FINITE OSCILLATOR MODELS DESCRIBED BY THE LIE SUPERALGEBRA $\mathfrak{sl}(2|1)$

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We investigate new models for a finite quantum oscillator based upon the Lie superalgebra $\mathfrak{sl}(2|1)$, where the position and momentum operators are represented as odd elements of the algebra. We discuss properties of the spectrum of the position operator, and of the (discrete) position wavefunctions given by (alternating) Krawtchouk polynomials.

Keywords: Algebraic oscillator models, discrete wavefunctions, Lie superalgebra $\mathfrak{sl}(2|1)$, Krawtchouk polynomials

1. Introduction and the Lie superalgebra $\mathfrak{sl}(2|1)$

The literature on quantum mechanics in a finite-dimensional Hilbert space is substantial. This paper is devoted to an algebraic model for a quantum oscillator allowing finite-dimensional representations, thus leading to a finite oscillator model.

The canonical one-dimensional quantum oscillator (in the convention $m = \omega = \hbar = 1$) is described by a position operator $\hat{q}$, a momentum operator $\hat{p}$ and a Hamiltonian $\hat{H}$ given by

$$H = \frac{\hat{p}^2}{2} + \frac{\hat{q}^2}{2}. \tag{1}$$

The *oscillator Lie algebra* is usually defined to be the Lie algebra generated by $\hat{q}, \hat{p}, \hat{H}$ and the identity operator 1, subject to the relations

$$[\hat{H}, \hat{q}] = -\mathrm{i}\,\hat{p}, \qquad [\hat{H}, \hat{p}] = \mathrm{i}\,\hat{q}, \tag{2}$$

$$[\hat{q}, \hat{p}] = \mathrm{i}. \tag{3}$$

Equation (3) is the canonical commutation relation (CCR); (2) is known as the Hamilton-Lie equations. In fact, (3) together with (1) imply the

Hamilton-Lie equations (2). The converse is not true. If one drops the CCR (3) but keeps (1) and (2), the resulting system is the *Wigner quantum oscillator*, also known as the *paraboson oscillator*.[1,2] The underlying algebra (generated by $\hat{q}$ and $\hat{p}$) is the Lie superalgebra $\mathfrak{osp}(1|2)$.

Most algebraic models for the quantum oscillator go even one step further, and also drop relation (1), thus keeping only the "dynamics" of the oscillator described by (2). In order to speak of *an algebraic oscillator model*, one requires[3]

- there should be three operators $\hat{q}$, $\hat{p}$, $\hat{H}$ belonging to some (Lie) algebra (or superalgebra) $\mathcal{A}$ such that (2) is satisfied,
- the spectrum of $\hat{H}$ in representations of $\mathcal{A}$ is equidistant.

The best known algebraic model for the quantum oscillator is based on the Lie algebra $\mathcal{A} = \mathfrak{su}(2)$.[3] $\hat{H}$, $\hat{q}$ and $\hat{p}$ are certain self-adjoint elements in $\mathfrak{su}(2)$, and the corresponding unitary representations are the common finite dimensional representations of dimension $2j+1$, with $2j$ a positive integer. The spectrum of $\hat{H}$ is of the form $n+\frac{1}{2}$ (with $n = 0, 1, \ldots, 2j$), and the spectrum of $\hat{q}$ is discrete and equidistant.The corresponding discrete position wavefunctions are expressed in terms of normalized Krawtchouk polynomials, tending to normalized Hermite polynomials when $j \to \infty$. Some interesting deformations of the $\mathfrak{su}(2)$ oscillator model were studied recently, in which case the discrete wavefunctions turn out to be normalized Hahn polynomials.[4,5]

Inspired by Wigner's work, where the paraboson oscillator is described by $\mathfrak{osp}(1|2)$, we have initiated the study of oscillator models based on a Lie superalgebra $\mathcal{A}$. We shall consider here an algebraic oscillator model based on $\mathcal{A} = \mathfrak{sl}(2|1)$. In order to do so, we need to consider an appropriate class of representations of $\mathfrak{sl}(2|1)$. Furthermore, we shall make a proper choice for $\hat{H}$, $\hat{q}$ and $\hat{p}$. In particular, $\hat{q}$ (and $\hat{p}$) should be odd elements of the Lie superalgebra. The main task is then to determine the spectrum of these operators and to construct the corresponding wavefunctions. In the current paper, we highlight just some of the main results for the $\mathfrak{sl}(2|1)$ oscillator. More detailed results are found in Ref. 6.

