be omitted. We find it useful to include them, just in order to visualize the θ-conditions. When there is an arrow $a \to b$ between labels a and b, it means that either $b = a$ or else $b = a + 1$ (a θ-condition). We will also refer to “rows” and “columns” of the GZ-pattern. Rows are counted from the bottom: row 1 is the bottom row in (28), and row $2n$ is the top row in (28). In an obvious way, columns 1, 2, 3,… refer to the columns to the right of the dashed line in (28), and columns -1, -2, $-3, \ldots$ (or $\bar{1}$, $\bar{2}$, $\bar{3}, \ldots$) to the columns to the left of this dashed line. For two consecutive rows in the GZ-pattern (28), about half of the labels involve θ-conditions, and the other half involves betweenness conditions.

It should already be clear from this construction that the GZ-patterns of $\mathfrak{gl}(n|n)$ consist of those of $\mathfrak{gl}(n-1|n-1)$ to which two rows are added at the top. Hence it will be possible to gradually increase n, and we are in a setting for which the limit $n \to \infty$ can be examined.

One of the main computational results of [13] is the determination of the action of the parastatistics operators $c_i^{\pm}$ on the GZ basis vectors $|m)^{2n}$. For this, it is necessary to note that the $2n$ elements c_i^+ themselves form a standard $\mathfrak{gl}(n|n)$ tensor. Thus every element of $(c_n^+, c_{-n}^+, \cdots, c_2^+, c_{-2}^+, c_1^+, c_{-1}^+)$ corresponds, in this order, to a GZ-pattern of type (28) consisting of k top rows of the form $10\cdots0$ and $2n-k$ bottom rows of the form $0\cdots0$ for $k = 1, 2, \ldots, 2n$. It will be convenient to introduce a notation for the order in which these $2n$ elements appear:

$$\rho(i) = \left\{ \begin{array}{r} 2i \text{ for } i \in [1,n] \\ -2i-1 \text{ for } i \in [\bar{n},\bar{1}] \end{array} \right. . \tag{30}$$

Then the pattern corresponding to c_i^+ has rows of the form $10\cdots0$ for each row index $j \in [\rho(i), 2n]$ and zero rows for each row index $j \in [1, \rho(i)-1]$.

Following standard methods [9, 18], and knowing the tensor product rule in $\mathfrak{gl}(n|n)$ for covariant representations, the matrix elements of c_i^+ in $V(p)$ can be written as follows:

$$
\begin{aligned}
{}^{2n}(m'|c_i^+|m)^{2n} &= \left(\begin{matrix} [m]^{2n}_{+(k)} \\ |m')^{2n-1} \end{matrix} \middle| c_i^+ \middle| \begin{matrix} [m]^{2n} \\ |m)^{2n-1} \end{matrix} \right) \\
&= \left(\begin{matrix} 10\cdots 00 \\ 10\cdots 0 \\ \cdots \\ 0 \end{matrix} ; \begin{matrix} [m]^{2n} \\ \\ |m)^{2n-1} \end{matrix} \middle| \begin{matrix} [m]^{2n}_{+(k)} \\ \\ |m')^{2n-1} \end{matrix} \right) \times ([m]^{2n}_{+(k)}||c^+||[m]^{2n}).
\end{aligned} \tag{31}
$$

Herein, the GZ-pattern with 0's and 1's is the one corresponding to c_i^+, as described earlier, and $[m]^{2n}_{\pm(k)}$ is the pattern obtained from $[m]^{2n}$ by the replacement of $m_{k,2n}$ by $m_{k,2n} \pm 1$. The first factor in the right hand side of (31) is a $\mathfrak{gl}(n|n)$ Clebsch-Gordan coefficient (CGC), where all patterns are of the form (28). These CGC's have been determined in the Appendix of [13], and will not be repeated here. The second factor in (31) is a *reduced matrix element* for the standard $\mathfrak{gl}(n|n)$ tensor. The possible values of the patterns $|m')^{2n}$ are determined by the $\mathfrak{gl}(n|n)$ tensor product rule and the first line of $|m')^{2n}$ is of the form $[m]^{2n}_{+(k)}$. The reduced matrix elements themselves depend only upon the $\mathfrak{gl}(n|n)$ highest weights $[m]^{2n}$ and $[m]^{2n}_{+k}$ (and not on the type of GZ basis that is being used.) These reduced matrix elements have actually been determined in [12, Proposition 4].

