

On the algebra of symmetries of Laplace and Dirac operators

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Abstract We consider a generalization of the classical Laplace operator, which includes the Laplace-Dunkl operator defined in terms of the differential-difference operators associated with finite reflection groups called Dunkl operators. For this Laplace-like operator, we determine a set of symmetries commuting with it, which are generalized angular momentum operators, and we present the algebraic relations for the symmetry algebra. In this context, the generalized Dirac operator is then defined as a square root of our Laplace-like operator. We explicitly determine a family of graded operators which commute or anticommute with our Dirac-like operator depending on their degree. The algebra generated by these symmetry operators is shown to be a generalization of the standard angular momentum algebra and the recently defined higher rank Bannai-Ito algebra.

Keywords Laplace operator · Dirac operator · Dunkl operator · Symmetry algebra · Bannai-Ito algebra

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1 Introduction

The study of solutions of the Laplace equation or of the Dirac equation, in any context or setting, is a major topic of investigation. For that purpose, a crucial role is played by the symmetries of the Laplace operator Δ or of the Dirac operator \underline{D} , i.e. operators commuting with Δ or (anti)commuting with \underline{D} . The symmetries involved and the algebras they generate lead to topics such as separation of variables and special functions. Without claiming completeness we refer the reader to [3, 12, 7, 8, 2, 20].

For this paper, the context we have in mind is that of Dunkl derivatives [11, 25], i.e. where the ordinary derivative $\frac{\partial}{\partial x_i}$ is replaced by a Dunkl derivative \mathcal{D}_i in the expression of the Laplace or Dirac operator. One often refers to these operators as the Laplace-Dunkl and the Dirac-Dunkl operator. The chief purpose of this paper is to determine the symmetries of the Laplace-Dunkl operator and of the Dirac-Dunkl operator, and moreover study the algebra generated by these symmetries.

In the process of this investigation, it occurred to us that it is advantageous to treat this problem in a more general context, which we shall describe here in the introduction. For this purpose, let us first turn to a standard topic in quantum mechanics: the description of the N -dimensional (isotropic) harmonic oscillator. The Hamiltonian \hat{H} of this oscillator (with the common convention $m = \omega = \hbar = 1$ for mass, frequency and the reduced Planck constant) is given by

$$\hat{H} = \frac{1}{2} \sum_{j=1}^N \hat{p}_j^2 + \frac{1}{2} \sum_{j=1}^N \hat{x}_j^2. \quad (1.1)$$

In canonical quantum mechanics, the coordinate operators \hat{x}_j and momentum operators \hat{p}_j are required to be (essentially self-adjoint) operators satisfying the canonical commutation relations

$$[\hat{x}_i, \hat{x}_j] = 0, \quad [\hat{p}_i, \hat{p}_j] = 0, \quad [\hat{x}_i, \hat{p}_j] = i\delta_{ij}. \quad (1.2)$$

So in the “coordinate representation”, where \hat{x}_j is represented by multiplication with the variable x_j , the operator \hat{p}_j is represented by $\hat{p}_j = -i\frac{\partial}{\partial x_j}$.

Because the canonical commutation relations are sometimes considered as “unphysical” or “imposed without a physical motivation”, more fundamental ways of quantization have been the topic of various research fields (such as geometrical quantization). One of the pioneers of a more fundamental quantization procedure was Wigner, who introduced a method that later became known as “Wigner quantization” [22, 23, 26, 28, 29]. Briefly said, in Wigner quantization one preserves all axioms of quantum mechanics, except that the canonical commutation relations are replaced by a more fundamental principle: the compatibility of the (classical) Hamilton equations with the Heisenberg equations of motion. Concretely, these compatibility conditions read

$$[\hat{H}, \hat{x}_j] = -i\hat{p}_j, \quad [\hat{H}, \hat{p}_j] = i\hat{x}_j \quad (j = 1, \dots, N). \quad (1.3)$$

Thus for the quantum oscillator, one keeps the Hamiltonian (1.1), but replaces the relations (1.2) by (1.3). When the canonical commutation relations (1.2) hold, the compatibility relations (1.3) are automatically valid (this is a version of the Ehrenfest theorem), but not vice versa. Hence Wigner quantization is a generalization of canonical quantization, and canonical quantization is just one possible solution of Wigner quantization. Note that in Wigner quantization the coordinate operators \hat{x}_j (and the momentum operators) in general do not commute, so this is of particular significance in the field of non-commutative quantum mechanics.

In a mathematical context, as in this paper, one usually replaces the physical momentum operator components \hat{p}_j by operators $p_j = i\hat{p}_j$, and one also denotes the coordinate operators \hat{x}_j by x_j . Then the operator \hat{H} takes the form

$$H = -\frac{1}{2} \sum_{j=1}^N p_j^2 + \frac{1}{2} \sum_{j=1}^N x_j^2. \quad (1.4)$$

So in the canonical case, where x_j stands for multiplication by the variable x_j , p_j is just the derivative $\frac{\partial}{\partial x_j}$, and the first term of H is (up to a factor $-1/2$) equal to the Laplace operator $\sum_{j=1}^N p_j^2 = \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} = \Delta$. In the more general case, the compatibility conditions (1.3) read

$$[H, x_j] = -p_j, \quad [H, p_j] = -x_j \quad (j = 1, \dots, N). \quad (1.5)$$

We are now in a position to describe our problem in a general framework:

Given N commuting operators x_1, \dots, x_N and N commuting operators p_1, \dots, p_N , consider the operator $H = -\frac{1}{2} \sum_{j=1}^N p_j^2 + \frac{1}{2} \sum_{j=1}^N x_j^2$, and suppose that the compatibility conditions (1.5) hold. Classify the symmetries of the generalized Laplace operator, i.e. classify the operators that commute with $\sum_{j=1}^N p_j^2$.

In other words, we are given N operators x_1, \dots, x_N and N operators p_1, \dots, p_N that satisfy

$$[x_i, x_j] = 0, \quad [p_i, p_j] = 0, \quad (1.6)$$

$$\left[\frac{1}{2} \sum_{i=1}^N p_i^2, x_j \right] = p_j, \quad \left[\frac{1}{2} \sum_{i=1}^N x_i^2, p_j \right] = -x_j. \quad (1.7)$$

Under these conditions, the *first problem* is: determine the operators that commute with the generalized Laplace operator

$$\Delta = \sum_{i=1}^N p_i^2. \quad (1.8)$$

Our two major examples of systems satisfying (1.6) and (1.7) are the “canonical case” and the “Dunkl case”. Another example can be found in the context of discrete counterparts of the Laplacian [10].

For the first example, x_i is just the multiplication by the variable x_i , and p_i is the derivative with respect to x_i : $p_i = \frac{\partial}{\partial x_i}$. Clearly, these operators satisfy (1.6) and (1.7), and the operator Δ in (1.8) coincides with the classical Laplace operator.

For the second example, x_i is again the multiplication by the variable x_i , but p_i is the Dunkl derivative $p_i = \mathcal{D}_i$, which is a certain differential-difference operator with an underlying reflection group determined by a root system (a precise definition follows later in this paper). Conditions (1.6) still hold: the commutativity of the operators x_i is trivial, but the commutativity of the operators p_i is far from trivial [11, 13]. Following [11], also the conditions (1.7) are valid in the Dunkl case. The operator Δ in (1.8) now takes the form $\sum_{i=1}^N \mathcal{D}_i^2$ and is known as the Dunkl Laplacian or the Laplace-Dunkl operator. By the way, it is no surprise that the operators x_i and \mathcal{D}_j do not satisfy the canonical commutation relations. It is, however, very surprising that they satisfy the more general Wigner quantization relations (for a Hamiltonian of oscillator type).

So the solution of the first problem in the general context will in particular lead to the determination of symmetries of the Laplace-Dunkl operator.

Since we are dealing with these operators in an algebraic context, it is worthwhile to move to a closely related operator, the Dirac operator. For this purpose, consider a set of N generators of a Clifford algebra, i.e. N elements e_i satisfying

$$\{e_i, e_j\} = \epsilon 2\delta_{ij}$$

where $\{a, b\} = ab + ba$ denotes the anticommutator and ϵ is $+1$ or -1 . The generators e_i are supposed to commute with the general operators x_j and p_j . Under the general conditions (1.6) and (1.7), the *second problem* is now: determine the operators that commute (or anticommute) with the generalized Dirac operator

$$\underline{D} = \sum_{i=1}^N e_i p_i. \quad (1.9)$$

Obviously, this is a refinement of the first problem, since $\underline{D}^2 = \epsilon \Delta$.

For our two major examples, in the canonical case the operator (1.9) is just the classical Dirac operator; in the Dunkl case, the operator (1.9) is known as the Dirac-Dunkl operator.

In the present paper we solve both problems in the general framework (1.6)–(1.7), and even go beyond it by determining the algebraic relations satisfied by the symmetries. In section 2 we consider the generalized Laplace operator Δ and determine all symmetries, i.e. all operators commuting with Δ (Theorem 2.3). Next, in Theorem 2.5 the algebraic relations satisfied by these symmetries are established. For the generalized Dirac operator \underline{D} , the determination of the symmetries is computationally far more involved. In section 3,

Theorem 3.7 classifies essentially all operators that commute or anticommute with \underline{D} . In the following subsections, we derive the quadratic relations satisfied by the symmetries of the Dirac operator. The computations of these relations are very intricate, and involve subtle techniques. Fortunately, there is a case to compare with. For the Dunkl case, in which the underlying reflection group is the simplest possible example (namely \mathbb{Z}_2^N), the symmetries and their algebraic relations have been determined in [7, 8] and give rise to the so-called (higher rank) Bannai-Ito algebra. Our results can be considered as an extension of these relations to an arbitrary underlying reflection group, in fact in an even more general context.

2 Symmetries of Laplace operators

We start by formally describing the operator algebra that will contain the desired symmetries of a generalized Laplace operator (1.8), as brought up in the introduction.

Definition 2.1. *We define the algebra \mathcal{A} to be the unital (over the field \mathbb{R} or \mathbb{C}) associative algebra generated by the $2N$ elements x_1, \dots, x_N and p_1, \dots, p_N subject to the following relations:*

$$\begin{aligned} [x_i, x_j] &= 0, & [p_i, p_j] &= 0, \\ \left[\frac{1}{2} \sum_{i=1}^N p_i^2, x_j \right] &= p_j, & \left[\frac{1}{2} \sum_{i=1}^N x_i^2, p_j \right] &= -x_j. \end{aligned}$$

Note that an immediate consequence of the relations of \mathcal{A} is

$$[x_i, p_j] = [x_i, -[H, x_j]] = -[[x_i, H], x_j] = -[p_i, x_j] = [x_j, p_i],$$

where H is given by (1.4). This reciprocity

$$[x_i, p_j] = [x_j, p_i] \tag{2.1}$$

will be useful for many ensuing calculations, starting with the following theorem.

Theorem 2.2. *The algebra \mathcal{A} contains a copy of the Lie algebra $\mathfrak{sl}(2)$ generated by the elements*

$$\frac{|x|^2}{2} = \frac{1}{2} \sum_{i=1}^N x_i^2, \quad -\frac{\Delta}{2} = -\frac{1}{2} \sum_{i=1}^N p_i^2, \quad \mathbb{E} = \frac{1}{2} \sum_{i=1}^N \{p_i, x_i\}, \tag{2.2}$$

satisfying the relations

$$\left[\mathbb{E}, \frac{|x|^2}{2} \right] = |x|^2, \quad \left[\mathbb{E}, -\frac{\Delta}{2} \right] = -\Delta, \quad \left[\frac{|x|^2}{2}, -\frac{\Delta}{2} \right] = \mathbb{E}.$$

Proof. By direct computation we have

$$\frac{1}{4}[\Delta, |x|^2] = \frac{1}{4} \sum_{i=1}^N [\Delta, x_i^2] = \frac{1}{4} \sum_{i=1}^N \{[\Delta, x_i], x_i\} = \frac{1}{2} \sum_{i=1}^N \{p_i, x_i\}.$$

Using the commutativity of p_1, \dots, p_N and relation (2.1), we have

$$\begin{aligned} [\mathbb{E}, \Delta] &= \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N [\{p_i, x_i\}, p_j^2] = \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \{[\{p_i, x_i\}, p_j], p_j\} \\ &= \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \{(p_i(x_i p_j - p_j x_i) + (x_i p_j - p_j x_i)p_i), p_j\} \\ &= \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \{(p_i(x_j p_i - p_i x_j) + (x_j p_i - p_i x_j)p_i), p_j\} \\ &= -\frac{1}{2} \sum_{j=1}^N \left\{ \left[\sum_{i=1}^N p_i^2, x_j \right], p_j \right\} = -\sum_{j=1}^N \{p_j, p_j\} = -2\Delta. \end{aligned}$$

In the same manner, using now the commutativity of x_1, \dots, x_N , we find $[\mathbb{E}, |x|^2] = 2|x|^2$. \square

In the spirit of Howe duality [18, 19], our objective is to determine the subalgebra of \mathcal{A} which commutes with the Lie algebra $\mathfrak{sl}(2)$ realized by Δ and $|x|^2$ as appearing in Theorem 2.2. As mentioned in the introduction, the element Δ corresponds to a generalized version of the Laplace operator, which reduces to the classical Laplace operator for a specific choice of the elements p_1, \dots, p_N . In the (Euclidean) coordinate representation, $|x|^2$ of course represents the norm squared.

2.1 Symmetries

As p_1, \dots, p_N are commuting operators, by definition they also commute with Δ . However, in general they are not symmetries of $|x|^2$. An immediate first example of an operator which does commute with both Δ and $|x|^2$ is given by the Casimir operator (in the universal enveloping algebra) of their $\mathfrak{sl}(2)$ realization

$$\Omega = \mathbb{E}^2 - 2\mathbb{E} - |x|^2 \Delta \in \mathcal{U}(\mathfrak{sl}(2)) \subset \mathcal{A}. \quad (2.3)$$

Note that this operator is of the same order in both x_1, \dots, x_N and p_1, \dots, p_N as it has to commute with both Δ and $|x|^2$. More precisely it is of fourth order in the generators of \mathcal{A} , being quadratic in x_1, \dots, x_N and quadratic in p_1, \dots, p_N . We now set out to consider the most elementary symmetries, those which are of second order in the generators of \mathcal{A} .

Theorem 2.3. *In the algebra \mathcal{A} , the elements which are quadratic in the generators x_1, \dots, x_N and p_1, \dots, p_N , and which commute with Δ and $|x|^2$ are spanned by*

$$L_{ij} = x_i p_j - x_j p_i, \quad C_{ij} = [p_i, x_j] = p_i x_j - x_j p_i \quad (i, j \in \{1, \dots, N\}). \quad (2.4)$$

Note that when $i = j$ we have $L_{ii} = 0$, while $C_{ii} = [p_i, x_i]$ does not necessarily vanish. Moreover, as $L_{ij} = -L_{ji}$, every symmetry L_{ij} is up to a sign equal to one of the $N(N-1)/2$ symmetries $\{L_{ij} \mid 1 \leq i < j \leq N\}$. By relation (2.1), $C_{ij} = C_{ji}$ and thus we have $N(N+1)/2$ symmetries $\{C_{ij} \mid 1 \leq i \leq j \leq N\}$. In total, this gives N^2 generically distinct symmetries.

Proof. We first show that L_{ij} and C_{ij} as defined by (2.4) are indeed symmetries of Δ and $|x|^2$. As Δ commutes with p_1, \dots, p_N and using condition (1.7), we have for $i, j \in \{1, \dots, N\}$

$$[\Delta, x_i p_j - x_j p_i] = x_i [\Delta, p_j] + [\Delta, x_i] p_j - x_j [\Delta, p_i] - [\Delta, x_j] p_i = 2p_i p_j - 2p_j p_i = 0.$$

In the same manner we have $[\Delta, p_i x_j - x_j p_i] = 0$. The relations for $|x|^2$ follow similarly.

Now, a general element $S \in \mathcal{A}$ quadratic in the generators x_1, \dots, x_N and p_1, \dots, p_N is of the form

$$S = \sum_{i,j} (a_{ij} x_i p_j + b_{ij} p_i x_j + c_{ij} x_i x_j + d_{ij} p_i p_j),$$

where i and j are summed over $\{1, \dots, N\}$ and $a_{ij}, b_{ij}, c_{ij}, d_{ij}$ are scalars. Using relations (1.6) and (1.7), we have

$$\begin{aligned} \frac{1}{2}[\Delta, S] &= \sum_{i,j} (a_{ij} p_i p_j + b_{ij} p_i p_j + c_{ij} (x_i p_j + p_i x_j)) \\ &= \sum_{i,j} c_{ij} (x_i p_j + p_i x_j) + \sum_i (a_{ii} + b_{ii}) p_i^2 + \sum_{i < j} (a_{ij} + a_{ji} + b_{ij} + b_{ji}) p_i p_j. \end{aligned}$$

In order for this to vanish, the coefficients must satisfy $a_{ii} + b_{ii} = 0$, $c_{ij} = 0$ for all i, j , and $a_{ij} + a_{ji} + b_{ij} + b_{ji} = 0$ for $i < j$, but by symmetry also for $i > j$, and by the previous case also for $i = j$, thus finally for all i, j . The condition $[|x|^2, S] = 0$ yields the additional requirements $d_{ij} = 0$ for $i \in \{1, \dots, N\}$. Hence, say $a_{ii} = -b_{ii}$ and $a_{ji} = -a_{ij} - b_{ij} - b_{ji}$, then the symmetry S is of the form

$$S = \sum_i b_{ii} (p_i x_i - x_i p_i) + \sum_{i < j} (a_{ij} (x_i p_j - x_j p_i) + b_{ij} (p_i x_j - x_j p_i) + b_{ji} (p_j x_i - x_i p_j))$$

where in the right-hand side we recognize C_{ii} , L_{ij} , C_{ij} and $C_{ji} + L_{ij}$. \square

2.2 Symmetry algebra

For the following results, we make explicit use of the symmetry $C_{ij} = [p_i, x_j]$ being symmetric in its two indices, by relation (2.1). This is the case for p_i corresponding to classical partial derivatives, but also for their generalization in the form of Dunkl operators. We will return in detail to these examples in section 2.3. Another consequence of relation (2.1) pertains to the form of the other symmetries of Theorem 2.3. By means of $x_i p_j - p_j x_i = x_j p_i - p_i x_j$ we readily observe that

$$L_{ij} = x_i p_j - x_j p_i = p_j x_i - p_i x_j. \quad (2.5)$$

Given these symmetry properties, the symmetries of Theorem 2.3 generate an algebraic structure within \mathcal{A} whose relations we present after the following lemma.

Lemma 2.4. *In the algebra \mathcal{A} , the symmetries (2.4) satisfy the following relations for all $i, j, k \in \{1, \dots, N\}$*

$$[C_{ij}, p_k] = [C_{kj}, p_i], \quad \text{and} \quad [C_{ij}, x_k] = [C_{kj}, x_i].$$

Moreover, we also have

$$L_{ij} p_k + L_{ki} p_j + L_{jk} p_i = 0 = p_k L_{ij} + p_j L_{ki} + p_i L_{jk},$$

and

$$x_k L_{ij} + x_j L_{ki} + x_i L_{jk} = 0 = L_{ij} x_k + L_{ki} x_j + L_{jk} x_i.$$

Proof. For the first relation, writing out the commutators in $[[p_k, x_j], p_i] - [[p_i, x_j], p_k]$ we find

$$p_k x_j p_i - x_j p_k p_i - p_i p_k x_j + p_i x_j p_k - p_i x_j p_k + x_j p_i p_k + p_k p_i x_j - p_k x_j p_i.$$

We see that all terms cancel due to the mutual commutativity of the operators p_1, \dots, p_N . The other relation of the first line follows in the same way.