In terms of 3×3 Weyl matrices e_{ij} $(i, j = 1, 2, 3)$, one can choose the following basis for $\mathfrak{sl}(2|1)$[7]

$$F^+ = e_{32},\ G^+ = e_{13},\ F^- = e_{31},\ G^- = e_{23}, \tag{4}$$

$$H = \frac{1}{2}(e_{11} - e_{22}),\ E^+ = e_{12},\ E^- = e_{21},\ Z = \frac{1}{2}(e_{11} + e_{22}) + e_{33}, \tag{5}$$

where the first four elements are odd, and the last four even. The even subalgebra of $\mathfrak{sl}(2|1)$ is just $\mathfrak{sl}(2)\oplus\mathfrak{u}(1)$, where $\mathfrak{sl}(2)$ is spanned by H, E^+, E^- and $\mathfrak{u}(1)$ by Z. This follows from the commutators

$$[H, E^\pm] = \pm E^\pm, \quad [E^+, E^-] = 2H, \quad [Z, H] = [Z, E^\pm] = 0. \tag{6}$$

For the odd basis elements, the mutual anti-commutators are

$$\begin{aligned} &\{F^\pm, G^\pm\} = E^\pm, \quad \{F^\pm, G^\mp\} = Z \mp H, \\ &\{F^\pm, F^\pm\} = \{G^\pm, G^\pm\} = \{F^\pm, F^\mp\} = \{G^\pm, G^\mp\} = 0. \end{aligned} \tag{7}$$

The "mixed" commutators are given by

$$\begin{aligned} &[H, F^\pm] = \pm\frac{1}{2}F^\pm,\ [Z, F^\pm] = \frac{1}{2}F^\pm,\ [E^\pm, F^\pm] = 0,\ [E^\mp, F^\pm] = -F^\mp; \\ &[H, G^\pm] = \pm\frac{1}{2}G^\pm,\ [Z, G^\pm] = -\frac{1}{2}G^\pm,\ [E^\pm, G^\pm] = 0,\ [E^\mp, G^\pm] = G^\mp. \end{aligned}$$

We shall consider here a known class of representations of $\mathfrak{sl}(2|1)$, but with basis and action rewritten in a nicer and more appropriate form. These are *atypical* irreducible representations labelled by a non-negative integer j, with representation space W_j and $\dim(W_j) = 2j+1$. The (orthonormal) basis of W_j is denoted by $|j,m\rangle$ ($m = -j, -j+1, \ldots, +j$). Using the practical notation $\mathcal{E}(n) = 1$ if n is even and 0 otherwise; and $\mathcal{O}(n) = 1$ if n is odd and 0 otherwise, the actions of the odd basis elements is easily written as:

$$\begin{aligned} F^\pm|j,m\rangle &= \pm\mathcal{O}(j-m)\sqrt{\frac{j\pm m+1}{2}}\ |j, m\pm 1\rangle, \\ G^\pm|j,m\rangle &= \pm\mathcal{E}(j-m)\sqrt{\frac{j\mp m}{2}}\ |j, m\pm 1\rangle. \end{aligned} \tag{8}$$

Those of the even basis elements read:

$$\begin{aligned} Z|j,m\rangle &= -\mathcal{E}(j-m)\frac{j}{2}\ |j,m\rangle - \mathcal{O}(j-m)(\frac{j+1}{2})\ |j,m\rangle, \\ H|j,m\rangle &= \frac{m}{2}\ |j,m\rangle, \\ E^\pm|j,m\rangle &= \frac{1}{2}\mathcal{E}(j-m)\sqrt{(j\mp m)(j\pm m+2)}\ |j, m\pm 2\rangle \\ &\quad + \frac{1}{2}\mathcal{O}(j-m)\sqrt{(j\mp m-1)(j\pm m+1)}\ |j, m\pm 2\rangle. \end{aligned} \tag{9}$$

