Clebsch-Gordan Coefficients for Covariant Representations of the Lie superalgebra $\mathfrak{gl}(n|n)$ in odd Gelfand-Zetlin basis

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Abstract. Clebsch-Gordan coefficients corresponding to the tensor product of the natural representation $V([1,0,\ldots,0])$ of $\mathfrak{gl}(n|n)$ with highest weight (1,0,...,0) with any covariant tensor representation of $\mathfrak{gl}(n|n)$ in the so called odd Gelfand-Zetlin basis are computed. The result is a step for the explicit construction of the parastatistics Fock space.

INTRODUCTION

In 1953, Green [1] introduced parabosons and parafermions as a natural generalization of bosons and fermions. Three decades later Palev [2] proved that the defining triple relations of k parafermions $f_j^{\pm}$ and n parabosons $b_j^{\pm}$ with so-called relative parafermion relations can be considered as relations which define the orthosymplectic Lie superalgebra $\mathfrak{osp}(2k+1|2n)$. And it took another three decades before the explicit construction of the corresponding parastatistics Fock space [3]. The latter is an infinite-dimensional unitary representation of $\mathfrak{osp}(2k+1|2n)$ and the branching $\mathfrak{B}(k|n) \equiv \mathfrak{osp}(2k+1|2n) \supset \mathfrak{gl}(k|n)$ is important for its construction. The states of the parastatistics Fock space are labeled by Gelfand-Zetlin (GZ) basis of covariant tensor representations of the subalgebra $\mathfrak{gl}(k|n)$.

The most interesting is however to investigate a mixed system of an infinite number of parafermions and parabosons ($k \to \infty$ and $n \to \infty$). Then the parastatistics Fock space would be a unitary representation of the infinite rank Lie superalgebra $\mathfrak{B}(\infty|\infty)$. The natural GZ-basis for $\mathfrak{gl}(k|n)$ [4] based on the decomposition

$$\mathfrak{gl}(k|n) \supset \mathfrak{gl}(k|n-1) \supset \mathfrak{gl}(k|n-2) \supset \cdots \supset \mathfrak{gl}(k|1) \supset \mathfrak{gl}(k) \supset \mathfrak{gl}(k-1) \supset \cdots \supset \mathfrak{gl}(2) \supset \mathfrak{gl}(1). \tag{1}$$

turned out to be not appropriate for generalization to $\mathfrak{gl}(\infty|\infty)$. Because of this problem, a new basis following another chain of subalgebras, namely

$$\mathfrak{gl}(n|n) \supset \mathfrak{gl}(n|n-1) \supset \mathfrak{gl}(n-1|n-1) \supset \mathfrak{gl}(n-1|n-2) \supset \mathfrak{gl}(n-2|n-2) \supset \cdots \supset \mathfrak{gl}(1|1) \supset \mathfrak{gl}(1) \tag{2}$$

and called odd Gelfand-Zetlin basis was introduced [5]. Since the defining relations of parabosons and parafermions imply that the set $(f_1^+, \ldots, f_n^+, b_1^+, \ldots, b_n^+)$ is a standard $\mathfrak{gl}(n|n)$ tensor of rank $(1,0,\ldots,0)$, for the construction of the parastatistics Fock space one needs the $\mathfrak{gl}(n|n)$ Clebsch-Gordan coefficients corresponding to the tensor product $V([1,0,\ldots,0]) \otimes V([m]^{2n})$, where $V([1,0,\ldots,0])$ is the $2n$-dimensional representation of $\mathfrak{gl}(n|n)$ with highest weight $(1,0,\ldots,0)$ and $V([m]^{2n})$ is any $\mathfrak{gl}(n|n)$ irreducible covariant tensor module. This paper deals with this problem. In the next section we define the Lie superalgebra $\mathfrak{gl}(n|n)$ and provide the reader with the action of the $\mathfrak{gl}(n|n)$ generators on the odd Gelfand-Zetlin basis for the covariant tensor representations. In the last section we compute the Clebsch-Gordan coefficients.