For the other two relations, the identities follow immediately by choosing the appropriate expression for L_{ij} of (2.5) and making use of the commutativity of either x_1, \dots, x_N or p_1, \dots, p_N . \square

Theorem 2.5. *In the algebra \mathcal{A} , the symmetries (2.4) satisfy the following relations for all $i, j, k, l \in \{1, \dots, N\}$,*

$$\begin{aligned} [L_{ij}, L_{kl}] &= L_{il} C_{jk} + L_{jk} C_{il} + L_{ki} C_{lj} + L_{lj} C_{ki} \\ &= C_{jk} L_{il} + C_{il} L_{jk} + C_{lj} L_{ki} + C_{ki} L_{lj}, \end{aligned} \quad (2.6)$$

$$\{L_{ij}, L_{kl}\} + \{L_{ki}, L_{jl}\} + \{L_{jk}, L_{il}\} = 0, \quad (2.7)$$

$$[L_{ij}, C_{kl}] + [L_{ki}, C_{jl}] + [L_{jk}, C_{il}] = 0, \quad (2.8)$$

and

$$L_{ij} L_{kl} + L_{ki} L_{jl} + L_{jk} L_{il} = L_{ij} C_{kl} + L_{ki} C_{jl} + L_{jk} C_{il}. \quad (2.9)$$

Proof. We will prove the first line of the first relation, i.e. (2.6), the second line follows in a similar manner. We have

$$\begin{aligned}
& [x_i p_j - x_j p_i, x_k p_l - x_l p_k] \\
&= [x_i p_j, x_k p_l] - [x_i p_j, x_l p_k] - [x_j p_i, x_k p_l] + [x_j p_i, x_l p_k] \\
&= x_i [p_j, x_k] p_l + x_k [x_i, p_l] p_j - x_i [p_j, x_l] p_k - x_l [x_i, p_k] p_j \\
&\quad - x_j [p_i, x_k] p_l - x_k [x_j, p_l] p_i + x_j [p_i, x_l] p_k + x_l [x_j, p_k] p_i \\
&= x_i C_{jk} p_l - x_k C_{li} p_j - x_i C_{jl} p_k + x_l C_{ki} p_j \\
&\quad - x_j C_{ik} p_l + x_k C_{lj} p_i + x_j C_{il} p_k - x_l C_{kj} p_i.
\end{aligned}$$

Swapping all operators p_l with C_{jk} , we find

$$\begin{aligned}
[L_{ij}, L_{kl}] &= x_i p_l C_{jk} + x_i [C_{jk}, p_l] - x_k p_j C_{li} - x_k [C_{li}, p_j] \\
&\quad - x_i p_k C_{jl} - x_i [C_{jl}, p_k] + x_l p_j C_{ki} + x_l [C_{ki}, p_j] \\
&\quad - x_j p_l C_{ik} - x_j [C_{ik}, p_l] + x_k p_i C_{lj} + x_k [C_{lj}, p_i] \\
&\quad + x_j p_k C_{il} + x_j [C_{il}, p_k] - x_l p_i C_{kj} - x_l [C_{kj}, p_i] \\
&= x_i p_l C_{jk} - x_k p_j C_{li} - x_i p_k C_{jl} + x_l p_j C_{ki} \\
&\quad - x_j p_l C_{ik} + x_k p_i C_{lj} + x_j p_k C_{il} - x_l p_i C_{kj} \\
&\quad + x_i ([C_{jk}, p_l] - [C_{jl}, p_k]) + x_k (-[C_{li}, p_j] + [C_{lj}, p_i]) \\
&\quad - x_l (-[C_{ki}, p_j] + [C_{kj}, p_i]) - x_j ([C_{ik}, p_l] - [C_{il}, p_k]) \\
&= L_{il} C_{jk} + L_{jk} C_{il} + L_{ki} C_{lj} + L_{lj} C_{ki},
\end{aligned}$$

where we used $C_{jk} = C_{kj}$ and Lemma 2.4.

The identities (2.7) and (2.8) follow by making explicit use of both expressions of (2.5) for L_{ij} . For the left-hand side of (2.7) we have

$$\begin{aligned}
& L_{ij} L_{kl} + L_{kl} L_{ij} + L_{ki} L_{jl} + L_{jl} L_{ki} + L_{jk} L_{il} + L_{il} L_{jk} \\
&= (x_i p_j - x_j p_i)(p_l x_k - p_k x_l) + (x_k p_l - x_l p_k)(p_j x_i - p_i x_j) \\
&\quad + (x_k p_i - x_i p_k)(p_l x_j - p_j x_l) + (x_j p_l - x_l p_j)(p_i x_k - p_k x_i) \\
&\quad + (x_j p_k - x_k p_j)(p_l x_i - p_i x_l) + (x_i p_l - x_l p_i)(p_k x_j - p_j x_k),
\end{aligned}$$

where again all terms vanish due to the commutativity of p_1, \dots, p_N .

Working out the commutators, the left-hand side of (2.8) becomes

$$\begin{aligned}
& L_{ij} [p_l, x_k] - [p_l, x_k] L_{ij} + L_{ki} [p_l, x_j] - [p_l, x_j] L_{ki} + L_{jk} [p_l, x_i] - [p_l, x_i] L_{jk} \\
&= L_{ij} p_l x_k - p_l x_k L_{ij} + L_{ki} p_l x_j - p_l x_j L_{ki} + L_{jk} p_l x_i - p_l x_i L_{jk} \\
&\quad - L_{ij} x_k p_l + x_k p_l L_{ij} - L_{ki} x_j p_l + x_j p_l L_{ki} - L_{jk} x_i p_l + x_i p_l L_{jk}.
\end{aligned}$$

Hence, plugging in suitable choices for the symmetries L_{ij} , this becomes

$$\begin{aligned}
& (x_i p_j - x_j p_i) p_l x_k - p_l x_k (x_i p_j - x_j p_i) + (x_k p_i - x_i p_k) p_l x_j \\
&\quad - p_l x_j (x_k p_i - x_i p_k) + (x_j p_k - x_k p_j) p_l x_i - p_l x_i (x_j p_k - x_k p_j) \\
&\quad - (p_j x_i - p_i x_j) x_k p_l + x_k p_l (p_j x_i - p_i x_j) - (p_i x_k - p_k x_i) x_j p_l \\
&\quad + x_j p_l (p_i x_k - p_k x_i) - (p_k x_j - p_j x_k) x_i p_l + x_i p_l (p_k x_j - p_j x_k).
\end{aligned}$$

One observes that all terms vanish due to the commutativity of x_1, \dots, x_N and p_1, \dots, p_N respectively.

For the final relation, using the definitions (2.4) we have

$$\begin{aligned} & L_{ij}(L_{kl} - C_{kl}) + L_{ki}(L_{jl} - C_{jl}) + L_{jk}(L_{il} - C_{il}) \\ &= L_{ij}(x_k p_l - p_k x_l) + L_{ki}(x_j p_l - p_j x_l) + L_{jk}(x_i p_l - p_i x_l) \\ &= (L_{ij}x_k + L_{ki}x_j + L_{jk}x_i)p_l - (L_{ij}p_k + L_{ki}p_j + L_{jk}p_i)x_l, \end{aligned}$$

which vanishes by Lemma 2.4. \square

2.3 Examples

Example 2.1. As a first example, we consider N mutually commuting variables x_1, \dots, x_N , doubling as operators acting on functions by left multiplication with the respective variable and p_j being just the derivative $\partial/\partial x_j$ for $j \in \{1, \dots, N\}$. In this case obviously p_1, \dots, p_N mutually commute and the operators of interest are

$$\Delta = \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2}, \quad |x|^2 = \sum_{i=1}^N x_i^2, \quad H = -\frac{1}{2}\Delta + \frac{1}{2}|x|^2,$$

which satisfy

$$\frac{1}{2}[\Delta, x_i] = \frac{\partial}{\partial x_i} = p_i, \quad \frac{1}{2}[|x|^2, p_i] = -x_i.$$

By Theorem 2.3, we have the following symmetries:

$$L_{ij} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}, \quad C_{ij} = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

While C_{ij} is a scalar for every i, j , the L_{ij} symmetries are the standard angular momentum operators whose symmetry algebra is the Lie algebra $\mathfrak{so}(N)$:

$$[L_{ij}, L_{kl}] = L_{il}\delta_{jk} + L_{jk}\delta_{il} + L_{ki}\delta_{lj} + L_{lj}\delta_{ik}.$$

This is in accordance with Theorem 2.5 as in this case evidently $C_{ij} = C_{ji}$.

Note that Δ and $|x|^2$ are also invariant under $O(N)$, the group of orthogonal transformations on \mathbb{R}^N , but these transformations are not contained in the algebra \mathcal{A} .

In the discrete setting of [10], the symmetry algebra generated by the discrete angular momentum operators was also shown to be $\mathfrak{so}(N)$.

Example 2.2. A more intriguing example is given by a generalization of partial derivatives to differential-difference operators associated to a Coxeter or Weyl group W . Let R be a (reduced) root system and k a multiplicity

function which is invariant under the natural action of the Weyl group W consisting of all reflections associated to R ,

$$\sigma_\alpha(x) = x - 2\langle x, \alpha \rangle \alpha / \|\alpha\|^2, \quad \alpha \in R, x \in \mathbb{R}^N.$$

For $\xi \in \mathbb{R}^N$, the Dunkl operator [11, 25] is defined as

$$\mathcal{D}_\xi f(x) := \frac{\partial}{\partial \xi} f(x) + \sum_{\alpha \in R_+} k(\alpha) \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle} \langle \alpha, \xi \rangle,$$

where the summation is taken over all roots in R_+ , a fixed positive subsystem of R . For a fixed root system and function k , the Dunkl operators associated to any two vectors commute, see [11]. Hence, they form potential candidates for the operators p_1, \dots, p_N satisfying condition (1.6). The operator of interest is the Laplace-Dunkl operator Δ_k , which can be written as

$$\Delta_k = \sum_{i=1}^N (\mathcal{D}_{\xi_i})^2$$

for any orthonormal basis $\{\xi_1, \dots, \xi_N\}$ of \mathbb{R}^N . For the orthonormal basis associated to the coordinates x_1, \dots, x_N we use the notation

$$\mathcal{D}_i f(x) := \frac{\partial}{\partial x_i} f(x) + \sum_{\alpha \in R_+} k(\alpha) \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle} \alpha_i \quad (i \in \{1, \dots, N\}), \quad (2.10)$$

where $\alpha_i = \langle \alpha, \xi_i \rangle$.

For our purpose, let x_j again stand for multiplication by the variable x_j and take now $p_j = \mathcal{D}_j$ for $j \in \{1, \dots, N\}$. Besides condition (1.6), condition (1.7) is also satisfied (see for instance [11, 25]). We note that the $\mathfrak{sl}(2)$ relations in this context were already obtained by [16].

By Theorem 2.3 we have as symmetries, on the one hand, the Dunkl version of the angular momentum operators

$$L_{ij} = x_i \mathcal{D}_j - x_j \mathcal{D}_i.$$

On the other hand, the symmetries

$$C_{ij} = [\mathcal{D}_i, x_j] = \delta_{ij} + \sum_{\alpha \in R_+} 2k(\alpha) \alpha_i \alpha_j \sigma_\alpha$$

consist of linear combinations of the reflections in the Weyl group, with coefficients determined by the multiplicity function k and the roots of the root system. This is of course in agreement with Δ_k being W -invariant [25]. The Weyl group is a subgroup of $O(N)$, and in this case the algebra \mathcal{A} does contain these reflections in W .

Note that indeed $C_{ij} = C_{ji}$, in accordance with relation (2.1). Theorem 2.5 now yields the Dunkl version of the angular momentum algebra:

$$[L_{ij}, L_{kl}] = L_{il} C_{jk} + L_{jk} C_{il} + L_{ki} C_{lj} + L_{lj} C_{ki}$$

$$\begin{aligned}
&= L_{il}\delta_{jk} + L_{jk}\delta_{il} + L_{ki}\delta_{lj} + L_{lj}\delta_{ki} \\
&\quad + \sum_{\alpha \in R_+} 2k(\alpha) (L_{il}\alpha_j\alpha_k + L_{jk}\alpha_i\alpha_l + L_{ki}\alpha_l\alpha_j + L_{lj}\alpha_k\alpha_i) \sigma_\alpha.
\end{aligned}$$

This relation states the interaction of the L_{ij} symmetries amongst one another. The interaction between the symmetries C_{kl} is governed by the group multiplication of the Weyl group, while the relations for the symmetries L_{ij} and C_{kl} follow immediately from the action of a reflection $\sigma_\alpha \in W$ on the coordinate variables and the Dunkl operators:

$$\sigma_\alpha \xi = \sigma_\alpha(\xi) \sigma_\alpha, \quad \sigma_\alpha \mathcal{D}_\xi = \mathcal{D}_{\sigma_\alpha(\xi)} \sigma_\alpha \quad (\xi \in \mathbb{R}^N).$$

Let $\{\xi_1, \dots, \xi_N\}$ denote the orthonormal basis associated to the coordinates x_1, \dots, x_N , then

$$\sigma_\alpha L_{ij} = L_{\sigma_\alpha(\xi_i)\sigma_\alpha(\xi_j)} \sigma_\alpha,$$

where for $\xi, \eta \in \mathbb{R}^N$ we define

$$L_{\xi\eta} = \langle x, \xi \rangle \mathcal{D}_\eta - \langle x, \eta \rangle \mathcal{D}_\xi = \sum_{k,l} \langle \xi, \xi_k \rangle \langle \eta, \xi_l \rangle L_{kl}.$$

This allows us to interchange any two symmetries of the form L_{ij} and C_{kl} .

These results have been considered before, namely for the specific case $W = (\mathbb{Z}_2)^3$ in [15], and for $W = S_N$, and also for arbitrary Coxeter group, in [14]. Furthermore, relation (2.9) has been obtained already in the Dunkl case, and dubbed “the crossing relation” [14].

3 Symmetries of Dirac operators

We now turn to a closely related operator of the generalized Laplace operator considered in the preceding section, namely the Dirac operator. For an operator of the form (1.8), one can construct a “square root” by introducing a set of elements e_1, \dots, e_N which commute with x_i and p_i for all $i \in \{1, \dots, N\}$ and which satisfy the following relations

$$\{e_i, e_j\} = e_i e_j + e_j e_i = \epsilon 2\delta_{ij}, \quad (3.1)$$

where $\epsilon = \pm 1$, or thus for $i \neq j$

$$(e_i)^2 = \epsilon = \pm 1, \quad e_i e_j + e_j e_i = 0.$$

We use these elements to define the following two operators

$$\underline{D} = \sum_{i=1}^N e_i p_i, \quad \underline{x} = \sum_{i=1}^N e_i x_i,$$

whose squares equal

$$\underline{D}^2 = \epsilon \sum_{i=1}^N (p_i)^2 = \epsilon \Delta, \quad \underline{x}^2 = \epsilon \sum_{i=1}^N (x_i)^2 = \epsilon |x|^2,$$

by means of the anticommutation relations (3.1) of e_1, \dots, e_N and condition (1.6). For the classical case where p_i is the i th partial derivative, the operator \underline{D} is the standard Dirac operator.

The elements e_1, \dots, e_N in fact generate what is known as a Clifford algebra [24], which we will denote as $\mathcal{C} = \mathcal{Cl}(\mathbb{R}^N)$. A general element in this algebra is a linear combination of products of e_1, \dots, e_N . The standard convention is to denote for instance $e_1 e_2 e_3$ simply as e_{123} . Hereto, we introduce the concept of a ‘list’ for use as index of Clifford numbers.

Definition 3.1. *We define a list to indicate a finite sequence of distinct elements of a given set, in our case the set $\{1, \dots, N\}$. For a list $A = a_1 \cdots a_n$ of $\{1, \dots, N\}$ with $0 \leq n \leq N$, we will use the notation*

$$e_A = e_{a_1} e_{a_2} \cdots e_{a_n}. \quad (3.2)$$

Remark 3.2. *Note that in a list the order matters as the Clifford generators e_1, \dots, e_N anticommute. Moreover, duplicate elements would cancel out as they square to $\epsilon = \pm 1$, so we consider only lists containing distinct elements. For a set $A = \{a_1, \dots, a_n\} \subset \{1, \dots, N\}$, the notation e_A stands for $e_{a_1} e_{a_2} \cdots e_{a_n}$ with $a_1 < a_2 < \cdots < a_n$.*

The collection $\{e_A \mid A \subset \{1, \dots, N\}\}$ forms a basis of the Clifford algebra \mathcal{C} , where for the empty set we put $e_\emptyset = 1$.

Remark 3.3. *In general, the square of each individual element e_i where $i \in \{1, \dots, N\}$ can independently be chosen equal to either $+1$ or -1 . This corresponds to an underlying space with arbitrary signature defined by the specified signs. The original Dirac operator was constructed as a square root of the wave operator by means of the gamma or Dirac matrices which form a matrix realization of the Clifford algebra for $N = 4$ with Minkowski signature.*

To simplify notations in the following, we have chosen the square of all e_i ($i \in \{1, \dots, N\}$) to be equal to ϵ which can be either $+1$ or -1 . One can generalize all results to arbitrary signature by making the appropriate substitutions.

In order to consider symmetries of the generalized Dirac operator (1.9) we will work in the tensor product $\mathcal{A} \otimes \mathcal{C}$ with the algebra \mathcal{A} as defined in Definition 2.1. To avoid overloading on notations, we will omit the tensor symbol \otimes when writing down elements of $\mathcal{A} \otimes \mathcal{C}$ and use regular product notation. In this notation e_1, \dots, e_N indeed commutes with x_i and p_i for all $i \in \{1, \dots, N\}$.

Akin to the realization of the Lie algebra $\mathfrak{sl}(2)$ in the algebra \mathcal{A} given by Theorem 2.2, we have something comparable in this case.

Theorem 3.4. *The algebra $\mathcal{A} \otimes \mathcal{C}$ contains a copy of the Lie superalgebra $\mathfrak{osp}(1|2)$ generated by the (odd) elements \underline{D} and \underline{x} satisfying the relations*

$$\begin{aligned} \{\underline{x}, \underline{x}\} &= \epsilon 2|x|^2 & \{\underline{D}, \underline{D}\} &= \epsilon 2\Delta & \{\underline{x}, \underline{D}\} &= \epsilon 2\underline{\mathbb{E}} \\ [|\underline{x}|^2, \underline{x}] &= 0 & [|\underline{x}|^2, \underline{D}] &= -2\underline{D} & [\underline{\mathbb{E}}, \underline{x}] &= \underline{x} \\ [\Delta, \underline{x}] &= 2\underline{x} & [\Delta, \underline{D}] &= 0 & [\underline{\mathbb{E}}, \underline{D}] &= -\underline{D} \end{aligned}$$

and containing as an even subalgebra the Lie algebra $\mathfrak{sl}(2)$ in the algebra \mathcal{A} given by Theorem 2.2 with relations

$$\left[\mathbb{E}, \frac{|x|^2}{2}\right] = |x|^2 \quad \left[\mathbb{E}, -\frac{\Delta}{2}\right] = -\Delta \quad \left[\frac{|x|^2}{2}, -\frac{\Delta}{2}\right] = \mathbb{E}.$$

Proof. The relations follow by straightforward computations. By means of the anticommutation relations (3.1) one finds that

$$\begin{aligned} \{\underline{x}, \underline{D}\} &= \sum_{i=1}^N x_i e_i \sum_{j=1}^N p_j e_j + \sum_{j=1}^N p_j e_j \sum_{i=1}^N x_i e_i \\ &= \sum_{i=1}^N \epsilon(x_i p_i + p_i x_i) + \sum_{1 \leq i < j \leq N} (x_i p_j - p_j x_i - x_j p_i + p_i x_j) e_i e_j. \end{aligned}$$

Looking back at (2.2), the first summation is precisely $\epsilon 2\mathbb{E}$, while the second summation vanishes by relation (2.1). Moreover, by relation (2.1) we have

$$\begin{aligned} [\mathbb{E}, \underline{D}] &= \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \{p_i, x_i\}, p_j e_j = \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \{p_i, [x_i, p_j]\} e_j \\ &= \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \{p_i, [x_j, p_i]\} e_j = -\frac{1}{2} \sum_{j=1}^N \left[\sum_{i=1}^N p_i^2, x_j \right] e_j = -\underline{D}, \end{aligned}$$

and in the same manner $[\mathbb{E}, \underline{x}] = \underline{x}$. \square

3.1 Symmetries

We wish to determine symmetries in the algebra $\mathcal{A} \otimes \mathcal{C}$ for the Dirac operator \underline{D} which are linear in both x_1, \dots, x_N and p_1, \dots, p_N . Given the Lie superalgebra framework, it is natural to consider operators which either commute or anticommute with \underline{D} . Indeed, the Lie superalgebra $\mathfrak{osp}(1|2)$ has both a Scasimir and a Casimir element in its universal enveloping algebra [1]. The Scasimir operator

$$\mathcal{S} = \frac{1}{2} ([\underline{D}, \underline{x}] - \epsilon) \in \mathcal{U}(\mathfrak{osp}(1|2)) \subset \mathcal{A} \otimes \mathcal{C}, \quad (3.3)$$

anticommutes with odd elements and commutes with even elements. In the classical case, the Scasimir operator is up to a constant term equal to the angular Dirac operator Γ , i.e. \underline{D} restricted to the sphere. The Scasimir \mathcal{S} is a symmetry which is linear in both x_1, \dots, x_N and p_1, \dots, p_N and we will get back to it before the end of this subsection. Finally, the square of the Scasimir element yields the Casimir element $\mathcal{C} = \mathcal{S}^2$, which commutes with all elements of $\mathfrak{osp}(1|2)$.

Note that another symmetry is obtained by means of the anticommutation relations (3.1) of the Clifford algebra. The so-called pseudo-scalar $e_1 \cdots e_N$ is

easily seen to commute with \underline{D} for N odd and anticommute with \underline{D} for N even.

The Dirac operator is defined such that it squares to the Laplace operator, $\underline{D}^2 = \epsilon \Delta$. This allows us to readily make use of the properties of Δ by means of the following straightforward relations. For an operator Z , we have that

$$[\underline{D}, \{\underline{D}, Z\}] = \underline{D}(\underline{D}Z + Z\underline{D}) - (\underline{D}Z + Z\underline{D})\underline{D} = [\underline{D}^2, Z] \quad (3.4)$$

and

$$\{\underline{D}, [\underline{D}, Z]\} = \underline{D}(\underline{D}Z - Z\underline{D}) + (\underline{D}Z - Z\underline{D})\underline{D} = [\underline{D}^2, Z]. \quad (3.5)$$

A direct consequence of these relations is that every symmetry of the Laplace operator Δ yields symmetries of the Dirac operator \underline{D} .