Note furthermore that by the Hermiticity requirement one has

$$
{}^{2n}(m'|c_i^-|m)^{2n} = {}^{2n}(m|c_i^+|m')^{2n}. \tag{32}
$$

So in this way, one obtains a complete action of all parastatistics operators:

$$
c_i^+|m)^{2n} = \sum_{m'} C^+ \left[i, |m)^{2n}, |m')^{2n}\right] \; |m')^{2n}, \tag{33}
$$

$$
c_i^-|m)^{2n} = \sum_{m'} C^- \left[i, |m)^{2n}, |m')^{2n}\right] \; |m')^{2n}, \tag{34}
$$

where $C^+ \left[i, |m)^{2n}, |m')^{2n}\right]$ is just a shorthand notation for the element ${}^{2n}(m'|c_i^+|m)^{2n}$ computed in (31), and similarly for $C^- \left[i, |m)^{2n}, |m')^{2n}\right]$.

Examining the action of the creation operators c_i^+ in detail, one deduces the following property [13]: the action of c_i^+ on $|m)^{2n}$ yields vectors $|m')^{2n}$ such that rows $1, 2, \ldots, \rho(i) - 1$ of $|m')^{2n}$ are the same as those of $|m)^{2n}$. And in rows $\rho(i), \ldots, 2n$ there is a change by one unit for just one particular column index s: $[m']^j = [m]^j + [0, \ldots, 0, 1, 0, \ldots, 0]$ for $j \in [\rho(i), 2n]$. The increase can be in any possible column, as long as the remaining pattern is still valid, i.e. as long as (29) is satisfied.

An important observation is a certain stability property. For this, one introduces the following definition: the pattern, or equivalently the associated basis vector, $|m)^{2n}$ is *row-stable* with respect to row s if there exists a partition ν such that all rows $s, s+1, \ldots, 2n$ are of the form

$$[\nu_1, \nu_2, \ldots, 0; 0, 0, \ldots].$$

In that case, s is called a *stability index* of $|m)^{2n}$.

The following properties were proven in [13]:

- The action of a consecutive number of c_i^+'s on the vacuum vector produces row-stable patterns if n is sufficiently large. More precisely, if $k < n$, then all basis vectors appearing in

$$c_{i_k}^+ \cdots c_{i_2}^+ c_{i_1}^+ |0\rangle \qquad (\text{each } i_r \in [\bar{n}, n]^*) \tag{35}$$

  are row-stable with respect to some row index s.
- Row-stable patterns remain row-stable under the action of c_i^+'s (but the stability index might increase). Specifically, let $|m)^{2n}$ be row-stable with respect to row s, where $s < 2n-1$. Then the vectors $|m')^{2n}$ appearing in $c_i^+|m)^{2n}$ are row-stable with respect to row $\max\{s+2, \rho(i)+1\}$.
- Row-stable patterns remain row-stable under the action of c_i^-'s for the same stability index.

Also the matrix elements (33)-(34) satisfy a stability property. To specify this, one defines a map from GZ-patterns with $2n$ rows to GZ-patterns with $2n+2$ rows. For this, suppose that the top row of $|m)^{2n}$ has the zero partition as second part, i.e. it is of the form

$$[m]^{2n} = [\nu_1, \nu_2, \ldots; 0, \ldots, 0]$$

with ν a partition. Define the map $\phi_{2n,+2}$ from the set of GZ-patterns $|m)^{2n}$ with zero second part to the set of GZ-patterns $|m)^{2n+2}$ with stability index $2n$ by:

$$|m)^{2n+2} = \phi_{2n,+2}\left(|m)^{2n}\right), \text{ where} \tag{36}$$
$$[m]^{2n+1} = [\nu_1, \nu_2, \ldots, 0, 0; 0, \ldots, 0], \quad [m]^{2n+2} = [\nu_1, \nu_2, \ldots, 0, 0; 0, \ldots, 0, 0].$$

In other words, the top row of $|m)^{2n}$ is just repeated twice, with the extra addition of zeros in order to have sufficient entries for the pattern $|m)^{2n+2}$. Clearly, the action of $\phi_{2n,+2}$ can also be extended by linearity, on a linear combination of vectors $|m)^{2n}$ with zero second part.