These are the known "dispin" representations:[8,9] W_j decomposes as $(\frac{j}{2}; -\frac{j}{2}) \oplus (\frac{j-1}{2}; -\frac{j+1}{2})$, where $(l; b)$ denotes the $\mathfrak{su}(2) \oplus u(1)$ representation "with isospin l and hypercharge b". Note that W_j is a *unitary* representation, for the adjoint operation

$$Z^\dagger = Z,\ H^\dagger = H,\ (E^\pm)^\dagger = E^\mp,\ (F^\pm)^\dagger = -G^\mp,\ (G^\pm)^\dagger = -F^\mp. \tag{10}$$

2. The $\mathfrak{sl}(2|1)$ oscillator model

Let us now make a proper choice for position, momentum and Hamiltonian operator in $\mathfrak{sl}(2|1)$. Following the second requirement, we should take:

$$\hat{H} = 2H + j + \frac{1}{2}, \tag{11}$$

having spectrum $n + \frac{1}{2}$ $(n = 0, 1, \ldots, 2j)$. As concerns $\hat{q}$, an arbitrary self-adjoint odd element is of the form $\hat{q} = AF^+ + BG^+ - BF^- - AG^-$, with A and B arbitrary real numbers. An overall factor does not play a crucial role, however, so we can take $A^2 + B^2 = 1$. Assuming that A and B have the same sign (the other case is analogous), it follows that the most general form for $\hat{q}$ is

$$\hat{q} = \sqrt{p}\, F^+ + \sqrt{1-p}\, G^+ - \sqrt{1-p}\, F^- - \sqrt{p}\, G^-, \qquad (0 \le p \le 1). \tag{12}$$

Herein, p is a free parameter, and we shall treat only the generic case $0 < p < 1$. For the momentum operator, we have

$$\hat{p} = \mathrm{i}(\sqrt{p}\, F^+ + \sqrt{1-p}\, G^+ + \sqrt{1-p}\, F^- + \sqrt{p}\, G^-). \tag{13}$$

The expressions (11), (12) and (13) do indeed satisfy the Hamilton-Lie equations (2).

Next, we need to determine the eigenvalues and eigenvectors of the operators $\hat{q}$ and $\hat{p}$ in the representation W_j (we will consider only $\hat{q}$ here). From the actions of $F^\pm$ and $G^\pm$ in the (ordered) basis $\{|j,j\rangle, |j,j-1\rangle, \ldots, |j,-j\rangle\}$ of W_j, it follows that the matrix M_q of $\hat{q}$ is a symmetric tridiagonal matrix, with zero diagonal and off diagonal elements given by

$$R_1, S_1, R_2, S_2, \ldots, R_j, S_j,$$

where

$$R_k = \sqrt{p}\sqrt{j+1-k}, \qquad S_k = \sqrt{1-p}\sqrt{k} \qquad (k = 1, 2, \ldots, j).$$

The eigenvectors of this matrix M_q are described in terms of Krawtchouk polynomials.[10] These polynomials $K_n(x; p, N)$, of degree n $(n = 0, 1, \ldots, N)$ in the variable x, with parameter p, satisfy a discrete orthogonality relation:

$$\sum_{x=0}^{N} w(x; p, N) K_n(x; p, N) K_{n'}(x; p, N) = h(n; p, N)\, \delta_{nn'},$$

where $w(x; p, N) = \binom{N}{x} p^x (1-p)^{N-x}$ and $h(n; p, N) = \frac{n!(N-n)!}{N!} \left(\frac{1-p}{p}\right)^n$. The corresponding orthonormal Krawtchouk polynomials are denoted by

$$\tilde{K}_n(x; p, N) \equiv \frac{\sqrt{w(x; p, N)}}{\sqrt{h(n; p, N)}}\, K_n(x; p, N).$$

In terms of these, let U be the $(2j+1)\times(2j+1)$-matrix:

$$\begin{aligned}
&U_{2n,j} = (-1)^n \tilde{K}_0(n;p,j), \quad U_{2m+1,j}=0,\\
&U_{2n,j-k} = U_{2n,j+k} = \frac{(-1)^n}{\sqrt{2}}\tilde{K}_k(n;p,j),\\
&U_{2n'+1,j-k} = -U_{2n'+1,j+k} = -\frac{(-1)^{n'}}{\sqrt{2}}\tilde{K}_{k-1}(n';p,j-1),
\end{aligned}$$

where $n\in\{0,1,\ldots,j\}$, $n'\in\{0,\ldots,j-1\}$ and $k\in\{1,\ldots,j\}$. Then the orthogonality of Krawtchouk polynomials implies that U is an orthogonal matrix: $UU^T=U^TU=I$. Furthermore, from forward and backward shift operator formulas[10] for Krawtchouk polynomials one can deduce that $M_qU=UD$, with D a diagonal matrix:

$$D = \mathrm{diag}(-\sqrt{j},-\sqrt{j-1},\ldots,-\sqrt{2},-1,0,1,\sqrt{2},\ldots,\sqrt{j-1},\sqrt{j}).$$