Proposition 3.5. *If Z commutes with Δ and $\underline{D}^2 = \epsilon \Delta$, then the operator $\{\underline{D}, Z\}$ commutes with \underline{D} , while the operator $[\underline{D}, Z]$ anticommutes with \underline{D} .*

Letting Z be one of the symmetries of Theorem 2.3, we indeed obtain symmetries of \underline{D} , but they are not of the same order in x_1, \dots, x_N as in p_1, \dots, p_N . These symmetries are in fact combinations of the obvious symmetries p_1, \dots, p_N and symmetries which are linear in x_1, \dots, x_N and in p_1, \dots, p_N . We set forth to determine the latter explicitly. Hereto, a first observation is that the elements of the Clifford algebra also commute with the Laplace operator, by definition as it is a scalar (non-Clifford) operator. For A a list of distinct elements of $\{1, \dots, N\}$, we have

$$\underline{D}(\underline{D}e_A \pm e_A \underline{D}) \mp (\underline{D}e_A \pm e_A \underline{D})\underline{D} = [\underline{D}^2, e_A] = 0.$$

The explicit expression of these operators follow from the anticommutation relations (3.1) as

$$e_A \underline{D} = e_A \sum_{l=1}^N p_l e_l = (-1)^{|A|-1} \sum_{a \in A} p_a e_a e_A + (-1)^{|A|} \sum_{a \notin A} p_a e_a e_A, \quad (3.6)$$

with $|A|$ denoting the number of elements of the list A , so

$$\underline{D}e_A - (-1)^{|A|} e_A \underline{D} = \sum_{a \in A} 2p_a e_a e_A \quad \text{and} \quad \underline{D}e_A + (-1)^{|A|} e_A \underline{D} = \sum_{a \notin A} 2p_a e_a e_A, \quad (3.7)$$

where (here and throughout the paper) the summation index $a \notin A$ is meant to run over all elements of $\{1, \dots, N\} \setminus A$. Note that for a list of one element $A = i$, we have $\underline{D}e_i + e_i \underline{D} = \epsilon 2p_i$. With this information, the relations (3.4) and (3.5) now also lend themselves to the construction of more intricate symmetries of both \underline{D} and \underline{x} .

Theorem 3.6. *In the algebra $\mathcal{A} \otimes \mathcal{C}$, for $i \in \{1, \dots, N\}$, the operator*

$$O_i = \frac{\epsilon}{2} ([\underline{D}, x_i] - e_i) = \frac{\epsilon}{2} ([p_i, \underline{x}] - e_i) = \frac{\epsilon}{2} \left(\sum_{l=1}^N e_l C_{li} - e_i \right) \quad (3.8)$$

anticommutes with \underline{D} and \underline{x} .

Proof. The equalities in (3.8) follow immediately from relation (2.1), that is $C_{ij} = [p_i, x_j] = [p_j, x_i] = C_{ji}$. By direct computation, using (3.5) and the anticommutation relations (3.1), we have

$$\{\underline{D}, O_i\} = \frac{\epsilon}{2} \{\underline{D}, [\underline{D}, x_i]\} - \epsilon \frac{1}{2} \{\underline{D}, e_i\} = \frac{1}{2} [\underline{D}, x_i] - p_i = 0.$$

In the same manner, one finds that $O_i = \frac{\epsilon}{2} ([p_i, \underline{x}] - e_i)$ anticommutes with \underline{x} . \square

The symmetries O_i with one index $i \in \{1, \dots, N\}$ defined in (3.8) can be generalized to symmetries with multiple indices. Hereto, we define the operators

$$\underline{D}_A = \sum_{a \in A} p_a e_a \quad \text{and} \quad \underline{x}_A = \sum_{a \in A} x_a e_a, \quad (3.9)$$

for A a subset of $\{1, \dots, N\}$, and by extension for A a list of $\{1, \dots, N\}$ as the order does not matter in the summation. By means of the operators (3.9) we state the following result.

Theorem 3.7. *In the algebra $\mathcal{A} \otimes \mathcal{C}$, for A a list of distinct elements of $\{1, \dots, N\}$, the operator*

$$O_A = \frac{1}{2} (\underline{D} \underline{x}_A e_A - e_A \underline{x}_A \underline{D} - \epsilon e_A) \quad (3.10)$$

$$= \frac{1}{2} (e_A \underline{D}_A \underline{x} - \underline{x} \underline{D}_A e_A - \epsilon e_A) \quad (3.11)$$

$$= \frac{1}{2} \left(-\epsilon + \sum_{j \in A} \sum_{i \notin A \setminus \{j\}} C_{ij} e_i e_j - \sum_{\{i,j\} \subset A} 2L_{ij} e_i e_j \right) e_A \quad (3.12)$$

satisfies

$$\underline{D} O_A = (-1)^{|A|} O_A \underline{D} \quad \text{and} \quad \underline{x} O_A = (-1)^{|A|} O_A \underline{x}.$$

Proof. We first show the equivalence of the three expressions (3.10)–(3.12). The common term $-(\epsilon/2) e_A$ will be left out in the following manipulations.

Starting from (3.10), up to a factor $1/2$, and using $C_{ij} = C_{ji}$, we have

$$\begin{aligned} \underline{D} \underline{x}_A e_A - e_A \underline{x}_A \underline{D} &= \sum_{l=1}^N p_l e_l \sum_{a \in A} x_a e_a e_A - e_A \sum_{a \in A} x_a e_a \sum_{l=1}^N p_l e_l \\ &= \sum_{a \in A} \left(\epsilon (p_a x_a - x_a p_a) + \sum_{l \in A \setminus \{a\}} (p_l x_a + x_a p_l) e_l e_a + \sum_{l \notin A} (p_l x_a - x_a p_l) e_l e_a \right) e_A \\ &= \sum_{a \in A} \left(\epsilon (p_a x_a - x_a p_a) + \sum_{l \in A \setminus \{a\}} (p_a x_l + x_l p_a) e_a e_l + \sum_{l \notin A} (p_a x_l - x_l p_a) e_l e_a \right) e_A \\ &= e_A \sum_{a \in A} p_a e_a \sum_{l=1}^N x_l e_l - \sum_{l=1}^N x_l e_l \sum_{a \in A} p_a e_a e_A = e_A \underline{D}_A \underline{x} - \underline{x} \underline{D}_A e_A. \end{aligned}$$

Again starting from (3.10), up to a factor $1/2$, we have

$$\begin{aligned}
\frac{D}{2} \underline{x}_A e_A - e_A \underline{x}_A \frac{D}{2} &= \sum_{l=1}^N p_l e_l \sum_{a \in A} x_a e_a e_A - e_A \sum_{a \in A} x_a e_a \sum_{l=1}^N p_l e_l \\
&= \left(\sum_{l=1}^N p_l e_l \sum_{a \in A} x_a e_a - \sum_{a \in A} x_a e_a \sum_{l \in A} p_l e_l + \sum_{a \in A} x_a e_a \sum_{l \notin A} p_l e_l \right) e_A \\
&= \left(\epsilon \sum_{a \in A} (p_a x_a - x_a p_a) + \sum_{a \in A} \sum_{l \notin A} (p_l x_a - x_a p_l) e_l e_a \right. \\
&\quad \left. + \sum_{a \in A} \sum_{l \in A \setminus \{a\}} (p_l x_a + x_a p_l) e_l e_a \right) e_A \\
&= \left(\sum_{a \in A} \sum_{l \notin A \setminus \{a\}} C_{la} e_l e_a + \sum_{\{a, l\} \subset A} ((p_l x_a + x_a p_l) e_l e_a + (p_a x_l + x_l p_a) e_a e_l) \right) e_A
\end{aligned}$$

which, when using $L_{ij} = x_i p_j - x_j p_i = p_j x_i - p_i x_j$ and $C_{ij} = C_{ji}$, equals (3.12), up to a factor $1/2$.

Now for the actual proof, the case where A is the empty set is trivial, as $O_\emptyset = -\epsilon/2$ obviously commutes with \underline{D} and \underline{x} . For A a singleton the result is given by Theorem 3.6 so let now $|A| \geq 2$. Using $\underline{x}_A e_A = (-1)^{|A|-1} e_A \underline{x}_A$ and (3.6), we have

$$\begin{aligned}
&\frac{D}{2} O_A - (-1)^{|A|} O_A \frac{D}{2} \\
&= \frac{1}{2} \underline{D} (\underline{D} \underline{x}_A e_A - e_A \underline{x}_A \underline{D} - \epsilon e_A) - (-1)^{|A|} \frac{1}{2} (\underline{D} \underline{x}_A e_A - e_A \underline{x}_A \underline{D} - \epsilon e_A) \underline{D} \\
&= \frac{1}{2} (\underline{D}^2 \underline{x}_A e_A - \underline{x}_A e_A \underline{D}^2) - \frac{\epsilon}{2} (\underline{D} e_A - (-1)^{|A|} e_A \underline{D}) \\
&= \frac{\epsilon}{2} \sum_{a \in A} [\Delta, x_a] e_a e_A - \epsilon \sum_{a \in A} p_a e_a e_A,
\end{aligned}$$

which vanishes because of condition (1.7). In the same manner, using now the form (3.11) for O_A , the expression $\underline{x} O_A - (-1)^{|A|} O_A \underline{x}$ vanishes. \square

Remark 3.8. Note that if the order of the list A is altered, O_A changes but only in sign. Say π is a permutation on the list A , we have $O_A = \text{sign}(\pi) O_{\pi(A)}$, where $\text{sign}(\pi)$ is positive for an even permutation π and negative for an odd one. Hence up to a sign all the symmetries of this form are given by $\{O_A \mid A \subset \{1, \dots, N\}\}$ where the elements of A are in ascending order in accordance with the standard order for natural numbers.

For the special case where $A = \{1, \dots, N\}$, the operator (3.10) is seen to correspond precisely to the Scasimir element (3.3) of $\mathfrak{osp}(1|2)$ multiplied by the pseudo-scalar

$$O_{1\dots N} = \frac{1}{2} ([\underline{D}, \underline{x}] - \epsilon) e_1 \cdots e_N.$$

For a list A of $\{1, \dots, N\}$, the operator O_A either commutes or anticommutes with \underline{D} and \underline{x} . The subsequent corollary is useful if one is interested solely in commuting symmetries.

Corollary 3.9. *In the algebra $\mathcal{A} \otimes \mathcal{C}$, for A a list of distinct elements of $\{1, \dots, N\}$, we have*

$$\left[\underline{D}, O_A \prod_{i \in A} O_i \right] = 0, \quad \left[\underline{x}, O_A \prod_{i \in A} O_i \right] = 0.$$

Note that the order of the product matters in general, but not for the result.

Proof. Follows immediately from Theorem 3.6 and Theorem 3.7. \square

Expression (3.12) shows that the symmetries O_A are constructed using the symmetries C_{ij} and L_{ij} from the previous section, together with the Clifford algebra generators e_1, \dots, e_N . The factor $1/2$ is chosen such that for a list of $\{1, \dots, N\}$ consisting of just two distinct elements i and j , the symmetry O_{ij} corresponds to the generalized angular momentum symmetry L_{ij} , up to additive terms. Indeed, we have that

$$O_{ij} = L_{ij} - \epsilon \frac{1}{2} e_i e_j + \epsilon \frac{1}{2} \sum_{l \neq j} C_{li} e_l e_j - \epsilon \frac{1}{2} \sum_{l \neq i} C_{lj} e_l e_i.$$

This can be written more compactly by means of the explicit expression (3.8) for O_i as

$$O_{ij} = L_{ij} + \epsilon \frac{1}{2} e_i e_j + O_i e_j - O_j e_i \quad (3.13)$$

Together with the Clifford algebra generators e_1, \dots, e_N , the symmetries O_i with one index and O_{ij} with two indices in fact suffice to build up all other symmetries O_A . Indeed, plugging in the expression (3.8) for O_i , one easily verifies that

$$O_A = \left(\epsilon \frac{|A| - 1}{2} + \epsilon \sum_{i \in A} O_i e_i - \sum_{\{i, j\} \subset A} L_{ij} e_i e_j \right) e_A \quad (3.14)$$

reduces to (3.12). Using now the form (3.13) to substitute L_{ij} , the operator O_A can also be written as

$$O_A = \left(-\epsilon \frac{(|A| - 1)(|A| - 2)}{4} - \epsilon(|A| - 2) \sum_{i \in A} O_i e_i - \sum_{\{i, j\} \subset A} O_{ij} e_i e_j \right) e_A. \quad (3.15)$$

Finally, the symmetries can also be constructed recursively. If we denote by $A \setminus \{a\}$ the list A with the element a omitted, and we define $\text{sign}(A, a)$ such that $\text{sign}(A, a) e_{A \setminus \{a\}} e_a = e_A$, then it follows that

$$\sum_{a \in A} \text{sign}(A, a) O_{A \setminus \{a\}} e_a = \sum_{a \in A} \left(\epsilon \frac{|A| - 2}{2} + \epsilon \sum_{i \in A \setminus \{a\}} O_i e_i - \sum_{\{i, j\} \subset A \setminus \{a\}} L_{ij} e_i e_j \right) e_A$$

$$\begin{aligned}
&= \left(\epsilon |A| \frac{|A|-2}{2} + \epsilon (|A|-1) \sum_{i \in A} O_i e_i - (|A|-2) \sum_{\{i,j\} \subset A} L_{ij} e_i e_j \right) e_A \\
&= \epsilon \frac{|A|-2}{2} e_A + \epsilon \sum_{i \in A} O_i e_i e_A + (|A|-2) O_A.
\end{aligned}$$

Using this relation, Theorem 3.7 can also be proved by induction on the cardinality of A , starting from $|A| = 3$.

3.2 Symmetry algebra

Before establishing the algebraic structure generated by the symmetries O_A , we first introduce some helpful relations with Clifford numbers. From the definition (3.8) of O_j we have

$$\{e_i, O_j\} = e_i O_j + O_j e_i = [p_i, x_j] - \delta_{ij}.$$

The property $[p_i, x_j] = [p_j, x_i]$ then implies that $\{e_i, O_j\} = \{e_j, O_i\}$, or by a reordering of terms $O_i e_j - O_j e_i = e_i O_j - e_j O_i$. This is in fact a special case of the following useful result.

Lemma 3.10. *In the algebra $\mathcal{A} \otimes \mathcal{C}$, for A a list of distinct elements of $\{1, \dots, N\}$, we have*

$$\sum_{a \in A} O_a e_a e_A = e_A \sum_{a \in A} e_a O_a.$$

Proof. The identity follows by direct calculation using the definition (3.8) of O_a and the commutation relations of e_1, \dots, e_N :

$$\begin{aligned}
\sum_{a \in A} O_a e_a e_A &= \sum_{a \in A} \epsilon \frac{1}{2} \sum_{l=1}^N [p_l, x_a] e_l e_a e_A - \sum_{a \in A} \epsilon \frac{1}{2} e_a e_a e_A \\
&= \sum_{a \in A} \epsilon \frac{1}{2} \sum_{l \in A} [p_l, x_a] e_A e_l e_a - \sum_{a \in A} \epsilon \frac{1}{2} \sum_{l \notin A} [p_l, x_a] e_A e_l e_a - \sum_{a \in A} \frac{1}{2} e_A \\
&= \sum_{a \in A} \epsilon \frac{1}{2} \sum_{l \in A} [p_a, x_l] e_A e_a e_l + \sum_{a \in A} \epsilon \frac{1}{2} \sum_{l \notin A} [p_l, x_a] e_A e_a e_l - \sum_{a \in A} \epsilon \frac{1}{2} e_A e_a e_a \\
&= e_A \sum_{a \in A} e_a O_a. \quad \square
\end{aligned}$$

Note that by means of this lemma, the symmetry O_A , in the form (3.14), can equivalently be written with e_A in front, that is

$$O_A = e_A \left(\epsilon \frac{|A|-1}{2} + \epsilon \sum_{i \in A} e_i O_i - \sum_{\{i,j\} \subset A} L_{ij} e_i e_j \right).$$

3.2.1 Relations for symmetries with one or two indices

Next, we present some relations which hold for symmetries with one or two indices.

Theorem 3.11. *In the algebra $\mathcal{A} \otimes \mathcal{C}$, for $i, j, k \in \{1, \dots, N\}$ we have*

$$[O_{ij}, O_k] + [O_{jk}, O_i] + [O_{ki}, O_j] = 0.$$

Proof. If any of the indices are equal, the identity becomes trivial as $O_{ij} = -O_{ji}$. For distinct i, j, k , we have by (3.8), (3.13) and using $O_i e_j - O_j e_i = e_i O_j - e_j O_i$

$$\begin{aligned} & [O_{ij}, O_k] + [O_{jk}, O_i] + [O_{ki}, O_j] \\ &= \epsilon \frac{1}{2} \sum_{l=1}^N e_l ([L_{ij}, C_{lk}] + [L_{jk}, C_{li}] + [L_{ki}, C_{lj}]) \\ & \quad + \epsilon \frac{1}{2} ([e_i e_j, O_k] + [e_j e_k, O_i] + [e_k e_i, O_j]) \\ & \quad + (O_i e_j - O_j e_i) O_k - O_k (e_i O_j - e_j O_i) + (O_j e_k - O_k e_j) O_i \\ & \quad - O_i (e_j O_k - e_k O_j) + (O_k e_i - O_i e_k) O_j - O_j (e_k O_i - e_i O_k). \end{aligned}$$

The first line of the right-hand side vanishes by Theorem 2.5, while the second line does so by direct calculation plugging in the definition of O_i and using $C_{ij} = C_{ji}$. Finally, the remaining terms of the last two lines cancel out pairwise. \square

For the next result, we first write out the form (3.14) of O_A for A a list of three elements, say i, j, k which are all distinct:

$$O_{ijk} = \epsilon e_i e_j e_k + O_i e_j e_k - O_j e_i e_k + O_k e_i e_j + L_{ij} e_k - L_{ik} e_j + L_{jk} e_i. \quad (3.16)$$

The commutation relations for symmetries with two indices are as follows.

Theorem 3.12. *In the algebra $\mathcal{A} \otimes \mathcal{C}$, for $i, j, k, l \in \{1, \dots, N\}$ the symmetries satisfy*

$$\begin{aligned} [O_{ij}, O_{kl}] &= (O_{il} + \epsilon [O_i, O_l]) \delta_{jk} + (O_{jk} + \epsilon [O_j, O_k]) \delta_{il} \\ & \quad + (O_{ki} + \epsilon [O_k, O_i]) \delta_{lj} + (O_{lj} + \epsilon [O_l, O_j]) \delta_{ki} \\ & \quad + \frac{1}{2} (\{O_i, O_{jkl}\} - \{O_j, O_{ikl}\} - \{O_{ijl}, O_k\} + \{O_{ijk}, O_l\}). \end{aligned}$$

Proof. For the cases where $i = j$ or $k = l$ or $\{i, j\} = \{k, l\}$, both sides of the equation reduce to zero, so from now on we assume that $i \neq j$ and $k \neq l$ and $\{i, j\} \neq \{k, l\}$. Plugging in (3.13) we have

$$[O_{ij}, O_{kl}] = \left[L_{ij} + \epsilon \frac{1}{2} e_i e_j + O_i e_j - O_j e_i, L_{kl} + \epsilon \frac{1}{2} e_k e_l + O_k e_l - O_l e_k \right]$$

$$\begin{aligned}
&= [L_{ij}, L_{kl}] + [L_{ij}, O_k e_l - O_l e_k] + [O_i e_j - O_j e_i, L_{kl}] \\
&\quad + [O_i e_j - O_j e_i, O_k e_l - O_l e_k] + \frac{1}{4} [e_i e_j, e_k e_l] + \epsilon \frac{1}{2} [e_i e_j, O_k e_l] \\
&\quad + \epsilon \frac{1}{2} [e_i e_j, -O_l e_k] + \epsilon \frac{1}{2} [O_i e_j, e_k e_l] + \epsilon \frac{1}{2} [-O_j e_i, e_k e_l].
\end{aligned}$$

By Theorem 2.5, and using $\{e_i, O_j\} = [p_i, x_j] - \delta_{ij}$, we have

$$\begin{aligned}
[L_{ij}, L_{kl}] &= L_{il} \delta_{jk} + L_{jk} \delta_{il} + L_{ki} \delta_{lj} + L_{lj} \delta_{ki} \\
&\quad + L_{il} \{e_j, O_k\} + L_{jk} \{e_i, O_l\} + L_{ki} \{e_l, O_j\} + L_{lj} \{e_k, O_i\}.
\end{aligned}$$

Using $\{e_i, O_j\} = \{e_j, O_i\} = 1/2\{e_i, O_j\} + 1/2\{e_j, O_i\}$, the terms in the last line can be rewritten as

$$\begin{aligned}
&\frac{1}{2}((L_{il} e_j + L_{lj} e_i) O_k + (L_{jk} e_i + L_{ki} e_j) O_l + (L_{lj} e_k + L_{jk} e_l) O_i \\
&\quad + (L_{ki} e_l + L_{il} e_k) O_j + (L_{jk} O_l + L_{lj} O_k) e_i + (L_{il} O_k + L_{ki} O_l) e_j \\
&\quad + (L_{lj} O_i + L_{il} O_j) e_k + (L_{ki} O_j + L_{jk} O_i) e_l).
\end{aligned}$$

Together with

$$\begin{aligned}
[L_{ij}, O_k e_l - O_l e_k] &= [L_{ij}, O_k] e_l - [L_{ij}, O_l] e_k = \{L_{ij} e_k, O_l\} - \{L_{ij} e_l, O_k\} \\
[O_i e_j - O_j e_i, L_{kl}] &= [O_i, L_{kl}] e_j - [O_j, L_{kl}] e_i = \{L_{kl} e_j, O_i\} - \{L_{kl} e_i, O_j\},
\end{aligned}$$

we find that $[L_{ij}, L_{kl}] + [L_{ij}, O_k e_l - O_l e_k] + [O_i e_j - O_j e_i, L_{kl}]$ equals

$$\begin{aligned}
&L_{il} \delta_{jk} + L_{jk} \delta_{il} + L_{ki} \delta_{lj} + L_{lj} \delta_{ki} \\
&\quad + \frac{1}{2}((L_{il} e_j + L_{lj} e_i) O_k + \{-L_{ij} e_l, O_k\} + (L_{jk} e_i + L_{ki} e_j) O_l + \{L_{ij} e_k, O_l\} \\
&\quad + (L_{lj} e_k + L_{jk} e_l) O_i + \{O_i, L_{kl} e_j\} + (L_{ki} e_l + L_{il} e_k) O_j + \{O_j, -L_{kl} e_i\}) \\
&\quad + \frac{1}{2}((L_{jk} O_l + L_{lj} O_k + [L_{kl}, O_j]) e_i + (L_{il} O_k + L_{ki} O_l - [L_{kl}, O_i]) e_j \\
&\quad + (L_{lj} O_i + L_{il} O_j - [L_{ij}, O_l]) e_k + (L_{ki} O_j + L_{jk} O_i + [L_{ij}, O_k]) e_l).
\end{aligned}$$

This simplifies to

$$\begin{aligned}
&L_{il} \delta_{jk} + L_{jk} \delta_{il} + L_{ki} \delta_{lj} + L_{lj} \delta_{ki} \\
&\quad + \frac{1}{2}(\{L_{il} e_j + L_{lj} e_i + L_{ji} e_l, O_k\} + \{L_{jk} e_i + L_{ki} e_j + L_{ij} e_k, O_l\} \\
&\quad + \{O_i, L_{lj} e_k + L_{jk} e_l + L_{kl} e_j\} + \{O_j, L_{ki} e_l + L_{il} e_k + L_{lk} e_i\}),
\end{aligned}$$

by means of

$$\begin{aligned}
&([L_{jk}, O_l] + [L_{lj}, O_k] + [L_{kl}, O_j]) e_i + ([L_{il}, O_k] + [L_{ki}, O_l] - [L_{kl}, O_i]) e_j \\
&\quad + ([L_{lj}, O_i] + [L_{il}, O_j] - [L_{ij}, O_l]) e_k + ([L_{ki}, O_j] + [L_{jk}, O_i] + [L_{ij}, O_k]) e_l = 0,
\end{aligned}$$

which is a direct consequence of Theorem 2.5 after plugging in the definition (3.8) of O_i .