THE LIE SUPERALGEBRA $\mathfrak{gl}(n|n)$ AND ITS COVARIANT REPRESENTATIONS

The vector space for the Lie superalgebra $\mathfrak{gl}(n|n)$ consists of the space of $(2n \times 2n)$-matrices and the Lie superalgebra $\mathfrak{gl}(n|n)$ can be defined [6, 7] through the matrix realization

$$\mathfrak{gl}(n|n) = \{x = \begin{pmatrix} A & B \\ C & D \end{pmatrix} | A, B, C, D \in M_{n\times n}\}, \tag{3}$$

where $M_{n\times n}$ is the space of all $n \times n$ complex matrices. The even subalgebra $\mathfrak{gl}(n|n)_{\bar 0}$ has $B = 0$ and $C = 0$; the odd subspace $\mathfrak{gl}(n|n)_{\bar 1}$ has $A = 0$ and $D = 0$.

We will use the ordered set $\{-n, \dots, -2, -1; 1, 2, \dots n\}$ as indices for the rows and columns of the matrices in (3) throughout the paper and sometimes it will be convenient to write the minus sign of an index as an overlined number. So with this convention the indices $\dots, \bar 3, \bar 2, \bar 1; 1, 2, 3, \dots$ stand for $\dots, -3, -2, -1; 1, 2, 3 \dots$.

A basis for $\mathfrak{gl}(n|n)$ consists of matrices E_{ij} $(i, j = -n, \dots, -2, -1; 1, 2, \dots, n)$, with entry 1 at position (i, j) and 0 elsewhere. The bracket for the basis elements is given by

$$[\![E_{ab}, E_{cd}]\!] = \delta_{bc}E_{ad} - (-1)^{\deg(E_{ab})\deg(E_{cd})}\delta_{ad}E_{cb}. \tag{4}$$

A Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is spanned by the elements E_{jj} $(j = -n, \dots, -2, -1; 1, 2, \dots, n)$. The space dual to $\mathfrak{h}$ is $\mathfrak{h}^*$ and is described by the forms ϵ_i $(i = -n, \dots, -2, -1; 1, 2, \dots, n)$ where $\epsilon_i : x \to A_{ii}$ $(i = -n, -n+1, \dots, -1)$ and $\epsilon_i : x \to D_{ii}$ $(i = 1, 2, \dots, n)$, and where x is given as in (3). The components of an element $\Lambda \in \mathfrak{h}^*$ will be written as

$$\begin{aligned} [m]^{2n} &= [m_{-n,2n}, \dots, m_{-2,2n}, m_{-1,2n}; m_{1,2n}, m_{2,2n}, \dots, m_{n,2n}] \\ &= [m_{\bar n,2n}, \dots, m_{\bar 2,2n}, m_{\bar 1,2n}; m_{1,2n}, m_{2,2n}, \dots, m_{n,2n}], \end{aligned} \tag{5}$$

where

$$\Lambda = \sum_{i=-n(i\neq 0)}^{n} m_{i,2n}\epsilon_i, \tag{6}$$

$m_{i,2n}$ are complex numbers and are called the weights.

The covariant tensor $\mathfrak{gl}(n|n)$ representations are finite-dimensional, irreducible and unitary highest weight representations. Any such $\mathfrak{gl}(n|n)$ module $V([m]^{2n})$ [8, 9, 10] is determined by its highest weight, given by the nonnegative integer $2n$-tuple

$$[m]^{2n} = [m_{-n,2n}, \dots, m_{-2,2n}, m_{-1,2n}; m_{1,2n}, m_{2,2n}, \dots, m_{n,2n}], \tag{7}$$

such that

$$m_{i,2n} - m_{i+1,2n} \in \mathbb{Z}_+, \quad i = -n, \dots, -2; 1, \dots, n-1 \tag{8}$$

and

$$m_{-1,2n} \geq \#\{i : m_{i,2n} > 0, \ 1 \leq i \leq n\}. \tag{9}$$

Within a given $\mathfrak{gl}(n|n)$ module $V([m]^{2n})$ the numbers (7) are fixed.