The final important stability property can now be formulated: let $|m)^{2n}$ be row-stable with respect to row $2n$, and $|m)^{2n+2} = \phi_{2n,+2}\left(|m)^{2n}\right)$ Then for all i with $\rho(i) \le 2n$ (or equivalently, $i \in [-n, n]^*$):

$$c_i^+|m)^{2n+2} = \phi_{2n,+2}\left(c_i^+|m)^{2n}\right).$$

5 The Fock representations $V(p)$ of $\mathfrak{B}(\infty,\infty)$

Due to the stability properties just described, we can extend both the parastatistics algebra $\mathfrak{B}(n,n)$ and its Fock representations $V(p)$ to the infinite rank case $\mathfrak{B}(\infty,\infty)$.

The infinite rank Lie superalgebra $\mathfrak{B}(\infty,\infty)$ consists of infinite matrices X of the form (22) with $n\to\infty$ but with a finite number of non-zero elements, see [13] for a more precise definition. The indices of the matrices X now belong to $\mathbb{Z}$ instead of $[-n,n]$. The matrices e_{ij} consist of zeros everywhere except a 1 on position (i,j), where the row and column indices belong to $\mathbb{Z}$. A basis of a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{B}(\infty,\infty)$ consists of the elements $h_i = e_{2i-1,2i-1} - e_{2i,2i}$ $(i\in\mathbb{Z}_+^*)$ and $h_i = e_{2i,2i} - e_{2i+1,2i+1}$ $(i\in\mathbb{Z}_-^*)$. The corresponding dual basis of $\mathfrak{h}^*$ is denoted by ϵ_i $(i\in\mathbb{Z}^*)$. As in the finite rank case, we can identify the following even root vectors with roots ϵ_{-i} and $-\epsilon_{-i}$ respectively $(i\in\mathbb{Z}_+^*)$:

$$
\begin{aligned}
c_{-i}^+ &\equiv f_{-i}^+ = \sqrt{2}(e_{-2i,0} - e_{0,-2i+1}),\\
c_{-i}^- &\equiv f_{-i}^- = \sqrt{2}(e_{0,-2i} - e_{-2i+1,0}),
\end{aligned}
\tag{37}
$$

and odd root vectors with roots ϵ_i and $-\epsilon_i$ respectively $(i\in\mathbb{Z}_+^*)$:

$$
\begin{aligned}
c_i^+ &\equiv b_i^+ = \sqrt{2}(e_{0,2i} + e_{2i-1,0}),\\
c_i^- &\equiv b_i^- = \sqrt{2}(e_{0,2i-1} - e_{2i,0}).
\end{aligned}
\tag{38}
$$

The operators c_i^+ can be chosen as positive root vectors, and the c_i^- as negative root vectors.

The operators introduced here satisfy the triple relations of parastatistics. But now we are dealing with an infinite number of parafermions and an infinite number of parabosons, satisfying the mutual relative parafermion relations. In other words, the triple relations (11) are satisfied, but now with $j,k,l\in\mathbb{Z}^*$. We also have: as a Lie superalgebra defined by generators and relations, $\mathfrak{B}(\infty,\infty)$ is generated by the elements $c_i^\pm$ $(i\in\mathbb{Z}^*)$ subject to the relations (11).