Now, the other terms appearing in $[O_{ij}, O_{kl}]$ can be expanded as follows. First,

$$[e_i e_j, e_k e_l] = 2\epsilon(\delta_{jk} e_i e_l - \delta_{il} e_k e_j - \delta_{lj} e_i e_k + \delta_{ik} e_l e_j).$$

Moreover, we have

$$\begin{aligned} [e_i e_j, O_k e_l] &= e_i e_j O_k e_l - O_k e_l e_i e_j \\ &= e_i e_j \{e_l, O_k\} - e_i e_j e_l O_k + O_k e_i e_l e_j - O_k (\epsilon 2 \delta_{il}) e_j \\ &= e_i e_j \{e_l, O_k\} - e_i e_j e_l O_k - O_k e_i e_j e_l + O_k e_i (\epsilon 2 \delta_{jl}) - O_k (\epsilon 2 \delta_{il}) e_j. \end{aligned}$$

As $\{e_k, O_l\} = \{e_l, O_k\}$, after interchanging k and l in this result, subtraction yields the following for $[e_i e_j, O_k e_l - O_l e_k]$

$$\delta_{jl} \epsilon 2 O_k e_i - \delta_{il} \epsilon 2 O_k e_j - \delta_{jk} \epsilon 2 O_l e_i + \delta_{ik} \epsilon 2 O_l e_j + \{e_i e_j e_k, O_l\} - \{e_i e_j e_l, O_k\}.$$

For the last term, $[O_i e_j - O_j e_i, O_k e_l - O_l e_k]$, we use Lemma 3.10 to find

$$\begin{aligned} &2[O_i e_j - O_j e_i, O_k e_l - O_l e_k] \\ &= 2(O_i e_j - O_j e_i)(e_k O_l - e_l O_k) - 2(O_k e_l - O_l e_k)(e_i O_j - e_j O_i) \\ &= (O_k e_l e_j - O_l e_k e_j) O_i + O_i (e_j e_k O_l - e_j e_l O_k) \\ &\quad + (-O_k e_l e_i + O_l e_k e_i) O_j + O_j (-e_i e_k O_l + e_i e_l O_k) \\ &\quad + (-O_i e_j e_l + O_j e_i e_l) O_k + O_k (-e_l e_i O_j + e_l e_j O_i) \\ &\quad + (O_i e_j e_k - O_j e_i e_k) O_l + O_l (e_k e_i O_j - e_k e_j O_i). \end{aligned}$$

We first consider the case where one of i, j is equal to either k or l , for instance say $i = l$:

$$\begin{aligned} &(O_k e_i e_j - O_i e_k e_j) O_i + O_i (e_j e_k O_i - e_j e_i O_k) \\ &\quad + (-O_k \epsilon + O_i e_k e_i) O_j + O_j (-e_i e_k O_i + \epsilon O_k) \\ &\quad + (-O_i e_j e_i + O_j \epsilon) O_k + O_k (-\epsilon O_j + e_i e_j O_i) \\ &\quad + (O_i e_j e_k - O_j e_i e_k) O_i + O_i (e_k e_i O_j - e_k e_j O_i) \\ &= (O_k e_i e_j - O_i e_k e_j - O_j e_i e_k + O_i e_j e_k - O_j e_i e_k + O_k e_i e_j) O_i \\ &\quad + O_i (e_j e_k O_i - e_j e_i O_k + e_k e_i O_j + e_k e_i O_j - e_k e_j O_i - e_j e_i O_k) + \epsilon 2[O_j, O_k] \end{aligned}$$

Using Lemma 3.10 we find

$$\begin{aligned} &= (O_k e_i e_j - O_i e_k e_j - O_j e_i e_k + O_i e_j e_k - O_j e_i e_k + O_k e_i e_j) O_i \\ &\quad + O_i (O_i e_j e_k - O_k e_j e_i + O_j e_k e_i + O_j e_k e_i - O_i e_k e_j - O_k e_j e_i) + \epsilon 2[O_j, O_k] \\ &= 2\{O_k e_i e_j - O_j e_i e_k + O_i e_j e_k, O_i\} + \epsilon 2[O_j, O_k]. \end{aligned}$$

Finally, when i, j, k, l are all distinct, $2[O_i e_j - O_j e_i, O_k e_l - O_l e_k]$ equals

$$\begin{aligned} &(O_k e_l e_j - O_l e_k e_j) O_i + O_i (-e_k e_j O_l + e_l e_j O_k + e_k e_l O_j - e_k e_l O_j) \\ &\quad + (-O_k e_l e_i + O_l e_k e_i) O_j + O_j (e_k e_i O_l - e_l e_i O_k + e_l e_k O_i - e_l e_k O_i) \\ &\quad + (-O_i e_j e_l + O_j e_i e_l) O_k + O_k (e_i e_l O_j - e_j e_l O_i + e_j e_i O_l - e_j e_i O_l) \\ &\quad + (O_i e_j e_k - O_j e_i e_k) O_l + O_l (-e_i e_k O_j + e_j e_k O_i + e_i e_j O_k - e_i e_j O_k) \end{aligned}$$

$$\begin{aligned}
&= (O_k e_l e_j - O_l e_k e_j - O_j e_l e_k) O_i + O_i (-O_l e_k e_j + O_k e_l e_j + O_j e_k e_l) \\
&\quad + (-O_k e_l e_i + O_l e_k e_i - O_i e_k e_l) O_j + O_j (O_l e_k e_i - O_k e_l e_i + O_i e_l e_k) \\
&\quad + (-O_i e_j e_l + O_j e_i e_l - O_l e_i e_j) O_k + O_k (O_j e_i e_l - O_i e_j e_l + O_l e_j e_i) \\
&\quad + (O_i e_j e_k - O_j e_i e_k - O_k e_j e_i) O_l + O_l (-O_j e_i e_k + O_i e_j e_k + O_k e_i e_j) \\
&= \{O_k e_l e_j - O_l e_k e_j + O_j e_k e_l, O_i\} + \{O_l e_k e_i - O_k e_l e_i + O_i e_l e_k, O_j\} \\
&\quad + \{O_j e_i e_l - O_i e_j e_l + O_l e_j e_i, O_k\} + \{O_i e_j e_k - O_j e_i e_k + O_k e_i e_j, O_l\}.
\end{aligned}$$

This boils down to $[O_i e_j - O_j e_i, O_k e_l - O_l e_k]$ being equal to

$$\begin{aligned}
&\epsilon \delta_{jk} [O_i, O_l] + \epsilon \delta_{il} [O_j, O_k] + \epsilon \delta_{lj} [O_k, O_i] + \epsilon \delta_{ki} [O_l, O_j] \\
&\quad + \frac{1}{2} (\{O_k e_l e_j - O_l e_k e_j + O_j e_k e_l, O_i\} + \{O_l e_k e_i - O_k e_l e_i + O_i e_l e_k, O_j\} \\
&\quad + \{O_j e_i e_l - O_i e_j e_l + O_l e_j e_i, O_k\} + \{O_i e_j e_k - O_j e_i e_k + O_k e_i e_j, O_l\}).
\end{aligned}$$

Hence, combining all of the above, we find

$$\begin{aligned}
[O_{ij}, O_{kl}] &= L_{il} \delta_{jk} + L_{jk} \delta_{il} + L_{ki} \delta_{lj} + L_{lj} \delta_{ki} + \frac{1}{2} (\{L_{il} e_j + L_{lj} e_i + L_{ji} e_l, O_k\} \\
&\quad + \{L_{jk} e_i + L_{ki} e_j + L_{ij} e_k, O_l\} + \{L_{ki} e_l + L_{il} e_k + L_{lk} e_i, O_j\} \\
&\quad + \{L_{lj} e_k + L_{jk} e_l + L_{kl} e_j, O_i\}) + \epsilon \frac{1}{2} (\delta_{jk} e_i e_l - \delta_{il} e_k e_j - \delta_{lj} e_i e_k + \delta_{ik} e_l e_j) \\
&\quad + \epsilon \delta_{jk} [O_i, O_l] + \epsilon \delta_{il} [O_j, O_k] + \epsilon \delta_{lj} [O_k, O_i] + \epsilon \delta_{ki} [O_l, O_j] \\
&\quad + \frac{1}{2} (\{O_k e_l e_j - O_l e_k e_j + O_j e_k e_l, O_i\} + \{O_l e_k e_i - O_k e_l e_i + O_i e_l e_k, O_j\} \\
&\quad + \{O_j e_i e_l - O_i e_j e_l + O_l e_j e_i, O_k\} + \{O_i e_j e_k - O_j e_i e_k + O_k e_i e_j, O_l\}) \\
&\quad + \epsilon \frac{1}{2} (\delta_{jl} \epsilon 2 O_k e_i - \delta_{il} \epsilon 2 O_k e_j - \{e_i e_j e_l, O_k\} - \delta_{jk} \epsilon 2 O_l e_i + \delta_{ik} \epsilon 2 O_l e_j \\
&\quad + \{e_i e_j e_k, O_l\}) - \epsilon \frac{1}{2} (\delta_{lj} \epsilon 2 O_i e_k - \delta_{kj} \epsilon 2 O_i e_l - \{e_k e_l e_j, O_i\} - \delta_{li} \epsilon 2 O_j e_k \\
&\quad + \delta_{ki} \epsilon 2 O_j e_l + \{e_k e_l e_i, O_j\}).
\end{aligned}$$

Collecting the appropriate terms, we recognize all ingredients to make symmetries with three indices (3.16) and we arrive at the desired result

$$\begin{aligned}
[O_{ij}, O_{kl}] &= (O_{il} + \epsilon [O_i, O_l]) \delta_{jk} + (O_{jk} + \epsilon [O_j, O_k]) \delta_{il} \\
&\quad + (O_{ki} + \epsilon [O_k, O_i]) \delta_{lj} + (O_{lj} + \epsilon [O_l, O_j]) \delta_{ki} \\
&\quad + \frac{1}{2} (\{O_i, O_{jkl}\} - \{O_j, O_{ikl}\} - \{O_{ijl}, O_k\} + \{O_{ijk}, O_l\}). \quad \square
\end{aligned}$$

In summary, for i, j, k, l all distinct elements of $\{1, \dots, N\}$ we have

$$\begin{aligned}
[O_{ij}, O_{ki}] &= O_{jk} + \epsilon [O_j, O_k] + \{O_{ijk}, O_i\} \\
[O_{ij}, O_{kl}] &= \frac{1}{2} (\{O_i, O_{jkl}\} - \{O_j, O_{ikl}\} - \{O_{ijl}, O_k\} + \{O_{ijk}, O_l\}).
\end{aligned}$$

3.2.2 Relations for symmetries with general index

Now, we are interested in relations for general symmetries with an arbitrary index list A , i.e.

$$O_A = \frac{1}{2}(\underline{D}\underline{x}_A e_A - e_A \underline{x}_A \underline{D} - \epsilon e_A).$$

Before doing so, we make a slight detour clearing up some conventions and notations. An important fact to take into account is the interaction of the appearing Clifford numbers with different lists as index (recall Definition 3.1). For instance, if A and B denote two lists of $\{1, \dots, N\}$, the following properties are readily shown to hold by direct computation:

$$e_B e_A = (-1)^{|A||B| - |A \cap B|} e_A e_B, \quad (3.17)$$

and

$$(e_A)^2 = \epsilon^{|A|} (-1)^{\frac{|A|^2 - |A|}{2}}. \quad (3.18)$$

The product $e_A e_B$ in fact reduces (up to a sign) to contain only e_i with i an index in the set $(A \cup B) \setminus (A \cap B)$, since all indices in $A \cap B$ appear twice and cancel out (for these set operations we disregard the order of the lists A and B and view them just as sets). The remaining indices are what is called the symmetric difference of the sets A and B . We will denote this associative operation by $A \triangle B = (A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$. When applied to two lists A and B , we view them as sets and the resultant $A \triangle B$ is a set. Note that $|A \triangle B| = |A| + |B| - 2|A \cap B|$.

When dealing with interactions between symmetries O_A and O_B for two lists A and B , products of the kind $e_A e_B$ are exactly what we encounter. Not wanting to overburden notations, but still taking into account all resulting signs due to the anticommutation relations (3.1) if one were to work out the reduction $e_A e_B$, we introduce the following notation

$$O_{A,B} = \frac{1}{2}(\underline{D}\underline{x}_{A \triangle B} e_A e_B - e_A e_B \underline{x}_{A \triangle B} \underline{D} - \epsilon e_A e_B). \quad (3.19)$$

Note that the order of A and B matters, as by (3.17) we have $O_{A,B} = (-1)^{|A||B| - |A \cap B|} O_{B,A}$. Moreover, up to a sign $O_{A,B}$ is equal to $O_{A \triangle B}$, where the elements of $A \triangle B$ are in ascending order when used as a list, see Remark 3.2. Since the symmetric difference operation on sets is associative, one easily extends this definition to an arbitrary number of lists, e.g.

$$O_{A,B,C} = \frac{1}{2}(\underline{D}\underline{x}_{A \triangle B \triangle C} e_A e_B e_C - e_A e_B e_C \underline{x}_{A \triangle B \triangle C} \underline{D} - \epsilon e_A e_B e_C).$$

As an example, if we consider the lists $A = 234$ and $B = 31$, then the set $A \triangle B$ contains the elements 2, 4, 1 but not 3 as 3 appears in both A and B , so we have

$$O_{234,31} = \frac{1}{2}(\underline{D}\underline{x}_{\{2,4,1\}} e_{234} e_{31} - e_{234} e_{31} \underline{x}_{\{2,4,1\}} \underline{D} - \epsilon e_{234} e_{31})$$

$$\begin{aligned}
&= -\epsilon \frac{1}{2} (\underline{D}(x_2 e_2 + x_4 e_4 + x_1 e_1) e_{241} - e_{241} (x_2 e_2 + x_4 e_4 + x_1 e_1) \underline{D} - \epsilon e_{241}) \\
&= -\epsilon O_{241} = -\epsilon O_{124}.
\end{aligned}$$

We elaborate on one final convention. If A and B denote two lists of $\{1, \dots, N\}$, then when viewed as ordinary sets, the intersection $A \cap B$ contains all elements of A that also belong to B (or equivalently, all elements of B that also belong to A). As a list appearing as index of a Clifford number, we distinguish between $A \cap B$ and $B \cap A$ in the sense that we understand the elements of $A \cap B$ to be in the sequential order of A , while those of $B \cap A$ are in the sequential order of B . If $A = 124$ and $B = 231$, then $e_{A \cap B} = e_{12}$, while $e_{B \cap A} = e_{21}$.

The framework where operators of the form (3.19) make their appearance is one where both commutators and anticommutators are considered. Inspired by property (3.17), we define the “supercommutators”

$$[[O_A, O_B]]_- = O_A O_B - (-1)^{|A||B| - |A \cap B|} O_B O_A, \quad (3.20)$$

$$[[O_A, O_B]]_+ = O_A O_B + (-1)^{|A||B| - |A \cap B|} O_B O_A. \quad (3.21)$$

The algebraic relations we obtained in Theorem 3.12 can now be generalized to higher index versions. We start with a generalization of Theorem 3.11.

Theorem 3.13. *In the algebra $\mathcal{A} \otimes \mathcal{C}$, for A a list of distinct elements of $\{1, \dots, N\}$, we have*

$$\sum_{a \in A} [[O_a, O_{a,A}]]_- = 0.$$

Note that $O_{a,A}$ is up to a sign equal to $O_{A \setminus \{a\}}$ since $a \in A$, where $A \setminus \{a\}$ is the list A with the element a removed.

Proof. By direct calculation and using (3.8), (3.14) and Lemma 3.10 to arrive at the second line we have

$$\begin{aligned}
&\sum_{a \in A} (O_a O_{a,A} - (-1)^{|A|-1} O_{a,A} O_a) \\
&= \epsilon \frac{|A|-2}{2} \left(\sum_{a \in A} O_a e_a e_A - (-1)^{|A|-1} \sum_{a \in A} e_a e_A O_a \right) \\
&\quad + \epsilon \sum_{a \in A} \sum_{b \in A \setminus \{a\}} O_a e_a e_A e_b O_b - (-1)^{|A|-1} \epsilon \sum_{a \in A} \sum_{b \in A \setminus \{a\}} O_b e_b e_a e_A O_a \\
&\quad + \epsilon \frac{1}{2} \sum_{a \in A} \sum_{\{i,j\} \subset A \setminus \{a\}} \sum_{l \notin A \setminus \{a,i,j\}} [L_{ij}, C_{la}] e_l e_i e_j e_a e_A \\
&\quad + \epsilon \frac{1}{2} \sum_{a \in A} \sum_{\{i,j\} \subset A \setminus \{a\}} \sum_{l \in A \setminus \{a,i,j\}} \{L_{ij}, C_{la}\} e_l e_i e_j e_a e_A \\
&\quad - \epsilon \frac{1}{2} \sum_{a \in A} \sum_{\{i,j\} \subset A \setminus \{a\}} L_{ij} (e_a e_i e_j e_a e_A - (-1)^{|A|-1} e_i e_j e_a e_A e_a).
\end{aligned}$$

As $(-1)^{|A|-1}e_a e_A = e_A e_a$ for $a \in A$, the last line vanishes identically. Similarly, the first line of the right-hand side vanishes by Lemma 3.10. Moreover, the second line then vanishes by interchanging the summations. In the third line, we write the first two summations as one summation running over all three-element subsets of A as follows

$$\frac{\epsilon}{2} \sum_{\{a,i,j\} \subset A} \sum_{l \notin A \setminus \{a,i,j\}} ([L_{ij}, C_{la}] + [L_{ai}, C_{lj}] + [L_{ja}, C_{li}]) e_l e_i e_j e_a e_A,$$

which vanishes by Theorem 2.5. Finally, rewriting the four summations in the fourth line as one summation over all four-element subsets of A , one sees that all terms in this summation cancel out using $C_{ij} = C_{ji}$ and $L_{ij} = -L_{ji}$. \square

Using Theorem 3.13 for the list $A = ijkl$ yields

$$\{O_i, O_{jkl}\} - \{O_j, O_{ikl}\} + \{O_k, O_{ijl}\} - \{O_l, O_{ijk}\} = 0.$$

By means of this identity, the relation of Theorem 3.12 for four distinct indices can be cast also in two other formats:

$$[[O_{ij}, O_{kl}]_- = [O_{ij}, O_{kl}] = \{O_i, O_{jkl}\} - \{O_j, O_{ikl}\} = -\{O_{ijl}, O_k\} + \{O_{ijk}, O_l\}. \quad (3.22)$$

The results of Theorem 3.12 are actually special cases of two different more general relations. By means of the following three theorems and the supercommutators (3.20)–(3.21), one is able to swap two symmetry operators O_A and O_B for A, B arbitrary lists of $\{1, \dots, N\}$. Moreover, the supercommutators reduce to explicit expressions in terms of the symmetries and supercommutators containing (at least) one symmetry with just one index. First, we generalize (3.22) to arbitrary disjoint lists.