In other words, the columns of U are the eigenvectors of M_q, and the eigenvalues of M_q (and hence of $\hat{q}$ in the representation W_j) are of the form $q_{\pm k}=\pm\sqrt{k}$ $(k=0,1,\ldots,j)$. Note that in the columns of U, Krawtchouk polynomials with parameter j and with parameter $j-1$ alternate.

Let us denote the eigenvector for the eigenvalue q_k by $|j,q_k)$:

$$|j,q_k) = \sum_{m=-j}^{j} U_{j+m,j+k}|j,-m\rangle = \sum_{m=-j}^{j}\phi^{(p)}_{j+m}(q_k)|j,-m\rangle. \qquad (14)$$

The reason to denote this coefficient by $\phi^{(p)}_{j+m}(q_k)$ is because overlaps between the normalized eigenstates of the position operator and the eigenstates of the Hamiltonian have an interpretation as position wavefunctions. Hence position wavefunctions are given in terms of (alternating) Krawtchouk polynomials, e.g. when $q_k>0$:

$$\phi^{(p)}_{2n}(q_k) = \frac{(-1)^n}{\sqrt{2}}\tilde{K}_k(n;p,j), \qquad \phi^{(p)}_{2n+1}(q_k) = \frac{(-1)^n}{\sqrt{2}}\tilde{K}_{k-1}(n;p,j-1).$$

In Ref. 6 we have given some plots of these position wavefunctions, for various values of p, j and n. Their behaviour shows similarities with canonical oscillator wavefunctions, but of course they are a discrete version. When the dimension parameter j is large, the $\mathfrak{sl}(2|1)$ wavefunctions tend to paraboson wavefunctions (which is also confirmed by a limit calculation).

3. Further results

The determination of the eigenvalues and eigenvectors of $\hat{p}$ is very similar. The spectrum of $\hat{p}$ is the same as that of $\hat{q}$: $p_k=\pm\sqrt{k}$. The momentum wavefunctions are, up to a power of i, the same as the position wavefunctions:

$$\psi^{(p)}_{j+m}(p_k) = (\mathrm{i})^{j+m-1}\phi^{(p)}_{j+m}(p_k).$$

So one can determine the matrix F that transforms position wavefunctions into momentum wavefunctions; this is the $\mathfrak{sl}(2|1)$ discrete Fourier transform F. Since the wavefunctions have such a simple form, the matrix elements of F can be determined explicitly using the known bilinear generating function for Krawtchouk polynomials. This $\mathfrak{sl}(2|1)$ discrete Fourier transform has many properties similar to the standard Discrete Fourier Transform.

Choosing a different adjoint operation for $\mathfrak{sl}(2|1)$, the corresponding unitary representations are infinite dimensional. These discrete series representations of $\mathfrak{sl}(2|1)$, labelled by a positive number $\beta > 0$, have been determined recently.[11] In these representations, $\hat{H}$ has the same spectrum as the canonical oscillator. General forms for $\hat{q}$ (and $\hat{p}$) involve one parameter $\gamma > 0$ (like our p in the finite dimensional case). The spectrum of $\hat{q}$ is continuous ($= \mathbb{R}$) for $\gamma = 1$ and infinite discrete ($\pm\sqrt{\gamma^2 - 1}\sqrt{k}$, $k \in \mathbb{Z}_+$) for $\gamma > 1$ (similar for $0 < \gamma < 1$). The position wavefunctions $\Phi_n^{(\beta,\gamma)}(x)$ coincide with paraboson wavefunctions for $\gamma = 1$, and are related to (discrete) *Meixner polynomials* for $\gamma \neq 1$; the canonical oscillator is recovered for $\beta = 1/2$.

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