The parastatistics Fock space of order p, with p a positive integer, can be defined as before, and will correspond to a lowest weight representation $V(p)$ of the algebra $\mathfrak{B}(\infty,\infty)$. $V(p)$ is the Hilbert space generated by a vacuum vector $|0\rangle$ and the parastatistics creation and annihilation operators, i.e. subject to $\langle 0|0\rangle = 1$, $c_j^-|0\rangle = 0$, $(c_j^\pm)^\dagger = c_j^\mp$,

$$
[\![c_j^-, c_k^+]\!]|0\rangle = p\delta_{jk}\,|0\rangle \quad (j,k\in\mathbb{Z}^*)
\tag{39}
$$

and which is irreducible under the action of the algebra $\mathfrak{B}(\infty,\infty)$. Clearly $|0\rangle$ is a lowest weight vector of $V(p)$ with weight $(\ldots,-\frac{p}{2},-\frac{p}{2}|\frac{p}{2},\frac{p}{2},\ldots)$ in the basis $\{\ldots,\epsilon_{-2},\epsilon_{-1};\epsilon_1,\epsilon_2,\ldots\}$.

The basis vectors of $V(p)$ will consist of infinite GZ-patterns. Not all possible infinite GZ-patterns will appear, but only row-stable ones. Such row-stable infinite GZ-patterns consist of an infinite number of rows, of the type introduced in (28), but such that from a certain row index s all rows $s, s+1, s+2, \ldots$ are of the same form. As an example,

$$|m)^\infty = \left|\begin{array}{r:l} \ddots \;\; \ddots \qquad & \qquad \iddots \;\; \iddots \\ 4\;\;3\;1\;0 & 0\;0\;\;0\;\;\;0 \\ 4\;\;3\;1\;0 & 0\;0\;\;0 \\ 4\;3\;1 & 0\;0\;\;0 \\ 4\;3\;1 & 0\;0 \\ 3\;3 & 1\;0 \\ 3\;2 & 1 \\ 2 & 2 \\ 1 & \end{array}\right) \tag{40}$$

where the row $(4,3,1,0,\ldots)$ is repeated up to infinity.

The basis of $V(p)$ is described as follows.

Proposition 1. *A basis of $V(p)$ is given by all infinite row-stable GZ-patterns $|m)^\infty$ of the form (28) with $n \to \infty$ where for each $|m)^\infty$ there should exist a row index s (depending on $|m)^\infty$) such that row s is of the form*

$$[m]^s = [\nu_1, \nu_2, \ldots, 0; 0, 0, \ldots]$$

with ν a partition, all rows above s are of the same form (up to extra zeros), and $\nu_1 \le p$. Furthermore all $m_{ij} \in \mathbb{Z}_+$ and the usual GZ-conditions should be satisfied:

1. $m_{-i,2r} - m_{-i,2r-1} \equiv \theta_{-i,2r-1} \in \{0,1\}, \quad 1 \le i \le r;$
2. $m_{i,2r} - m_{i,2r+1} \equiv \theta_{i,2r} \in \{0,1\}, \quad 1 \le i \le r;$
3. $m_{-1,2r} \ge \#\{i : m_{i,2r} > 0,\ i \in [1,r]\},\ r \in \mathbb{Z}_+^*;$
4. $m_{-1,2r+1} \ge \#\{i : m_{i,2r+1} > 0,\ i \in [1,r]\},\ r \in \mathbb{Z}_+^*;$
5. $m_{i,2r+2} - m_{i,2r+1} \in \mathbb{Z}_+$ *and* $m_{i,2r+1} - m_{i+1,2r+2} \in \mathbb{Z}_+, \quad 1 \le i \le r;$
6. $m_{-i-1,2r+1} - m_{-i,2r} \in \mathbb{Z}_+$ *and* $m_{-i,2r} - m_{-i,2r+1} \in \mathbb{Z}_+, \quad 1 \le i \le r.$