Theorem 3.14. *In the algebra $\mathcal{A} \otimes \mathcal{C}$, for two lists of $\{1, \dots, N\}$, denoted by A and B , such that $A \cap B = \emptyset$ as sets, we have*

$$[[O_A, O_B]]_- = \epsilon \sum_{a \in A} [[O_a, O_{a,A,B}]]_-.$$

Note that in this case $a \triangle A \triangle B = (A \setminus \{a\}) \cup B$. Moreover, following a similar strategy (or using Theorem 3.13) one also obtains

$$[[O_A, O_B]]_- = \epsilon \sum_{b \in B} [[O_{A,B,b}, O_b]]_-.$$

Proof. A practical property for this proof and the following ones is (3.7). By definition, plugging in (3.10), we find

$$\begin{aligned} [[O_A, O_B]]_- &= O_A O_B - (-1)^{|A||B| - |A \cap B|} O_B O_A \\ &= \frac{1}{4} (\underline{D} \underline{x}_A e_A - e_A \underline{x}_A \underline{D} - \epsilon e_A) (\underline{D} \underline{x}_B e_B - e_B \underline{x}_B \underline{D} - \epsilon e_B) \\ &\quad - (-1)^{|A||B|} \frac{1}{4} (\underline{D} \underline{x}_B e_B - e_B \underline{x}_B \underline{D} - \epsilon e_B) (\underline{D} \underline{x}_A e_A - e_A \underline{x}_A \underline{D} - \epsilon e_A). \end{aligned}$$

First, note that for the terms of $\llbracket O_A, O_B \rrbracket_-$ which do not contain \underline{D} , we have

$$e_A e_B - (-1)^{|A||B|} e_B e_A = 0.$$

For the terms with a single occurrence of \underline{D} , we have (up to a factor $-\epsilon/4$)

$$\begin{aligned} & (\underline{D} x_A e_A - e_A x_A \underline{D}) e_B - (-1)^{|A||B|} e_B (\underline{D} x_A e_A - e_A x_A \underline{D}) \\ & - (-1)^{|A||B|} (\underline{D} x_B e_B - e_B x_B \underline{D}) e_A + e_A (\underline{D} x_B e_B - e_B x_B \underline{D}) \\ = & \sum_{a \in A} (\underline{D} x_a e_a e_A e_B - x_a e_A e_a \underline{D} e_B - (-1)^{|A||B|} e_B \underline{D} x_a e_a e_A \\ & + (-1)^{|A||B|} e_B e_A e_a x_a \underline{D}) - (-1)^{|A||B|} \underline{D} x_B e_B e_A + (-1)^{|A||B|} e_B x_B \underline{D} e_A \\ & + e_A \underline{D} x_B e_B - e_A e_B x_B \underline{D} \end{aligned}$$

Making use of (3.7), $\{\underline{D}, e_a\} = \epsilon 2p_a$ and $x_{a \triangle A \triangle B} = x_{A \setminus \{a\}} + x_B$ for $a \in A$ and $A \cap B = \emptyset$, this equals

$$\begin{aligned} & \sum_{a \in A} (\underline{D} x_a e_a e_A e_B - (-1)^{|A|-1} x_a \underline{D} e_A e_a e_B - (-1)^{|A||B|+|A|-1} e_B e_a e_A \underline{D} x_a \\ & + e_A e_B e_a x_a \underline{D}) + (-1)^{|A|-1} \sum_{a \in A} x_a \sum_{l \in A \setminus \{a\}} 2p_l e_l e_A e_a e_B \\ & - (-1)^{|A||B|} \sum_{a \in A} e_B \sum_{l \in A \setminus \{a\}} 2p_l e_l e_a e_A x_a - \underline{D} x_B e_A e_B \\ & + (-1)^{|A||B|+|A|} e_B x_B e_A \underline{D} + (-1)^{|A|} \underline{D} e_A x_B e_B - e_A e_B x_B \underline{D} \\ & + (-1)^{|A||B|} e_B x_B \sum_{a \in A} 2p_a e_a e_A - (-1)^{|A|} \sum_{a \in A} 2p_a e_a e_A x_B e_B \\ = & \sum_{a \in A} (\underline{D} x_a e_a e_A e_B - x_a \underline{D} e_a e_A e_B - (-1)^{|B|+|A|-1} e_a e_A e_B \underline{D} x_a \\ & + (-1)^{|A|+|B|-1} e_a e_A e_B x_a \underline{D}) + \sum_{a \in A} \epsilon (\underline{D} e_a + e_a \underline{D}) \sum_{l \in A \setminus \{a\}} e_l x_l e_a e_A e_B \\ & + (-1)^{|A|+|B|-1} \sum_{a \in A} e_a e_A e_B \sum_{l \in A \setminus \{a\}} x_l e_l \epsilon (\underline{D} e_a + e_a \underline{D}) \\ & + \sum_{a \in A} (-1)^{|B|+|A|-1} e_a e_A e_B x_B \epsilon (\underline{D} e_a + e_a \underline{D}) + \sum_{a \in A} \epsilon (\underline{D} e_a + e_a \underline{D}) x_B e_a e_A e_B \\ = & \sum_{a \in A} (\underline{D} x_a e_a e_A e_B - x_a \underline{D} e_a e_A e_B - (-1)^{|B|+|A|-1} e_a e_A e_B \underline{D} x_a \\ & + (-1)^{|A|+|B|-1} e_a e_A e_B x_a \underline{D} + \epsilon (-1)^{|A|+|B|-1} e_a e_A e_B x_{a \triangle A \triangle B} \underline{D} e_a \\ & - \epsilon e_a e_a e_A e_B x_{a \triangle A \triangle B} \underline{D} - \epsilon (-1)^{|A|+|B|-1} \underline{D} x_{a \triangle A \triangle B} e_a e_A e_B e_a \\ & + \epsilon e_a \underline{D} x_{a \triangle A \triangle B} e_a e_A e_B). \end{aligned}$$

Next, we work out the terms of $\llbracket O_A, O_B \rrbracket_-$ containing two occurrences of \underline{D} , that is

$$\begin{aligned} & (\underline{D} \underline{x}_A e_A - e_A \underline{x}_A \underline{D}) (\underline{D} \underline{x}_B e_B - e_B \underline{x}_B \underline{D}) \\ & - (-1)^{|A||B|} (\underline{D} \underline{x}_B e_B - e_B \underline{x}_B \underline{D}) (\underline{D} \underline{x}_A e_A - e_A \underline{x}_A \underline{D}). \end{aligned}$$

For the terms having $\underline{D}\underline{D}$ in the middle we readily find

$$\begin{aligned} & - e_A \underline{x}_A \underline{D}\underline{D} \underline{x}_B e_B + (-1)^{|A||B|} e_B \underline{x}_B \underline{D}\underline{D} \underline{x}_A e_A \\ & = - \sum_{a \in A} (e_A x_a e_a \underline{D}\underline{D} \underline{x}_B e_B - (-1)^{|A||B|} e_B \underline{x}_B \underline{D}\underline{D} x_a e_a e_A) \\ & = - \sum_{a \in A} (x_a \underline{D}\underline{D} \underline{x}_B e_a e_A e_B - (-1)^{|A|-1+|B|} e_a e_A e_B \underline{x}_B \underline{D}\underline{D} x_a) \\ & = - \sum_{a \in A} (x_a \underline{D}\underline{D} \underline{x}_{a \triangle A \triangle B} e_a e_A e_B - (-1)^{|A|-1+|B|} e_a e_A e_B \underline{x}_{a \triangle A \triangle B} \underline{D}\underline{D} x_a), \end{aligned}$$

since

$$\begin{aligned} & \sum_{a \in A} (x_a \underline{D}\underline{D} \underline{x}_{a \triangle A} e_a e_A e_B - (-1)^{|A|-1+|B|} e_a e_A e_B \underline{x}_{a \triangle A} \underline{D}\underline{D} x_a) \\ & = \sum_{a \in A} \sum_{l \in A \setminus \{a\}} x_a \underline{D}\underline{D} x_l e_l e_a e_A e_B - \sum_{a \in A} \sum_{l \in A \setminus \{a\}} x_l \underline{D}\underline{D} x_a e_a e_l e_A e_B = 0. \end{aligned}$$

Similarly, we have

$$\begin{aligned} & - \underline{D} \underline{x}_A e_A e_B \underline{x}_B \underline{D} + (-1)^{|A||B|} \underline{D} \underline{x}_B e_B e_A \underline{x}_A \underline{D} \\ & = - \sum_{a \in A} (\underline{D} x_a e_a e_A e_B \underline{x}_B \underline{D} - (-1)^{|A||B|} \underline{D} \underline{x}_B e_B e_A x_a e_a \underline{D}) \\ & = - \sum_{a \in A} (\underline{D} x_a e_a e_A e_B \underline{x}_{a \triangle A \triangle B} \underline{D} - (-1)^{|A|-1+|B|} \underline{D} \underline{x}_{a \triangle A \triangle B} e_a e_A e_B x_a \underline{D}). \end{aligned}$$

Finally, we manipulate the remaining terms as follows

$$\begin{aligned} & \underline{D} \underline{x}_A e_A \underline{D} \underline{x}_B e_B + e_A \underline{x}_A \underline{D} e_B \underline{x}_B \underline{D} \\ & - (-1)^{|A||B|} (\underline{D} \underline{x}_B e_B \underline{D} \underline{x}_A e_A + e_B \underline{x}_B \underline{D} e_A \underline{x}_A \underline{D}) \\ & = \sum_{a \in A} (\underline{D} x_a e_a e_A \underline{D} \underline{x}_B e_B + e_A x_a e_a \underline{D} e_B \underline{x}_B \underline{D} \\ & \quad - (-1)^{|A||B|} (\underline{D} \underline{x}_B e_B \underline{D} x_a e_a e_A + e_B \underline{x}_B \underline{D} e_A x_a e_a \underline{D})) \\ & = \sum_{a \in A} ((-1)^{|A|-1} \underline{D} x_a \underline{D} e_a e_A \underline{x}_B e_B + (-1)^{|A|-1} x_a \underline{D} e_A e_a e_B \underline{x}_B \underline{D} \\ & \quad - (-1)^{|A||B|+|A|-1} \underline{D} \underline{x}_B e_B e_a e_A \underline{D} x_a - (-1)^{|A||B|+|A|-1} e_B \underline{x}_B e_A e_a \underline{D} x_a \underline{D}) \\ & \quad + \sum_{a \in A} \sum_{l \in A \setminus \{a\}} (-(-1)^{|A|-1} \underline{D} x_a 2p_l e_l e_a e_A \underline{x}_B e_B \\ & \quad - (-1)^{|A|-1} x_a 2p_l e_l e_A e_a e_B \underline{x}_B \underline{D} - (-1)^{|A||B|} \underline{D} \underline{x}_B e_B 2p_l e_l e_a e_A x_a \\ & \quad - (-1)^{|A||B|} e_B \underline{x}_B 2p_l e_l e_A e_a x_a \underline{D}) \end{aligned}$$

$$\begin{aligned}
&= \sum_{a \in A} (\underline{D} x_a \underline{D} \underline{x}_{a \triangle A \triangle B} e_a e_A e_B + x_a \underline{D} e_a e_A e_B \underline{x}_{a \triangle A \triangle B} \underline{D} \\
&\quad - (-1)^{|B|+|A|-1} \underline{D} \underline{x}_{a \triangle A \triangle B} e_a e_A e_B \underline{D} x_a \\
&\quad - (-1)^{|A|-1+|B|} e_a e_A e_B \underline{x}_{a \triangle A \triangle B} \underline{D} x_a \underline{D}).
\end{aligned}$$

In the last step we used the following (and a similar result for the other two terms)

$$\begin{aligned}
&\sum_{a \in A} \underline{D} x_a \underline{D} \sum_{l \in A \setminus \{a\}} x_l e_l e_a e_A e_B - (-1)^{|B|+|A|-1} \sum_{a \in A} \underline{D} \sum_{l \in A \setminus \{a\}} x_l e_l e_a e_A e_B \underline{D} x_a \\
&= \sum_{a \in A} \sum_{l \in A \setminus \{a\}} \underline{D} x_a \underline{D} x_l e_l e_a e_A e_B + \sum_{a \in A} \sum_{l \in A \setminus \{a\}} \underline{D} x_l \underline{D} e_l e_a e_A e_B x_a \\
&\quad - \sum_{a \in A} \sum_{l \in A \setminus \{a\}} \underline{D} x_l \sum_{b \in A \setminus \{a, l\} \cup B} 2p_b e_b e_l e_a e_A e_B x_a \\
&= - \sum_{a \in A} \sum_{l \in A \setminus \{a\}} \underline{D} x_l \sum_{b \in A \setminus \{a, l\}} 2p_b e_b e_l e_a e_A e_B x_a \\
&\quad - \sum_{a \in A} \sum_{l \in A \setminus \{a\}} \underline{D} x_l \sum_{b \in B} 2p_b e_b e_l e_a e_A e_B x_a \\
&= - \sum_{\{a, l, b\} \subset A} 2\underline{D} (x_l p_b x_a - x_a p_b x_l + x_b p_a x_l - x_b p_l x_a + x_a p_l x_b \\
&\quad - x_l p_a x_b) e_b e_l e_a e_A e_B - \sum_{\{a, l\} \subset A} \sum_{b \in B} 2\underline{D} (x_l p_b x_a - x_a p_b x_l) e_b e_l e_a e_A e_B \\
&= - \sum_{\{a, l\} \subset A} \sum_{b \in B} 2\underline{D} (x_l (p_b x_a - x_a p_b) - x_a (p_b x_l - x_l p_b)) e_b e_l e_a e_A e_B \\
&= - \sum_{\{a, l\} \subset A} \sum_{b \in B} 2\underline{D} (x_l (p_a x_b - x_b p_a) - x_a (p_l x_b - x_b p_l)) e_b e_l e_a e_A e_B \\
&= - \sum_{\{a, l\} \subset A} \sum_{b \in B} 2\underline{D} (x_l p_a - x_a p_l) x_b e_b e_l e_a e_A e_B \\
&\quad - \sum_{\{a, l\} \subset A} \sum_{b \in B} x_b (x_a p_l - x_l p_a) e_b e_l e_a e_A e_B \\
&= \sum_{a \in A} \sum_{l \in A \setminus \{a\}} \underline{D} x_a 2p_l \underline{x}_B e_l e_a e_A e_B - \sum_{a \in A} \sum_{l \in A \setminus \{a\}} \underline{D} \underline{x}_B 2p_l x_a e_l e_a e_A e_B.
\end{aligned}$$

To arrive at this result we made use of property (3.7), Lemma 2.4, the commutativity of x_1, \dots, x_N and $C_{ij} = C_{ji}$.

Putting everything together and comparing with

$$\begin{aligned}
\sum_{a \in A} \llbracket O_a, O_{a, A, B} \rrbracket_- &= \sum_{a \in A} \frac{\epsilon}{4} ((\underline{D} x_a - x_a \underline{D} - e_a) (\underline{D} \underline{x}_{a \triangle A \triangle B} e_a e_A e_B \\
&\quad - e_a e_A e_B \underline{x}_{a \triangle A \triangle B} \underline{D} - \epsilon e_a e_A e_B) - (-1)^{|A|+|B|-1} \frac{\epsilon}{4} (\underline{D} \underline{x}_{a \triangle A \triangle B} e_a e_A e_B \\
&\quad - e_a e_A e_B \underline{x}_{a \triangle A \triangle B} \underline{D} - \epsilon e_a e_A e_B) (\underline{D} x_a - x_a \underline{D} - e_a)),
\end{aligned}$$

the proof is completed. \square

Theorem 3.15. *In the algebra $\mathcal{A} \otimes \mathcal{C}$, for two lists of $\{1, \dots, N\}$, denoted by A and B , such that $A \subset B$ as sets, we have*

$$\llbracket O_A, O_B \rrbracket_- = \epsilon \sum_{a \in A} \llbracket O_a, O_{a,A,B} \rrbracket_-.$$

Note that in this case $a \triangle A \triangle B = \{a\} \cup (B \setminus A)$.

Proof. To prove this result, one is not limited to just the form (3.10) for O_A and O_B as we did in the proof of Theorem 3.14. One may also use for instance the form (3.14), and employ a strategy similar to the one used in the proof of Theorem 3.12. A proof in this style can be found in the Appendix A. \square

Theorem 3.16. *In the algebra $\mathcal{A} \otimes \mathcal{C}$, for two lists of $\{1, \dots, N\}$, denoted by A and B , we have*

$$\begin{aligned} \llbracket O_A, O_B \rrbracket_+ &= \epsilon O_{A,B} + (e_{A \cap B})^2 \llbracket O_{A,(A \cap B)}, O_{(A \cap B),B} \rrbracket_+ \\ &\quad + (e_{A \cap B})^2 \llbracket O_{A \cap B}, O_{(A \cap B),A,B} \rrbracket_+. \end{aligned}$$

Note that $A \triangle B = (A \setminus B) \cup (B \setminus A)$, while $A \triangle (A \cap B) = A \setminus B$ and $(A \cap B) \triangle B = B \setminus A$, and finally $(A \cap B) \triangle A \triangle B = A \cup B$.

Proof. Because of its length and as it employs a similar strategy as used already in the proof of Theorem 3.14, we have moved the proof of this result to the Appendix A. \square

Corollary 3.17. *In the algebra $\mathcal{A} \otimes \mathcal{C}$, for two lists of $\{1, \dots, N\}$, denoted by A and B , we have*

$$\begin{aligned} &\llbracket O_A, O_B \rrbracket_+ \\ &= \epsilon O_{A,B} + (e_{A \cap B})^2 2O_{A,(A \cap B)} O_{(A \cap B),B} + (e_{A \cap B})^2 2O_{A \cap B} O_{(A \cap B),A,B} \\ &\quad - \epsilon (e_{A \cap B})^2 \sum_{a \in A \setminus B} \llbracket O_a, O_{(A \cap B),B} \rrbracket_- - \epsilon (e_{A \cap B})^2 \sum_{a \in A \cap B} \llbracket O_a, O_{(A \cap B),A,B} \rrbracket_- . \end{aligned}$$

Proof. Note first that $\llbracket O_A, O_B \rrbracket_- + \llbracket O_A, O_B \rrbracket_+ = 2O_A O_B$. Now, combine Theorem 3.16 with Theorem 3.14 and Theorem 3.15, using $(A \setminus B) \cap (B \setminus A) = \emptyset$ and $(A \cap B) \subset (A \cup B)$. \square

By means of Theorem 3.14, Theorem 3.15 and Theorem 3.16 (or thus Corollary 3.17) we can swap any two operators O_A and O_B where A or B is not a list of just one element. We briefly explain the need for three such theorems. Theorem 3.16 yields an empty identity in two cases, when $A \cap B = \emptyset$ or when either A or B is contained in the other as sets. For example, say $A \cap B = \emptyset$, then we have

$$\begin{aligned} \llbracket O_A, O_B \rrbracket_+ &= \epsilon O_{A,B} + (e_{A \cap B})^2 \llbracket O_{A,(A \cap B)}, O_{(A \cap B),B} \rrbracket_+ \\ &\quad + (e_{A \cap B})^2 \llbracket O_{A \cap B}, O_{(A \cap B),A,B} \rrbracket_+ \end{aligned}$$

$$\begin{aligned}
&= \epsilon O_{A,B} + \llbracket O_A, O_B \rrbracket_+ + O_\emptyset O_{A,B} + O_{A,B} O_\emptyset \\
&= \llbracket O_A, O_B \rrbracket_+,
\end{aligned}$$

as $e_\emptyset = 1$ and $O_\emptyset = -\epsilon/2$. For these cases we can resort to Theorem 3.14 or Theorem 3.15. However, if A is a list of a single element a , Theorem 3.14 and Theorem 3.15 are empty identities:

$$\llbracket O_A, O_B \rrbracket_- = \epsilon \sum_{a \in A} \llbracket O_a, O_{a,A,B} \rrbracket_- = \epsilon \llbracket O_a, O_{a,a,B} \rrbracket_- = \llbracket O_a, O_B \rrbracket_-,$$

but so is Theorem 3.16 as either $a \in B$ or $a \cap B = \emptyset$. Now, for the case $a \notin B$ Theorem 3.13 yields

$$O_a O_B = (-1)^{|B|} O_B O_a - \epsilon \sum_{b \in B} \llbracket O_b, O_{b,a,B} \rrbracket_- ,$$

while by definition we also have

$$O_a O_B = \pm (-1)^{|B| - |a \cap B|} O_B O_a + \llbracket O_a, O_B \rrbracket_\mp.$$

We see that all expressions involving supercommutators can be reduced to sums of supercommutators containing (at least) one symmetry with just one index. In the following section we review again examples of specific realizations of the elements x_1, \dots, x_N and p_1, \dots, p_N of the algebra \mathcal{A} . For these examples, the one index symmetries in particular take on an explicit form whose interaction with other symmetries can be computed explicitly. This form depends on the makeup of the symmetries $C_{ij} = [p_i, x_j]$ as O_i is given by (3.8).