The process of adding an infinite number of identical rows (up to additional zeros) at the top of a finite GZ-pattern can now be formalized by means of a map, just as we did by adding two identical rows in the previous section. Let $|m)^{2n}$ be a finite GZ-pattern of type (28) with $2n$ rows, such that row $2n$ is of the form $[\nu_1, \nu_2, \ldots; 0, 0, \ldots, 0]$. Then $\phi_{2n,\infty}\left(|m)^{2n}\right)$ is the infinite GZ-pattern consisting of the rows of $|m)^{2n}$ to which an infinite number of rows $[\nu_1, \nu_2, \ldots; 0, 0, \ldots, 0]$ are added at the top (all identical, up to additional zeros). Conversely, if an infinite GZ-pattern $|m)^\infty$ is given, which is stable with respect to row $2s$, then one can restrict the infinite pattern to a finite GZ-pattern, and

$$|m)^{2s} = \phi^{-1}_{2s,\infty}\left(|m)^{\infty}\right).$$

Both maps can be extended by linearity. Then one can define the action of $c_i^{\pm}$ on vectors $|m)^{\infty}$:

Definition 1. Given a vector $|m)^{\infty}$ of $V(p)$ with stability index $2s$, and a generator $c_i^{\pm}$. Let $2n$ be such that $2n > \max\{2s, \rho(i)\}$. Then

$$c_i^{\pm}|m)^{\infty} = \phi_{2n,\infty}\left(c_i^{\pm}|m)^{2n}\right), \text{ where } |m)^{2n} = \phi^{-1}_{2n,\infty}\left(|m)^{\infty}\right). \tag{41}$$

The main theorem, proved in [13] is then

Theorem 1. *The vector space $V(p)$, with basis vectors all infinite row-stable GZ-patterns for which $\nu_1 \le p$, on which the action of the $\mathfrak{B}(\infty,\infty)$ generators $c_i^{\pm}$ ($i \in \mathbb{Z}^*$) is defined by* (41)*, is an irreducible unitary Fock representation of $\mathfrak{B}(\infty,\infty)$.*

To conclude, we have managed to give a description of parastatistics Fock spaces with an infinite number of parafermions and parabosons. Our developments in previous years had already led to such a description for m parafermions and n parabosons by means of representations of $\mathfrak{osp}(2m+1|2n)$. The GZ basis for these representations, determined in [12], is however not appropriate for the limit to an infinite number of parastatistics operators. We therefore constructed a new GZ basis for $\mathfrak{B}(n,n) = \mathfrak{osp}(2n+1|2n)$ representations. In this new basis, there is a natural limit for $n \to \infty$, and the corresponding infinite row-stable GZ-patterns label the basis vectors of the corresponding Fock space $V(p)$ of $\mathfrak{B}(\infty,\infty)$.

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Appendix

Although a low-rank example is not very instructive for the case $n \to \infty$, it is still useful to the reader to visualize the basic structure of the basis vectors (28) and the action (31) with matrix elements (33). This is why we include the basis of $V(p)$ for $n = 1$, i.e. for $\mathfrak{B}(n,n)$. Let

$$|m) \equiv \left| \begin{array}{c|c} m_{\bar{1}2} & m_{12} \\ m_{\bar{1}1} & \end{array} \right)$$

where

$$\begin{array}{l} 1.\ m_{ij} \in \mathbb{Z}_+,\ m_{\bar{1}2} \le p; \\ 2.\ m_{\bar{1}2} \in \{0,1,2,\cdots\} \text{ if } m_{12}=0;\ m_{\bar{1}2} \in \{1,2,\cdots\} \text{ if } m_{12} \neq 0;. \\ 3.\ m_{\bar{1}1} \in \{m_{\bar{1}2}, m_{\bar{1}2}-1\}. \end{array} \tag{42}$$

The action of the Cartan algebra elements is:

$$
\begin{aligned}
h_{\bar{1}}|m) &= \Big(-\frac{p}{2} + m_{\bar{1}1}\Big)\,|m),\\
h_1|m) &= \Big(\frac{p}{2} + m_{\bar{1}2} + m_{12} - m_{\bar{1}1}\Big)\,|m).
\end{aligned} \tag{43}
$$