4 Examples

We recall the two examples from the previous section.

Example 4.1. Let again Δ be the classical Laplace operator, then

$$\underline{D} = \sum_{i=1}^N e_i p_i = \sum_{i=1}^N e_i \frac{\partial}{\partial x_i}$$

is the classical Dirac operator. The $\mathfrak{osp}(1|2)$ structure of Theorem 3.4 here was obtained already in [17]. The commutator C_{li} in the definition (3.8) of O_i reduces to $[p_i, x_j] = \delta_{ij}$, so

$$O_i = \epsilon \frac{1}{2} \left(\sum_{l=1}^N e_l \delta_{li} - e_i \right) = \epsilon \frac{1}{2} (e_i - e_i) = 0.$$

Moreover, (3.13) then becomes

$$O_{ij} = L_{ij} + \epsilon \frac{1}{2} e_i e_j,$$

in accordance with results obtained in [9], while for a general subset $A \subset \{1, \dots, N\}$ one has

$$O_A = \left(\epsilon \frac{|A| - 1}{2} - \sum_{\{i,j\} \subset A} L_{ij} e_i e_j \right) e_A.$$

Since $[p_i, x_j] = 0$ for $i \neq j$, given a subset $A \subset \{1, \dots, N\}$ the operators \underline{x}_A and \underline{D}_A as defined by (3.9) in fact also generate a copy of the Lie superalgebra $\mathfrak{osp}(1|2)$ whose Scasimir element we will denote by

$$\mathcal{S}_A = \frac{1}{2} ([\underline{D}_A, \underline{x}_A] - \epsilon).$$

From (3.10), we see that in this case O_A equals $\mathcal{S}_A e_A$.

As in this case the one-index symmetries are identically zero, the algebraic relations simplify accordingly. The symmetries with two indices generate a realization of the Lie algebra $\mathfrak{so}(N)$, as seen from the relations of Theorem 3.12. For general symmetries Theorem 3.16 now yields

$$\begin{aligned} \llbracket O_A, O_B \rrbracket_+ &= \epsilon O_{A,B} + (e_{A \cap B})^2 2 O_{A, (A \cap B)} O_{(A \cap B), B} \\ &\quad + (e_{A \cap B})^2 2 O_{A \cap B} O_{(A \cap B), A, B}, \end{aligned}$$

as by Theorem 3.14 and Theorem 3.15 $\llbracket O_A, O_B \rrbracket_- = 0$ for $A \cap B = \emptyset$, or $A \subset B$, or $B \subset A$. This corresponds to (a special case of) the higher rank Bannai-Ito algebra of [8], which strictly speaking is not included in the results obtained there.

To conclude, we mention another group of symmetries of \underline{D} and \underline{x} which are not inside the algebra $\mathcal{A} \otimes \mathcal{C}$ in this case, but which will also be useful for the next example. For a normed vector α , its embedding in the Clifford algebra

$$\underline{\alpha} = \sum_{l=1}^N e_l \alpha_l$$

is an element of the so-called Pin group, which forms a double cover of the orthogonal group $O(N)$. These elements have the property that

$$\underline{\alpha} \underline{v} = -\underline{w} \underline{\alpha},$$

where $w = \sigma_\alpha(v) = v - 2\langle v, \alpha \rangle \alpha / \|\alpha\|^2$. Hence, if we define the operators S_α as follows

$$S_\alpha = \sum_{l=1}^N e_l \alpha_l \sigma_\alpha = \underline{\alpha} \sigma_\alpha, \quad (4.1)$$

then it is immediately clear that they satisfy the following properties

$$S_\alpha \underline{v} = -\underline{w} S_\alpha, \quad S_\alpha f(x) = f(\sigma_\alpha(x)) S_\alpha, \quad (S_\alpha)^2 = \epsilon.$$

where again $w = \sigma_\alpha(v)$ and f is a function or operator which does not interact with the Clifford generators. By direct computation, one can show that the

operator S_α anticommutes with the Dirac operator. Moreover, the interaction of S_α and a symmetry operator O_A is simply given by the action of the reflection associated to α on the coordinate vectors corresponding to the elements of A .

Example 4.2. We consider again the case where p_1, \dots, p_N are given by the Dunkl operators (2.10) associated to a given root system R and with multiplicity function k . Here, the $\mathfrak{osp}(1|2)$ structure of Theorem 3.4 was obtained already in [4, 21].

The commutator in the definition (3.8) of O_i is then given by

$$C_{ij} = [\mathcal{D}_i, x_j] = \delta_{ij} + \sum_{\alpha \in R_+} 2k(\alpha)\alpha_i\alpha_j\sigma_\alpha.$$

The symmetries of the Dunkl Dirac operator $\sum_{i=1}^N \mathcal{D}_i e_i$ with one index thus become

$$O_i = \epsilon \frac{1}{2} \left(\sum_{l=1}^N e_l \delta_{li} + \sum_{l=1}^N e_l \sum_{\alpha \in R_+} 2k(\alpha)\alpha_l\alpha_i\sigma_\alpha - e_i \right) = \epsilon \sum_{\alpha \in R_+} k(\alpha)\alpha_i \sum_{l=1}^N e_l \alpha_l \sigma_\alpha.$$

On the right-hand side we see the operators (4.1) appear for the roots $\alpha \in R_+$. By direct computation, one can show that for a root α , the operator S_α anticommutes with the Dirac-Dunkl operator. The one-index symmetry O_i thus consist of linear combinations of the operators (4.1) determined by the root system and by the multiplicity function k

$$O_i = \epsilon \sum_{\alpha \in R_+} k(\alpha)\alpha_i S_\alpha.$$

Higher-index symmetries O_A contain the Dunkl angular momentum operators, appended with the anticommuting symmetries S_α for $\alpha \in R_+$. For instance, if $A = \{i, j\}$ we have

$$O_{ij} = x_i \mathcal{D}_j - x_j \mathcal{D}_i + \epsilon \frac{1}{2} e_i e_j + \sum_{\alpha \in R_+} k(\alpha)(e_i \alpha_j - e_j \alpha_i) S_\alpha.$$

The algebraic relations of Theorem 3.14 and Theorem 3.15 can now be worked out explicitly as the action of the one-index symmetries is given by the reflections associated to the roots of the root system.

Example 4.2.1. For the root system with Weyl group $W = (\mathbb{Z}_2)^N$, our results are in accordance with what was already obtained in [7, 8]. Here, the Dunkl operators (2.10) are given by

$$\mathcal{D}_i = \frac{\partial}{\partial x_i} + \frac{\mu_i}{x_i}(1 - r_i) \quad i \in \{1, \dots, N\}$$

where r_i is the reflection operator in the $x_i = 0$ hyperplane and μ_i is the value of the multiplicity function on the conjugacy class of r_i . The one-index symmetry (3.8) in this case reduces to

$$O_i = \epsilon \sum_{\alpha \in R_+} k(\alpha)\alpha_i S_\alpha = \epsilon \mu_i r_i e_i.$$

Here, we also have $[p_i, x_j] = 0$ for $i \neq j$, so for a given subset $A \subset \{1, \dots, N\}$ the operators \underline{x}_A and \underline{D}_A as defined by (3.9) generate a copy of the Lie superalgebra $\mathfrak{osp}(1|2)$ with the Scasimir element

$$\mathcal{S}_A = \frac{1}{2} ([\underline{D}_A, \underline{x}_A] - \epsilon).$$

From (3.10), we see that in this case O_A equals $\mathcal{S}_A e_A$. The relation with the symmetries denoted by Γ_A in [8] is

$$\Gamma_A = \mathcal{S}_A \prod_{i \in A} r_i = \mathcal{S}_A \prod_{i \in A} \frac{1}{\mu_i} O_i e_i = \mathcal{S}_A e_A \prod_{i \in A} \frac{1}{\mu_i} O_i = O_A \prod_{i \in A} \frac{1}{\mu_i} O_i,$$

where the product over $i \in A$ is taken according to the order of A . The operator Γ_A commutes with the Dunkl Dirac operator by Corollary 3.9. The algebraic structure generated by the operators Γ_A and corresponding to the relations of Theorem 3.16 is the higher rank Bannai-Ito algebra of [8]. For the case $N = 3$, see [7, 15], this is the regular Bannai-Ito algebra [27].

Example 4.2.2. For the root system of type A_{N-1} , with positive subsystem given, for instance, by

$$R_+ = \left\{ \frac{1}{\sqrt{2}}(\xi_i - \xi_j) \mid 1 \leq i < j \leq N \right\}$$

where $\{\xi_1, \dots, \xi_N\}$ is an orthonormal basis of \mathbb{R}^N , the associated Weyl group is the symmetric group S_N of permutations on N elements. All permutations in S_N are conjugate so the multiplicity function $k(\alpha)$ has the same value for all roots, which we will denote by κ . The Dunkl operators (2.10) are then given by

$$\mathcal{D}_i = \frac{\partial}{\partial x_i} + \kappa \sum_{j \neq i} \frac{1 - g_{ij}}{x_i - x_j} \quad i \in \{1, \dots, N\}$$

where g_{ij} denotes the reflection corresponding to the root $1/\sqrt{2}(\xi_i - \xi_j)$. The related operator of the form (4.1) will be denoted as

$$G_{ij} = \frac{1}{\sqrt{2}}(e_i - e_j)g_{ij} = -G_{ji}.$$

In this case, the commutator $[p_i, x_j]$ does not reduce to zero for $i \neq j$. It is given by

$$[\mathcal{D}_i, x_j] = \begin{cases} 1 + \kappa \sum_{l \neq i} g_{il} & \text{if } i = j \\ -\kappa g_{ij} & \text{if } i \neq j \end{cases}$$

and the one-index symmetry (3.8) becomes

$$O_i = \epsilon \sum_{\alpha \in R_+} k(\alpha) \alpha_i S_\alpha = \epsilon \kappa \sum_{1 \leq l < i} \frac{-1}{\sqrt{2}} G_{li} + \epsilon \kappa \sum_{i < l \leq N} \frac{1}{\sqrt{2}} G_{il} = \frac{\epsilon \kappa}{\sqrt{2}} \sum_{l=1}^N G_{il},$$

where $G_{ii} = 0$, while for the symmetry (3.13) we have

$$O_{ij} = x_i \mathcal{D}_j - x_j \mathcal{D}_i + \epsilon \frac{1}{2} e_i e_j + \frac{\epsilon \kappa}{\sqrt{2}} \sum_{l=1}^N (e_i G_{jl} - e_j G_{il}).$$

The relations for two-index symmetries of Theorem 3.12 are

$$\begin{aligned} [O_{ij}, O_{kl}] = & \frac{\kappa}{\sqrt{2}} ((O_{lij} - O_{lik})(G_{jk}) + (O_{kji} - O_{kjl})(G_{il}) + (O_{ikl} - O_{ikj})(G_{lj}) \\ & + (O_{jlk} - O_{jli})(G_{ki})) \end{aligned}$$

for four distinct indices, and when $l = i$ we have

$$\begin{aligned} [O_{ij}, O_{ki}] = & O_{jk} + \frac{\kappa}{\sqrt{2}} 2O_{ijk}(G_{ij} - G_{ki}) + \frac{\kappa}{\sqrt{2}} \sum_{a \neq i, j, k} (O_{ijk} - O_{ajk})G_{ia} \\ & + \frac{\kappa^2}{2} \sum_{a=1}^N \sum_{b=1}^N [G_{ja}, G_{kb}]. \end{aligned}$$

5 Summary and outlook

The replacement of ordinary derivatives by Dunkl derivatives \mathcal{D}_i in the expressions of the Laplace and the Dirac operator in N dimensions gives rise to the Laplace-Dunkl Δ and Dirac-Dunkl operator \underline{D} . This paper was devoted to the study of the symmetry algebras of these two operators, i.e. to the algebraic relations satisfied by the operators commuting or anticommuting with Δ or \underline{D} .

In the case of Dunkl derivatives, the underlying object is the reflection group G acting in N -dimensional space, characterized by a reduced root system. The Dunkl derivatives themselves then consist of an ordinary derivative plus a number of difference operators depending on this reflection group. So it can be expected that the reflection group G plays an essential role in the structure of the symmetry algebra.

One of the leading examples was for $N = 3$ and $G = \mathbb{Z}_2^3$. Even for this quite simple reflection group, the study of the symmetry algebras was already non-trivial, and led to the celebrated Bannai-Ito algebra [15, 7]. Following this, the second case where the symmetry algebra could be determined was for general N and $G = \mathbb{Z}_2^N$ [8], leading to a “higher rank Bannai-Ito algebra”, of which the structure is already highly non-trivial.

The question that naturally arises is whether the symmetry algebras of the Laplace-Dunkl and Dirac-Dunkl operators for other reflection groups G can still be determined, and what their structure is. We considered it as a challenge to study this problem. Originally, we were hoping to solve the problem for the case $G = S_N$, which would already be a significant breakthrough. Herein, S_N is the symmetric group acting on the coordinates x_i by permuting the indices; as a reflection group it is associated with the root system of type A_{N-1} .

Our initial attempts and computations for the case $G = S_N$ were not promising, and the situation looked extremely complicated, particularly because the explicit actions of the Dunkl derivatives \mathcal{D}_i are already very complex. Fortunately, at that moment we followed the advice “if you cannot solve the problem, generalize it.” So we went back to the general case, with arbitrary reflection group G , and no longer focused on the explicit actions of the Dunkl derivatives \mathcal{D}_i , but on the algebraic relations among the coordinate operators x_i and the \mathcal{D}_i . Then we realized that we could still jump one level higher in the generalization, and just work with coordinate operators x_i and “momentum operators” p_i in the framework of a Wigner quantum system, by identifying the p_i with \mathcal{D}_i . As a consequence, we could forget about the actual meaning of the Dunkl derivatives, and just work and perform our computations in the associative algebra \mathcal{A} (Definition 2.1). This general or “more abstract” setting enabled us to determine the elements (anti)commuting with Δ or \underline{D} , and to construct the algebraic relations satisfied by these elements. The resulting symmetry algebra, obtained in the paper, is still quite complicated. But we managed to determine it (for general G), going far beyond our initial goal. For the general Laplace-Dunkl operator, the symmetries and the symmetry algebra are described in Theorem 2.3 and Theorem 2.5. For the general Dirac-Dunkl operator, the symmetries are determined in Theorem 3.6 and Theorem 3.7. The relations for these symmetries (i.e. the symmetry algebra) are established in Section 3.2, and follow from Theorems 3.14, 3.15 and 3.16.

The results of this paper open the way to several new investigations. In particular, one could now go back to the interesting case $G = S_N$, starting with $N = 3$ [5], and investigate how the symmetry algebra specializes. One of the purposes is to study representations of the symmetry algebra in that case, since this leads to null solutions of the Dirac operator. As in the case of $G = \mathbb{Z}_2^N$ [15, 7], one can expect that interesting families of orthogonal polynomials should arise. Furthermore, note that for the case of $G = \mathbb{Z}_2^3$ a superintegrable model on the two-sphere was obtained [6]. It is definitely worthwhile to investigate possible superintegrable systems for other groups G .

In a different direction, one can examine whether the context of Wigner quantum systems, as used here for rational Dunkl operators, is still of use for other types of operators. Possible examples are trigonometric Dunkl operators, or the Dunkl operators appearing in discrete function theory.

Finally, it is known that solutions of Wigner quantum systems with a Hamiltonian of type (1.1) can be described in terms of unitary representations of the Lie superalgebra $\mathfrak{osp}(1|2N)$ [26, 28]. The action and restriction of the coordinate operators x_i and the Dunkl operators \mathcal{D}_i in these representations should be studied further.

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A Appendix

This appendix contains the proofs of Theorem 3.15 and Theorem 3.16, which were omitted from the main text due to their length.

Proof of Theorem 3.15. We systematically go over every term appearing in $\llbracket O_A, O_B \rrbracket_-$, using now the form (3.14) for O_A and O_B , that is

$$\left\llbracket \left(\epsilon \frac{|A|-1}{2} + \epsilon \sum_{a \in A} O_a e_a - \sum_{\{i,j\} \subset A} L_{ij} e_i e_j \right) e_A, \left(\epsilon \frac{|B|-1}{2} + \epsilon \sum_{b \in B} O_b e_b - \sum_{\{k,l\} \subset B} L_{kl} e_k e_l \right) e_B \right\llbracket_-.$$

Starting with the terms which contain no O_i or L_{ij} , we have using property (3.17) and $A \subset B$

$$\frac{|A|-1}{2} \frac{|B|-1}{2} (e_A e_B - (-1)^{|A||B|-|A \cap B|} e_B e_A) = \frac{|A|-1}{2} \frac{|B|-1}{2} (e_A e_B - e_A e_B) = 0.$$

Next, using property (3.17) and Lemma 3.10 we have

$$\begin{aligned} \left\llbracket e_A, \sum_{b \in B} O_b e_b e_B \right\llbracket_- &= \sum_{b \in B} e_A e_B e_b O_b - (-1)^{|A||B|-|A \cap B|} \sum_{b \in B} O_b e_b e_B e_A \\ &= \sum_{b \in A} e_A e_B e_b O_b + \sum_{b \in B \setminus A} e_A e_B e_b O_b - \sum_{b \in B \setminus A} e_A e_B e_b O_b - \sum_{b \in A} O_b e_b e_A e_B \\ &= \sum_{b \in A} e_A e_B e_b O_b - \sum_{b \in A} O_b e_b e_A e_B, \end{aligned}$$

while

$$\begin{aligned} \left\llbracket \sum_{a \in A} O_a e_a e_A, e_B \right\llbracket_- &= \sum_{a \in A} O_a e_a e_A e_B - (-1)^{|A||B|-|A \cap B|} \sum_{a \in A} e_B e_A e_a O_a \\ &= \sum_{a \in A} O_a e_a e_A e_B - \sum_{a \in A} e_A e_B e_a O_a. \end{aligned}$$

Hence, we have

$$\begin{aligned} &\frac{|A|-1}{2} \left\llbracket e_A, \sum_{b \in B} O_b e_b e_B \right\llbracket_- + \frac{|B|-1}{2} \left\llbracket \sum_{a \in A} O_a e_a e_A, e_B \right\llbracket_- \\ &= \frac{|B|-1-(|A|-1)}{2} \left(\sum_{a \in A} O_a e_a e_A e_B - \sum_{a \in A} e_A e_B e_a O_a \right) \\ &= \frac{|B|-|A|+1-1}{2} \sum_{a \in A} (O_a e_a e_A e_B - (-1)^{|B|-|A|} e_a e_A e_B O_a) \\ &= \sum_{a \in A} \left\llbracket O_a, \frac{|B|-|A|+1-1}{2} e_a e_A e_B \right\llbracket_-. \end{aligned}$$

For the next part, using the notation $A' = A \setminus \{a\}$, we find

$$\begin{aligned} &\left\llbracket \sum_{a \in A} O_a e_a e_A, \sum_{b \in B} O_b e_b e_B \right\llbracket_- \\ &= \sum_{a \in A} \sum_{b \in B} (O_a e_a e_A e_B e_b O_b - (-1)^{|A||B|-|A \cap B|} O_b e_b e_B e_A e_a O_a) \\ &= \sum_{a \in A} \sum_{b \in B} (O_a e_a e_A e_B e_b O_b - O_b e_b e_A e_B e_a O_a) \end{aligned}$$

$$= \sum_{a \in A} \left[O_a, \sum_{b \in B \setminus A'} O_b e_b e_a e_A e_B \right]_- + \sum_{a \in A} \sum_{b \in A'} O_a e_a e_A e_B e_b O_b - \sum_{a \in A} \sum_{b \in A'} O_b e_b e_A e_B e_a O_a.$$

One easily sees that the summations in the last line cancel out.