The action of the parastatistics creation operators reads

$$
\begin{aligned}
c_{\bar{1}}^{+}\left|\begin{array}{l:l} m_{\bar{1}2} & m_{12} \\ m_{\bar{1}2} & \end{array}\right) &= G_{\bar{1}}(m_{\bar{1}2},m_{12})\left|\begin{array}{l:l} m_{\bar{1}2}+1 & m_{12} \\ m_{\bar{1}2}+1 & \end{array}\right),\\
c_{\bar{1}}^{+}\left|\begin{array}{l:l} m_{\bar{1}2} & m_{12} \\ m_{\bar{1}2}-1 & \end{array}\right) &= \sqrt{\frac{m_{\bar{1}2}+m_{12}}{m_{\bar{1}2}+m_{12}+1}}\,G_{\bar{1}}(m_{\bar{1}2},m_{12})\left|\begin{array}{l:l} m_{\bar{1}2}+1 & m_{12} \\ m_{\bar{1}2} & \end{array}\right)\\
&\quad - \sqrt{\frac{1}{m_{\bar{1}2}+m_{12}+1}}\,G_{1}(m_{\bar{1}2},m_{12})\left|\begin{array}{l:l} m_{\bar{1}2} & m_{12}+1 \\ m_{\bar{1}2} & \end{array}\right),\\
c_{1}^{+}\left|\begin{array}{l:l} m_{\bar{1}2} & m_{12} \\ m_{\bar{1}2} & \end{array}\right) &= \sqrt{\frac{1}{m_{\bar{1}2}+m_{12}+1}}\,G_{\bar{1}}(m_{\bar{1}2},m_{12})\left|\begin{array}{l:l} m_{\bar{1}2}+1 & m_{12} \\ m_{\bar{1}2} & \end{array}\right)\\
&\quad + \sqrt{\frac{m_{\bar{1}2}+m_{12}}{m_{\bar{1}2}+m_{12}+1}}\,G_{1}(m_{\bar{1}2},m_{12})\left|\begin{array}{l:l} m_{\bar{1}2} & m_{12}+1 \\ m_{\bar{1}2} & \end{array}\right),\\
c_{1}^{+}\left|\begin{array}{l:l} m_{\bar{1}2} & m_{12} \\ m_{\bar{1}2}-1 & \end{array}\right) &= -G_{1}(m_{\bar{1}2},m_{12})\left|\begin{array}{l:l} m_{\bar{1}2} & m_{12}+1 \\ m_{\bar{1}2}-1 & \end{array}\right).
\end{aligned} \tag{44}
$$

Herein, $G_{\bar{1}}$ and G_1 are shorthand notations for the reduced matrix elements in (31): $G_{\bar{1}}(m_{\bar{1}2},m_{12}) = (m_{\bar{1}2}+1, m_{12}||c^{+}||m_{\bar{1}2},m_{12})$ and $G_1(m_{\bar{1}2},m_{12}) = (m_{\bar{1}2}, m_{12}+1||c^{+}||m_{\bar{1}2},m_{12})$, explicitly given by

$$
\begin{aligned}
G_{\bar{1}}(m_{\bar{1}2},m_{12}) &= \sqrt{\frac{m_{\bar{1}2}(m_{\bar{1}2}+m_{12}+1)(p-m_{\bar{1}2})}{m_{\bar{1}2}+m_{12}}}, \quad \text{if } m_{12} \text{ is even},\\
G_{\bar{1}}(m_{\bar{1}2},m_{12}) &= \sqrt{m_{\bar{1}2}(p-m_{\bar{1}2})}, \quad \text{if } m_{12} \text{ is odd},\\
G_{1}(m_{\bar{1}2},m_{12}) &= \sqrt{m_{\bar{1}2}+m_{12}+1}, \quad \text{if } m_{12} \text{ is even},\\
G_{1}(m_{\bar{1}2},m_{12}) &= \sqrt{\frac{(m_{12}+1)(p+m_{12}+1)}{m_{\bar{1}2}+m_{12}}}, \quad \text{if } m_{12} \text{ is odd}.
\end{aligned} \tag{45}
$$

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