Now, for the parts containing “ L_{ij} -terms”, we have

$$\left[\sum_{\{i,j\} \subset A} L_{ij} e_i e_j e_A e_B \right]_- = \sum_{\{i,j\} \subset A} L_{ij} \left(e_i e_j e_A e_B - (-1)^{|A||B| - |A \cap B|} e_B e_i e_j e_A \right) = 0,$$

as, using property (3.17) and $A \subset B$,

$$\begin{aligned} (-1)^{|A||B| - |A \cap B|} e_B e_i e_j e_A &= (-1)^{|A||B| - |A \cap B| + |A \setminus \{i,j\}||B| - |(A \setminus \{i,j\}) \cap B|} e_i e_j e_A e_B \\ &= e_i e_j e_A e_B. \end{aligned}$$

Moreover, using Lemma 3.10 and property (3.17) while denoting $A' = A \setminus \{a\}$, we have

$$\begin{aligned} & \left[\epsilon \sum_{a \in A} O_a e_a e_A, - \sum_{\{k,l\} \subset B} L_{kl} e_k e_l e_B \right]_- \\ &= -\epsilon \sum_{a \in A} O_a \sum_{\{k,l\} \subset B \setminus A'} L_{kl} e_k e_l e_a e_A e_B + \epsilon \sum_{a \in A} \sum_{\{k,l\} \subset B \setminus A'} L_{kl} e_k e_l e_A e_B e_a O_a \\ & \quad - \epsilon \sum_{a \in A} O_a \sum_{\{k,l\} \subset A'} L_{kl} e_k e_l e_a e_A e_B + \epsilon \sum_{a \in A} \sum_{\{k,l\} \subset A'} L_{kl} e_k e_l e_A e_B e_a O_a \\ & \quad + \epsilon \sum_{a \in A} O_a \sum_{k \in A'} \sum_{l \in B \setminus A'} L_{kl} e_k e_l e_a e_A e_B + \epsilon \sum_{a \in A} \sum_{k \in A'} \sum_{l \in B \setminus A'} L_{kl} e_k e_l e_A e_B e_a O_a \\ &= -\epsilon \sum_{a \in A} \left[O_a, \sum_{\{k,l\} \subset B \setminus A'} L_{kl} e_k e_l e_a e_A e_B \right]_- \\ & \quad - \epsilon \sum_{a \in A} O_a \sum_{\{k,l\} \subset A'} L_{kl} e_k e_l e_a e_A e_B + (-1)^{|B| - |A|} \epsilon \sum_{a \in A} \sum_{\{k,l\} \subset A'} L_{kl} e_k e_l e_a e_A e_B O_a \\ & \quad + \epsilon \sum_{a \in A} O_a \sum_{k \in A'} \sum_{l \in B \setminus A'} L_{kl} e_k e_l e_a e_A e_B + (-1)^{|B| - |A|} \epsilon \sum_{a \in A} \sum_{k \in A'} \sum_{l \in B \setminus A'} L_{kl} e_k e_l e_a e_A e_B O_a. \end{aligned} \tag{A.1}$$

(A.2)

We see that the final part to make $\epsilon \sum_{a \in A} [O_a, O_{a \triangle A \triangle B}]_-$ appears here. To complete the proof we show that the last two lines, (A.1) and (A.2), cancel out with the remaining terms of $[O_A, O_B]_-$.

Hereto, we use Lemma 3.10, to find (denoting again $A' = A \setminus \{a\}$)

$$\begin{aligned} & \left[- \sum_{\{i,j\} \subset A} L_{ij} e_i e_j e_A, \epsilon \sum_{b \in B} O_b e_b e_B \right]_- \\ &= -\epsilon \sum_{\{i,j\} \subset A} L_{ij} e_i e_j e_A e_B \sum_{a \in A \setminus \{i,j\}} e_a O_a - \epsilon \sum_{\{i,j\} \subset A} L_{ij} e_i e_j e_A e_B \sum_{b \in B \setminus (A \setminus \{i,j\})} e_b O_b \\ & \quad + (-1)^{|A||B| - |A \cap B|} \epsilon \sum_{\{i,j\} \subset A} \sum_{a \in A \setminus \{i,j\}} O_a L_{ij} e_a e_i e_j e_B e_A \\ & \quad + (-1)^{|A||B| - |A \cap B|} \epsilon \sum_{\{i,j\} \subset A} \sum_{b \in B \setminus (A \setminus \{i,j\})} O_b L_{ij} e_b e_i e_j e_B e_A \\ &= -(-1)^{|B| - |A|} \epsilon \sum_{a \in A} \sum_{\{i,j\} \subset A'} L_{ij} e_i e_j e_a e_A e_B O_a + \epsilon \sum_{a \in A} \sum_{\{i,j\} \subset A'} O_a L_{ij} e_i e_j e_a e_A e_B \\ & \quad - \epsilon \sum_{\{i,j\} \subset A} L_{ij} \sum_{b \in B \setminus (A \setminus \{i,j\})} O_b e_b e_i e_j e_A e_B + \epsilon \sum_{\{i,j\} \subset A} \sum_{b \in B \setminus (A \setminus \{i,j\})} O_b L_{ij} e_b e_i e_j e_A e_B \end{aligned}$$

$$\begin{aligned}
&= \epsilon \sum_{a \in A} O_a \sum_{\{i,j\} \subset A'} L_{ij} e_i e_j e_a e_A e_B - (-1)^{|B|-|A|} \epsilon \sum_{a \in A} \sum_{\{i,j\} \subset A'} L_{ij} e_i e_j e_a e_A e_B O_a \\
&\quad - \epsilon \sum_{\{i,j\} \subset A} \sum_{b \in B \setminus (A \setminus \{i,j\})} [L_{ij}, O_b] e_b e_i e_j e_A e_B. \tag{A.3}
\end{aligned}$$

This already causes (A.1) to vanish, so (A.2) and (A.3) remain.

Next, we look at

$$\sum_{\{i,j\} \subset A} \sum_{\{k,l\} \subset B} L_{ij} L_{kl} e_i e_j e_A e_k e_l e_B - (-1)^{|A||B|-|A \cap B|} L_{kl} L_{ij} e_k e_l e_B e_i e_j e_A. \tag{A.4}$$

According to the sign of

$$\begin{aligned}
&(-1)^{|A||B|-|A \cap B|} e_k e_l e_B e_i e_j e_A \\
&= (-1)^{|A||B|-|A \cap B|+|A \setminus \{i,j\}||B \setminus \{k,l\}|-|(A \setminus \{i,j\}) \cap (B \setminus \{k,l\})|} e_i e_j e_A e_k e_l e_B \\
&= (-1)^{|A \cap B|-|(A \setminus \{i,j\}) \cap (B \setminus \{k,l\})|} e_i e_j e_A e_k e_l e_B,
\end{aligned}$$

the summation (A.4) reduces to a combination of commutators and anticommutators involving L_{ij} and L_{kl} . We first treat the anticommutators

$$\begin{aligned}
& - \sum_{k \in A} \sum_{\{i,j\} \subset A \setminus \{k\}} \sum_{l \in B \setminus A} \{L_{ij}, L_{kl}\} e_i e_j e_k e_l e_A e_B \\
& - \sum_{i \in A} \sum_{\{j,k\} \subset A \setminus \{i\}} \epsilon(\{L_{ik}, L_{ji}\} e_k e_j e_A e_B + \{L_{ij}, L_{ki}\} e_j e_k e_A e_B).
\end{aligned}$$

The last line reduces to

$$\sum_{i \in A} \sum_{\{j,k\} \subset A \setminus \{i\}} \epsilon(\{L_{ik}, L_{ji}\} - \{L_{ij}, L_{ki}\}) e_k e_j e_A e_B = 0,$$

while the first line vanishes by Theorem 2.5 as it can be rewritten as

$$\begin{aligned}
& \sum_{\{k,i,j\} \subset A} \sum_{l \in B \setminus A} (\{L_{ij}, L_{kl}\} e_i e_j e_k e_l + \{L_{kj}, L_{il}\} e_k e_j e_i e_l + \{L_{ik}, L_{jl}\} e_i e_k e_j e_l) e_A e_B \\
&= \sum_{\{k,i,j\} \subset A} \sum_{l \in B \setminus A} (\{L_{ij}, L_{kl}\} + \{L_{jk}, L_{il}\} + \{L_{ki}, L_{jl}\}) e_i e_j e_k e_l e_A e_B.
\end{aligned}$$

Next, we treat the remaining terms of the summation (A.4) which reduce to four different summations of commutators

$$\begin{aligned}
& \sum_{\{i,j\} \subset A} \sum_{\{k,l\} \subset B \setminus A} [L_{ij}, L_{kl}] e_i e_j e_k e_l e_A e_B + \sum_{\{i,j\} \subset A} \sum_{\{k,l\} \subset (B \cap A) \setminus \{i,j\}} [L_{ij}, L_{kl}] e_i e_j e_k e_l e_A e_B \\
& - \sum_{i \in A} \sum_{j \in A \setminus \{i\}} \sum_{l \in B \setminus A} [L_{ij}, L_{jl}] e_i e_j e_l e_A e_B + \sum_{\{i,j\} \subset A} [L_{ij}, L_{ij}] e_i e_j e_i e_j e_A e_B. \tag{A.5}
\end{aligned}$$

Note that while the last summation obviously vanishes, the second one does also as

$$\begin{aligned}
& \sum_{\{i,j\} \subset A \cap B} \sum_{\{k,l\} \subset (B \cap A) \setminus \{i,j\}} [L_{ij}, L_{kl}] e_i e_j e_k e_l e_A e_B = \sum_{\{i,j,k,l\} \subset A \cap B} ([L_{ij}, L_{kl}] + [L_{ik}, L_{lj}] \\
& \quad + [L_{il}, L_{jk}] + [L_{jk}, L_{il}] + [L_{jl}, L_{ki}] + [L_{kl}, L_{ij}]) e_i e_j e_k e_l e_A e_B.
\end{aligned}$$

By Theorem 2.5 and using $\{e_i, O_j\} = [p_i, x_j] - \delta_{ij}$, the summand of the first summation of (A.5) can be written as

$$\begin{aligned}
& L_{jk} \{e_i, O_l\} e_j e_k e_i e_l e_A e_B + L_{il} \{e_j, O_k\} e_i e_l e_j e_k e_A e_B \\
& + L_{lj} \{e_k, O_i\} e_l e_j e_i e_k e_A e_B + L_{ki} \{e_l, O_j\} e_k e_i e_j e_l e_A e_B.
\end{aligned}$$

Using $\{e_j, O_k\} = \{e_k, O_j\}$ and $\{e_i, O_l\} = \{e_l, O_i\}$ each term is altered to one containing O_b where $b \in B \setminus A$. In this way, we find

$$\begin{aligned} & \epsilon(L_{jk}O_l e_j e_k e_l e_A e_B + L_{lj}O_k e_l e_j e_k e_A e_B + L_{il}O_k e_i e_l e_k e_A e_B + L_{ki}O_l e_k e_i e_l e_A e_B) \\ & + L_{jk}e_i O_l e_j e_k e_i e_l e_A e_B + L_{lj}e_i O_k e_l e_j e_i e_k e_A e_B \\ & + L_{il}e_j O_k e_i e_l e_j e_k e_A e_B + L_{ki}e_j O_l e_k e_i e_j e_l e_A e_B. \end{aligned}$$

The summation, as in (A.5), of the first line reduces to

$$\begin{aligned} & \epsilon \sum_{j \in A} \sum_{k \in B \setminus A} \sum_{l \in (B \setminus A) \setminus \{k\}} \sum_{i \in A \setminus \{j\}} L_{jk} O_l e_j e_k e_l e_A e_B \\ & = \epsilon(|A| - 1) \sum_{j \in A} \sum_{k \in B \setminus A} \sum_{l \in (B \setminus A) \setminus \{k\}} L_{jk} O_l e_j e_k e_l e_A e_B, \end{aligned}$$

while, using Lemma 3.10 for the last line, we find

$$\begin{aligned} & - \sum_{j \in A} \sum_{k \in B \setminus A} \sum_{i \in A \setminus \{j\}} \sum_{l \in (B \setminus A) \setminus \{k\}} L_{jk} e_i O_l e_l e_i e_j e_k e_A e_B \\ & = \sum_{j \in A} \sum_{k \in B \setminus A} \sum_{i \in A \setminus \{j\}} L_{jk} e_i O_i e_i e_j e_k e_A e_B + \sum_{j \in A} \sum_{k \in B \setminus A} \sum_{i \in A \setminus \{j\}} L_{jk} e_i O_j e_j e_i e_j e_k e_A e_B \\ & \quad - \sum_{j \in A} \sum_{k \in B \setminus A} \sum_{i \in A \setminus \{j\}} \sum_{l \in \{i, j\} \cup (B \setminus A) \setminus \{k\}} L_{jk} e_i e_i e_j e_k e_A e_B e_l O_l \\ & = \epsilon \sum_{j \in A} \sum_{k \in B \setminus A} \sum_{i \in A \setminus \{j\}} L_{jk} e_i (O_i e_j - O_j e_i) e_k e_A e_B \\ & \quad - \epsilon \sum_{j \in A} \sum_{k \in B \setminus A} \sum_{i \in A \setminus \{j\}} L_{jk} e_j e_k e_A e_B e_i O_i \\ & \quad - \epsilon(|A| - 1) \sum_{j \in A} \sum_{k \in B \setminus A} \sum_{l \in \{j\} \cup (B \setminus A) \setminus \{k\}} L_{jk} O_l e_l e_j e_k e_A e_B \\ & = -\epsilon \sum_{j \in A} \sum_{k \in B \setminus A} \sum_{i \in A \setminus \{j\}} L_{jk} e_i e_j O_i e_k e_A e_B \\ & \quad - \epsilon(-1)^{|B| - |A|} \sum_{j \in A} \sum_{k \in B \setminus A} \sum_{i \in A \setminus \{j\}} L_{jk} e_j e_k e_i e_A e_B O_i \\ & \quad - \epsilon(|A| - 1) \sum_{j \in A} \sum_{k \in B \setminus A} \sum_{l \in (B \setminus A) \setminus \{k\}} L_{jk} O_l e_l e_j e_k e_A e_B. \end{aligned}$$

In the end, the first summation of (A.5) thus becomes

$$-\epsilon \sum_{j \in A} \sum_{k \in B \setminus A} \sum_{i \in A \setminus \{j\}} L_{jk} e_i e_j O_i e_k e_A e_B - (-1)^{|B| - |A|} \epsilon \sum_{j \in A} \sum_{k \in B} \sum_{i \in A \setminus \{j\}} L_{jk} e_j e_k e_i e_A e_B O_i. \quad (\text{A.6})$$

The final result is obtained following essentially the same strategy as used in the proof of Theorem 3.12 and Theorem 3.14, now applied to the third summation of (A.5):

$$-\epsilon \sum_{i \in A} \sum_{j \in A \setminus \{i\}} \sum_{l \in B \setminus A} [L_{ij}, L_{jl}] e_i e_l e_A e_B.$$

By Theorem 2.5, and using $\{e_i, O_j\} = [p_i, x_j] - \delta_{ij}$, the summand of this can be written as

$$-\epsilon L_{il} e_i e_l e_A e_B - \epsilon L_{il} \{e_j, O_j\} e_i e_l e_A e_B + \epsilon L_{ij} \{e_j, O_l\} e_i e_l e_A e_B - \epsilon L_{lj} \{e_i, O_j\} e_i e_l e_A e_B. \quad (\text{A.7})$$

Here, the summation of the first term cancels out with

$$\left[\epsilon \frac{|A| - 1}{2} e_A - \sum_{\{k,l\} \subset B} L_{kl} e_k e_l e_B \right]_- = \epsilon(|A| - 1) \sum_{k \in A} \sum_{l \in B \setminus A} L_{kl} e_k e_l e_A e_B,$$

which, using property (3.17) and $A \subset B$, follows from

$$\begin{aligned} (-1)^{|A||B| - |A \cap B|} e_k e_l e_B e_A &= (-1)^{|A||B| - |A| + |A||B \setminus \{k,l\}| - |A \cap (B \setminus \{k,l\})|} e_A e_k e_l e_B \\ &= (-1)^{|A| - |A \cap (B \setminus \{k,l\})|} e_A e_k e_l e_B. \end{aligned}$$

For the summation of the second and the fourth term of (A.7), using $O_i e_j - O_j e_i = e_i O_j - e_j O_i$, we have

$$\begin{aligned} & - \sum_{i \in A} \sum_{j \in A \setminus \{i\}} \sum_{l \in B \setminus A} (\epsilon L_{il} O_j e_j e_i + L_{lj} O_j) e_l e_A e_B \\ & - \epsilon \sum_{i \in A} \sum_{j \in A \setminus \{i\}} \sum_{l \in B \setminus A} (L_{il} e_j + L_{lj} e_i) O_j e_i e_l e_A e_B \\ &= - \sum_{\{i,j\} \in A} \sum_{l \in B \setminus A} (\epsilon L_{il} O_j e_j e_i + L_{lj} O_j + \epsilon L_{jl} O_i e_i e_j + L_{li} O_i) e_l e_A e_B \\ & - \epsilon \sum_{\{i,j\} \in A} \sum_{l \in B \setminus A} (L_{il} e_j + L_{lj} e_i) (O_j e_i - O_i e_j) e_l e_A e_B. \\ &= - \epsilon \sum_{\{i,j\} \in A} \sum_{l \in B \setminus A} (L_{il} O_j e_j e_i - L_{lj} O_i e_i e_j - L_{il} e_j e_i O_j + L_{lj} e_i e_j O_i) e_l e_A e_B \\ &= \epsilon \sum_{i \in A} \sum_{j \in A \setminus \{i\}} \sum_{l \in B \setminus A} L_{lj} (O_i e_i e_j - e_i e_j O_i) e_l e_A e_B. \end{aligned} \tag{A.8}$$

When summed over l in $B \setminus A$, the third term of (A.7) yields, using Lemma 3.10,

$$\begin{aligned} & - \epsilon \sum_{l \in B \setminus A} L_{ij} e_j O_l e_l e_i e_A e_B + \epsilon \sum_{l \in B \setminus A} L_{ij} O_l e_l e_j e_i e_A e_B \\ &= \epsilon L_{ij} e_j O_i e_i e_A e_B - \epsilon \sum_{l \in B \setminus A \cup \{i\}} L_{ij} e_j e_i e_A e_B e_l O_l \\ & - \epsilon L_{ij} O_i e_i e_j e_A e_B - \epsilon L_{ij} O_j e_j e_i e_A e_B + \epsilon \sum_{l \in B \setminus A \cup \{i,j\}} L_{ij} e_j e_i e_A e_B e_l O_l \\ &= L_{ij} e_i O_j e_A e_B + (-1)^{|B| - |A| + 1} L_{ij} e_i e_A e_B O_j. \end{aligned}$$

Summing this over i in A and j in $A \setminus \{i\}$ and combined with (A.2), (A.3), (A.6), and (A.8), one observes that all terms cancel out. \square

Proof of Theorem 3.16. By definition, plugging in (3.10), the left-hand side expands to

$$\begin{aligned} \llbracket O_A, O_B \rrbracket_+ &= O_A O_B + (-1)^{|A||B| - |A \cap B|} O_B O_A \\ &= \frac{1}{4} (\underline{D} \underline{x}_A e_A - e_A \underline{x}_A \underline{D} - \epsilon e_A) (\underline{D} \underline{x}_B e_B - e_B \underline{x}_B \underline{D} - \epsilon e_B) \\ & + (-1)^{|A||B| - |A \cap B|} \frac{1}{4} (\underline{D} \underline{x}_B e_B - e_B \underline{x}_B \underline{D} - \epsilon e_B) (\underline{D} \underline{x}_A e_A - e_A \underline{x}_A \underline{D} - \epsilon e_A). \end{aligned}$$

The idea of the proof is now as follows. We split up \underline{x}_A and \underline{x}_B into $\underline{x}_A = \underline{x}_{A \setminus B} + \underline{x}_{A \cap B}$ and $\underline{x}_B = \underline{x}_{B \setminus A} + \underline{x}_{A \cap B}$. We then combine the appropriate terms to make $\underline{x}_{A \triangle B} = \underline{x}_{A \setminus B} + \underline{x}_{B \setminus A}$ and $\underline{x}_{A \cup B} = \underline{x}_{A \setminus B} + \underline{x}_{B \setminus A} + \underline{x}_{A \cap B}$, and in turn all terms that make up the right-hand side.

In doing so, we continually make use of the following facts: by property (3.18) we have $(e_{A \cap B})^4 = 1$, hence $(e_{A \cap B})^2$ is just a sign;

$$\underline{D}e_A - (-1)^{|A|}e_A\underline{D} = \sum_{a \in A} 2p_a e_a e_A;$$

and for integer n one has $(-1)^n = (-1)^{n^2}$ and $(-1)^{n(n+1)} = 1$.

For the terms of $\llbracket O_A, O_B \rrbracket_+$ which do not contain \underline{D} , we have

$$\begin{aligned} \frac{1}{4}e_A e_B + (-1)^{|A||B| - |A \cap B|} \frac{1}{4}e_B e_A &= \frac{1}{2}e_A e_B \\ &= -\frac{1}{2}e_A e_B + \frac{1}{2}(e_{A \cap B})^4 e_A e_B + \frac{1}{2}(e_{A \cap B})^4 e_A e_B. \end{aligned}$$

Next, for the terms of $\llbracket O_A, O_B \rrbracket_+$ containing two occurrences of \underline{D} , we have

$$\begin{aligned} &(\underline{D} \underline{x}_A e_A - e_A \underline{x}_A \underline{D})(\underline{D} \underline{x}_B e_B - e_B \underline{x}_B \underline{D}) \\ &+ (-1)^{|A||B| - |A \cap B|} (\underline{D} \underline{x}_B e_B - e_B \underline{x}_B \underline{D})(\underline{D} \underline{x}_A e_A - e_A \underline{x}_A \underline{D}) \\ &= \underline{D} \underline{x}_A e_A \underline{D} \underline{x}_B e_B - e_A \underline{x}_A \underline{D} \underline{D} \underline{x}_B e_B - \underline{D} \underline{x}_A e_A e_B \underline{x}_B \underline{D} + e_A \underline{x}_A \underline{D} e_B \underline{x}_B \underline{D} \\ &+ (-1)^{|A||B| - |A \cap B|} (\underline{D} \underline{x}_B e_B \underline{D} \underline{x}_A e_A - e_B \underline{x}_B \underline{D} \underline{D} \underline{x}_A e_A - \underline{D} \underline{x}_B e_B e_A \underline{x}_A \underline{D} \\ &+ e_B \underline{x}_B \underline{D} e_A \underline{x}_A \underline{D}). \end{aligned}$$

We first look at the terms having $\underline{D} \underline{D}$ in the middle:

$$\begin{aligned} &-e_A \underline{x}_A \underline{D} \underline{D} \underline{x}_B e_B - (-1)^{|A||B| - |A \cap B|} e_B \underline{x}_B \underline{D} \underline{D} \underline{x}_A e_A \\ &= -e_A (\underline{x}_{A \setminus B} + \underline{x}_{A \cap B}) \underline{D} \underline{D} (\underline{x}_{B \setminus A} + \underline{x}_{A \cap B}) e_B \\ &- (-1)^{|A||B| - |A \cap B|} e_B (\underline{x}_{B \setminus A} + \underline{x}_{A \cap B}) \underline{D} \underline{D} (\underline{x}_{A \setminus B} + \underline{x}_{A \cap B}) e_A \\ &= -(e_{A \cap B})^2 (e_A e_{A \cap B} \underline{x}_{A \setminus B} \underline{D} \underline{D} \underline{x}_{B \setminus A} e_{A \cap B} e_B \\ &- (-1)^{|A||B| - |A \cap B| + |A \cap B|(|A| + |B|)} e_{A \cap B} e_B \underline{x}_{B \setminus A} \underline{D} \underline{D} \underline{x}_{A \setminus B} e_A e_{A \cap B} \\ &- (-1)^{1 + |A \cap B|(|A| + |B| - 2|A \cap B| + 1) + |A \cap B| - 1} e_{A \cap B} e_A e_B \underline{x}_{A \setminus B} \underline{D} \underline{D} \underline{x}_{A \cap B} e_{A \cap B} \\ &- e_{A \cap B} \underline{x}_{A \cap B} \underline{D} \underline{D} \underline{x}_{B \setminus A} e_{A \cap B} e_A e_B - e_{A \cap B} \underline{x}_{A \cap B} \underline{D} \underline{D} \underline{x}_{A \cap B} e_{A \cap B} e_A e_B \\ &- (-1)^{1 + |A \cap B|(|A| + |B| - 2|A \cap B| + 1) + |A \cap B| - 1} e_{A \cap B} e_A e_B \underline{x}_{B \setminus A} \underline{D} \underline{D} \underline{x}_{A \cap B} e_{A \cap B} \\ &- e_{A \cap B} \underline{x}_{A \cap B} \underline{D} \underline{D} \underline{x}_{A \setminus B} e_{A \cap B} e_A e_B \\ &- (-1)^{|A \cap B|(|A| + |B| - 2|A \cap B|)} e_{A \cap B} e_A e_B \underline{x}_{A \cap B} \underline{D} \underline{D} \underline{x}_{A \cap B} e_{A \cap B}) \\ &= -(e_{A \cap B})^2 (e_A e_{A \cap B} \underline{x}_{A \setminus B} \underline{D} \underline{D} \underline{x}_{B \setminus A} e_{A \cap B} e_B \\ &- (-1)^{(|A| - |A \cap B|)(|B| - |A \cap B|)} e_{A \cap B} e_B \underline{x}_{B \setminus A} \underline{D} \underline{D} \underline{x}_{A \setminus B} e_A e_{A \cap B} \\ &- e_{A \cap B} \underline{x}_{A \cap B} \underline{D} \underline{D} \underline{x}_{A \cup B} e_{A \cap B} e_A e_B \\ &- (-1)^{|A \cap B|(|A| + |B| - |A \cap B|) - |A \cap B|} e_{A \cap B} e_A e_B \underline{x}_{A \cup B} \underline{D} \underline{D} \underline{x}_{A \cap B} e_{A \cap B}). \end{aligned}$$

In exactly the same manner, we have

$$\begin{aligned} &\underline{D} \underline{x}_A e_A e_B \underline{x}_B \underline{D} + (-1)^{|A||B| - |A \cap B|} \underline{D} \underline{x}_B e_B e_A \underline{x}_A \underline{D} \\ &= \underline{D} (\underline{x}_{A \setminus B} + \underline{x}_{A \cap B}) e_A e_B (\underline{x}_{B \setminus A} + \underline{x}_{A \cap B}) \underline{D} \\ &+ (-1)^{|A||B| - |A \cap B|} \underline{D} (\underline{x}_{B \setminus A} + \underline{x}_{A \cap B}) e_B e_A (\underline{x}_{A \setminus B} + \underline{x}_{A \cap B}) \underline{D} \\ &= (e_{A \cap B})^2 (\underline{D} \underline{x}_{A \setminus B} e_A (e_{A \cap B})^2 e_B \underline{x}_{B \setminus A} \underline{D} - \underline{D} \underline{x}_{A \cap B} (e_{A \cap B})^2 e_A e_B \underline{x}_{A \cup B} \underline{D} \\ &- (-1)^{(|A| - |A \cap B|)(|B| - |A \cap B|)} \underline{D} \underline{x}_{B \setminus A} e_{A \cap B} e_B e_A e_{A \cap B} \underline{x}_{A \setminus B} \underline{D} \\ &- (-1)^{|A \cap B|(|A| + |B| - |A \cap B|) - |A \cap B|} \underline{D} \underline{x}_{A \cup B} e_{A \cap B} e_A e_B e_{A \cap B} \underline{x}_{A \cap B} \underline{D}) \end{aligned}$$

Next, for the remaining terms with two occurrences of \underline{D} , we first have

$$\begin{aligned}
& \underline{D} \underline{x}_A e_A \underline{D} \underline{x}_B e_B + (-1)^{|A||B|-|A \cap B|} \underline{D} \underline{x}_B e_B \underline{D} \underline{x}_A e_A \\
&= \underline{D} (\underline{x}_{A \setminus B} + \underline{x}_{A \cap B}) e_A \underline{D} (\underline{x}_{B \setminus A} + \underline{x}_{A \cap B}) e_B \\
&\quad + (-1)^{|A||B|-|A \cap B|} \underline{D} (\underline{x}_{B \setminus A} + \underline{x}_{A \cap B}) e_B \underline{D} (\underline{x}_{A \setminus B} + \underline{x}_{A \cap B}) e_A \\
&= (e_{A \cap B})^2 (\underline{D} \underline{x}_{A \setminus B} e_A e_{A \cap B} \underline{D} \underline{x}_{B \setminus A} e_{A \cap B} e_B \\
&\quad + (-1)^{|A \cap B|(|A|+|B|-|A \cap B|-1)} \underline{D} \underline{x}_{A \setminus B} e_{A \cap B} e_A e_B \underline{D} \underline{x}_{A \cap B} e_{A \cap B} \\
&\quad + \underline{D} \underline{x}_{A \cap B} e_{A \cap B} \underline{D} \underline{x}_{B \setminus A} e_{A \cap B} e_A e_B + \underline{D} \underline{x}_{A \cap B} e_{A \cap B} \underline{D} \underline{x}_{A \cap B} e_{A \cap B} e_A e_B \\
&\quad + (-1)^{|A||B|-|A \cap B|+|A \cap B|(|B|+|A|)} \underline{D} \underline{x}_{B \setminus A} e_{A \cap B} e_B \underline{D} \underline{x}_{A \setminus B} e_A e_{A \cap B} \\
&\quad + (-1)^{|A \cap B|(|A|+|B|)} \underline{D} \underline{x}_{B \setminus A} e_{A \cap B} e_A e_B \underline{D} \underline{x}_{A \cap B} e_{A \cap B} \\
&\quad + \underline{D} \underline{x}_{A \cap B} e_{A \cap B} \underline{D} \underline{x}_{A \setminus B} e_{A \cap B} e_A e_B \\
&\quad + (-1)^{|A \cap B|(|A|+|B|)} \underline{D} \underline{x}_{A \cap B} e_{A \cap B} e_A e_B \underline{D} \underline{x}_{A \cap B} e_{A \cap B}) \\
&\quad - \underline{D} \underline{x}_{A \setminus B} \sum_{l \in A \cap B} 2p_l e_l \underline{x}_{B \setminus A} e_A e_B - \underline{D} \underline{x}_{A \setminus B} \sum_{l \in B \setminus A} 2p_l e_l \underline{x}_{A \cap B} e_A e_B \\
&\quad - \underline{D} \underline{x}_{A \cap B} \sum_{l \in A \setminus B} 2p_l e_l \underline{x}_{B \setminus A} e_A e_B + \underline{D} \underline{x}_{A \cap B} \sum_{l \in A \setminus B} 2p_l e_l \underline{x}_{A \cap B} e_A e_B \\
&\quad - \underline{D} \underline{x}_{B \setminus A} \sum_{l \in A \cap B} 2p_l e_l \underline{x}_{A \setminus B} e_A e_B - \underline{D} \underline{x}_{B \setminus A} \sum_{l \in A \setminus B} 2p_l e_l \underline{x}_{A \cap B} e_A e_B \\
&\quad - \underline{D} \underline{x}_{A \cap B} \sum_{l \in B \setminus A} 2p_l e_l \underline{x}_{A \setminus B} e_A e_B - \underline{D} \underline{x}_{A \cap B} \sum_{l \in A \setminus B} 2p_l e_l \underline{x}_{A \cap B} e_A e_B \\
&= (e_{A \cap B})^2 (\underline{D} \underline{x}_{A \setminus B} e_A e_{A \cap B} \underline{D} \underline{x}_{B \setminus A} e_{A \cap B} e_B + \underline{D} \underline{x}_{A \cap B} e_{A \cap B} \underline{D} \underline{x}_{A \cup B} e_{A \cap B} e_A e_B \\
&\quad + (-1)^{(|A|-|A \cap B|)(|B|-|A \cap B|)} \underline{D} \underline{x}_{B \setminus A} e_{A \cap B} e_B \underline{D} \underline{x}_{A \setminus B} e_A e_{A \cap B} \\
&\quad + (-1)^{|A \cap B|(|A|+|B|-|A \cap B|)-|A \cap B|} \underline{D} \underline{x}_{A \cup B} e_{A \cap B} e_A e_B \underline{D} \underline{x}_{A \cap B} e_{A \cap B}) \\
&\quad - 2\underline{D} \sum_{a \in A \setminus B} \sum_{c \in A \cap B} \sum_{b \in B \setminus A} (-x_a(p_c x_b - p_b x_c) + x_c(p_a x_b - p_b x_a) \\
&\quad \quad \quad + x_b(p_c x_a - p_a x_c)) e_a e_b e_c e_A e_B
\end{aligned}$$

where the last two lines vanish using $L_{ij} = x_i p_j - x_j p_i = p_j x_i - p_i x_j$. Similarly one finds

$$\begin{aligned}
& e_A \underline{x}_A \underline{D} e_B \underline{x}_B \underline{D} + (-1)^{|A||B|-|A \cap B|} e_B \underline{x}_B \underline{D} e_A \underline{x}_A \underline{D} \\
&= e_A (\underline{x}_{A \setminus B} + \underline{x}_{A \cap B}) \underline{D} e_B (\underline{x}_{B \setminus A} + \underline{x}_{A \cap B}) \underline{D} \\
&\quad + (-1)^{|A||B|-|A \cap B|} e_B (\underline{x}_{B \setminus A} + \underline{x}_{A \cap B}) \underline{D} e_A (\underline{x}_{A \setminus B} + \underline{x}_{A \cap B}) \underline{D} \\
&= (e_{A \cap B})^2 (e_A e_{A \cap B} \underline{x}_{A \setminus B} \underline{D} e_{A \cap B} e_B \underline{x}_{B \setminus A} \underline{D} + e_{A \cap B} \underline{x}_{A \cap B} \underline{D} e_{A \cap B} e_A e_B \underline{x}_{A \cup B} \underline{D} \\
&\quad + (-1)^{(|A|-|A \cap B|)(|B|-|A \cap B|)} e_{A \cap B} e_B \underline{x}_{B \setminus A} \underline{D} e_A e_{A \cap B} \underline{x}_{A \setminus B} \underline{D} \\
&\quad + (-1)^{|A \cap B|(|A|+|B|-|A \cap B|)-|A \cap B|} e_{A \cap B} e_A e_B \underline{x}_{A \cup B} \underline{D} e_{A \cap B} \underline{x}_{A \cap B} \underline{D}).
\end{aligned}$$

Finally, for the terms with a single occurrence of \underline{D} , we have (up to a factor $-\epsilon$)

$$\begin{aligned}
& (\underline{D} \underline{x}_A e_A - e_A \underline{x}_A \underline{D}) e_B + (-1)^{|A||B|-|A \cap B|} e_B (\underline{D} \underline{x}_A e_A - e_A \underline{x}_A \underline{D}) \\
&\quad + e_A (\underline{D} \underline{x}_B e_B - e_B \underline{x}_B \underline{D}) + (-1)^{|A||B|-|A \cap B|} (\underline{D} \underline{x}_B e_B - e_B \underline{x}_B \underline{D}) e_A \\
&= -\underline{D} \underline{x}_{A \setminus B} e_A e_B - \underline{D} \underline{x}_{B \setminus A} e_A e_B + e_A e_B \underline{x}_{A \setminus B} \underline{D} + e_A e_B \underline{x}_{B \setminus A} \underline{D} \\
&\quad + (e_{A \cap B})^2 (\underline{D} \underline{x}_{A \setminus B} e_A (e_{A \cap B})^2 e_B + (-1)^{|A \cap B|(|A|+|B|)} \underline{D} \underline{x}_{A \setminus B} e_{A \cap B} e_A e_B e_{A \cap B})
\end{aligned}$$

$$\begin{aligned}
& + \underline{D} \underline{x}_{A \cap B} (e_{A \cap B})^2 e_A e_B - e_A e_{A \cap B} \underline{x}_{A \setminus B} \underline{D} e_{A \cap B} e_B - e_{A \cap B} \underline{x}_{A \cap B} \underline{D} e_{A \cap B} e_A e_B \\
& + (-1)^{|A||B|+|A \cap B|(|B|+|A|-1)} e_{A \cap B} e_B \underline{D} \underline{x}_{A \setminus B} e_A e_{A \cap B} \\
& + (-1)^{|A \cap B|(|A|+|B|)} e_{A \cap B} e_A e_B \underline{D} \underline{x}_{A \cap B} e_{A \cap B} \\
& - (-1)^{|A \cap B|(|A|+|B|)} e_{A \cap B} e_A e_B \underline{x}_{A \setminus B} \underline{D} e_{A \cap B} \\
& - (-1)^{|A||B|+|A \cap B|(|B|+|A|-1)} e_{A \cap B} e_B e_A e_{A \cap B} \underline{x}_{A \setminus B} \underline{D} \\
& - (e_{A \cap B})^2 e_A e_B \underline{x}_{A \cap B} \underline{D} + e_A e_{A \cap B} \underline{D} \underline{x}_{B \setminus A} e_{A \cap B} e_B + e_{A \cap B} \underline{D} \underline{x}_{A \cap B} e_{A \cap B} e_A e_B \\
& - e_A (e_{A \cap B})^2 e_B \underline{x}_{B \setminus A} \underline{D} - (e_{A \cap B})^2 e_A e_B \underline{x}_{B \setminus A} \underline{D} \\
& - (-1)^{|A \cap B|(|A|+|B|)} e_{A \cap B} e_A e_B e_{A \cap B} \underline{x}_{A \cap B} \underline{D} \\
& + (-1)^{|A \cap B|(|A|+|B|)} \underline{D} \underline{x}_{B \setminus A} e_{A \cap B} e_A e_B e_{A \cap B} \\
& + (-1)^{|A||B|+|A \cap B|(|A|+|B|-1)} \underline{D} \underline{x}_{B \setminus A} e_{A \cap B} e_B e_A e_{A \cap B} \\
& + (-1)^{|A \cap B|(|A|+|B|)} \underline{D} \underline{x}_{A \cap B} e_{A \cap B} e_A e_B e_{A \cap B} \\
& - (-1)^{|A||B|+|A \cap B|(|B|+|A|-1)} e_{A \cap B} e_B \underline{x}_{B \setminus A} \underline{D} e_A e_{A \cap B} \\
& - (-1)^{|A \cap B|(|A|+|B|)} e_{A \cap B} e_A e_B \underline{x}_{A \cap B} \underline{D} e_{A \cap B},
\end{aligned}$$

where, for instance, we made use of the following computation

$$\begin{aligned}
& - e_A \underline{x}_{A \cap B} \underline{D} e_B - (-1)^{|A||B|-|A \cap B|} e_B \underline{x}_{A \cap B} \underline{D} e_A \\
& = - (e_{A \cap B})^2 ((e_{A \cap B})^2 e_A \underline{x}_{A \cap B} \underline{D} e_B + (-1)^{|A||B|-|A \cap B|} e_B \underline{x}_{A \cap B} \underline{D} (e_{A \cap B})^2 e_A) \\
& = - (e_{A \cap B})^2 ((-1)^{|A|-|A \cap B|} e_{A \cap B} \underline{x}_{A \cap B} e_{A \cap B} e_A \underline{D} e_B \\
& \quad + (-1)^{|A||B|+|A \cap B||A|} e_B \underline{x}_{A \cap B} \underline{D} e_{A \cap B} e_A e_{A \cap B}) \\
& = - (e_{A \cap B})^2 (e_{A \cap B} \underline{x}_{A \cap B} \underline{D} e_{A \cap B} e_A e_B \\
& \quad + (-1)^{|A||B|+|A \cap B||A|+|A|-|A \cap B|} e_B \underline{x}_{A \cap B} e_{A \cap B} e_A \underline{D} e_{A \cap B} \\
& \quad - e_{A \cap B} \underline{x}_{A \cap B} \sum_{l \in A \setminus B} 2p_l e_l e_{A \cap B} e_A e_B \\
& \quad + (-1)^{|A||B|+|A \cap B||A|} e_B \underline{x}_{A \cap B} \sum_{l \in A \setminus B} 2p_l e_l e_{A \cap B} e_A e_{A \cap B}) \\
& = (e_{A \cap B})^2 (- e_{A \cap B} \underline{x}_{A \cap B} \underline{D} e_{A \cap B} e_A e_B - (-1)^{|A \cap B|(|A|+|B|)} e_{A \cap B} e_A e_B \underline{x}_{A \cap B} \underline{D} e_{A \cap B}).
\end{aligned}$$

Putting everything together and comparing with

$$O_{A,B} = \frac{1}{2} (\underline{D} \underline{x}_{A \triangle B} e_A e_B - e_A e_B \underline{x}_{A \triangle B} \underline{D} - \epsilon e_A e_B),$$

and

$$\begin{aligned}
\llbracket O_{A,(A \cap B)}, O_{(A \cap B),B} \rrbracket_+ & = \frac{1}{4} \left((\underline{D} \underline{x}_{A \setminus B} e_A e_{A \cap B} - e_A e_{A \cap B} \underline{x}_{A \setminus B} \underline{D} - \epsilon e_A e_{A \cap B}) \right. \\
& \quad \times (\underline{D} \underline{x}_{B \setminus A} e_{A \cap B} e_B - e_{A \cap B} e_B \underline{x}_{B \setminus A} \underline{D} - \epsilon e_{A \cap B} e_B) \\
& \quad + (-1)^{(|A|-|A \cap B|)(|B|-|A \cap B|)} \\
& \quad \times (\underline{D} \underline{x}_{B \setminus A} e_{A \cap B} e_B - e_{A \cap B} e_B \underline{x}_{B \setminus A} \underline{D} - \epsilon e_{A \cap B} e_B) \\
& \quad \left. \times (\underline{D} \underline{x}_{A \setminus B} e_A e_{A \cap B} - e_A e_{A \cap B} \underline{x}_{A \setminus B} \underline{D} - \epsilon e_A e_{A \cap B}) \right),
\end{aligned}$$

and

$$\begin{aligned} \llbracket O_{A \cap B}, O_{(A \cap B), A, B} \rrbracket_+ = & \frac{1}{4} \left((D \underline{x}_{A \cap B} e_{A \cap B} - e_{A \cap B} \underline{x}_{A \cap B} D - \epsilon e_{A \cap B}) \right. \\ & \times (\underline{D} \underline{x}_{A \cup B} e_{A \cap B} e_{A \cap B} - e_{A \cap B} e_{A \cap B} \underline{x}_{A \cup B} \underline{D} - \epsilon e_{A \cap B} e_{A \cap B}) \\ & + (-1)^{|A \cap B|(|A|+|B|-|A \cap B|)-|A \cap B|} \\ & \times (\underline{D} \underline{x}_{A \cup B} e_{A \cap B} e_{A \cap B} - e_{A \cap B} e_{A \cap B} \underline{x}_{A \cup B} \underline{D} - \epsilon e_{A \cap B} e_{A \cap B}) \\ & \left. \times (\underline{D} \underline{x}_{A \cap B} e_{A \cap B} - e_{A \cap B} \underline{x}_{A \cap B} D - \epsilon e_{A \cap B}) \right), \end{aligned}$$

the proof is completed. \square

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