In[5] it was proved that the set of vectors

$$|m)^{2n} = \left| \begin{array}{ccccc|cccccc}
m_{\bar n,2n} & m_{\overline{n-1},2n} & \cdots & m_{\bar 2,2n} & m_{\bar 1,2n} & m_{1,2n} & m_{2,2n} & \cdots & m_{n-2,2n} & m_{n-1,2n} & m_{n,2n} \\
\uparrow & \uparrow & \cdots & \uparrow & \uparrow & & & & & & \\
m_{\bar n,2n-1} & m_{\overline{n-1},2n-1} & \cdots & m_{\bar 2,2n-1} & m_{\bar 1,2n-1} & m_{1,2n-1} & m_{2,2n-1} & \cdots & m_{n-2,2n-1} & m_{n-1,2n-1} & \\
 & & & & & \downarrow & \downarrow & \cdots & \downarrow & \downarrow & \\
 & m_{\overline{n-1},2n-2} & \cdots & m_{\bar 2,2n-2} & m_{\bar 1,2n-2} & m_{1,2n-2} & m_{2,2n-2} & \cdots & m_{n-2,2n-2} & m_{n-1,2n-2} & \\
 & \uparrow & \cdots & \uparrow & \uparrow & & & & & & \\
 & m_{\overline{n-1},2n-3} & \cdots & m_{\bar 2,2n-3} & m_{\bar 1,2n-3} & m_{1,2n-3} & m_{2,2n-3} & \cdots & m_{n-2,2n-3} & & \\
 & & \ddots & \vdots & \vdots & \vdots & \vdots & \cdot^{\cdot^{\cdot}} & & & \\
 & & & m_{\bar 2 4} & m_{\bar 1 4} & m_{14} & m_{24} & & & & \\
 & & & \uparrow & \uparrow & & & & & & \\
 & & & m_{\bar 2 3} & m_{\bar 1 3} & m_{13} & & & & & \\
 & & & & & \downarrow & & & & & \\
 & & & & m_{\bar 1 2} & m_{12} & & & & & \\
 & & & & \uparrow & & & & & & \\
 & & & & m_{\bar 1 1} & & & & & &
\end{array} \right) \tag{10}$$

satisfying the conditions

$$
\begin{array}{ll}
1. & m_{i,2n}\in\mathbb{Z}_+ \text{ are fixed and } m_{j,2n}-m_{j+1,2n}\in\mathbb{Z}_+,\ j\neq -1,0,\ -n\le j\le n-1,\\
 & m_{-1,2n}\ge \#\{i: m_{i,2n}>0,\ 1\le i\le n\};\\
2. & m_{i,2p}-m_{i,2p-1}\equiv\theta_{i,2p-1}\in\{0,1\},\quad 1\le p\le n;\ -p\le i\le -1;\\
3. & m_{i,2p}-m_{i,2p+1}\equiv\theta_{i,2p}\in\{0,1\},\quad 1\le p\le n-1;\ 1\le i\le p;\\
4. & m_{-1,2p}\ge \#\{i: m_{i,2p}>0,\ 1\le p\le n,\ 1\le i\le p\};\\
5. & m_{-1,2p-1}\ge \#\{i: m_{i,2p-1}>0,\ 2\le p\le n,\ 1\le i\le p-1\};\\
6. & m_{i,2p}-m_{i,2p-1}\in\mathbb{Z}_+ \text{ and } m_{i,2p-1}-m_{i+1,2p}\in\mathbb{Z}_+,\\
 & 2\le p\le n,\quad 1\le i\le p-1;\\
7. & m_{i,2p+1}-m_{i+1,2p}\in\mathbb{Z}_+ \text{ and } m_{i+1,2p}-m_{i+1,2p+1}\in\mathbb{Z}_+,\\
 & 1\le p\le n-1,\quad -p-1\le i\le -2.
\end{array}
\tag{11}
$$

constitute a basis in $V([m]^{2n})$. Furthermore the transformation of the irreducible covariant tensor module $V([m]^{2n})$ under the action of a set of generating elements of $\mathfrak{gl}(n|n)$ is given by:

$$
E_{-i,-i}|m)^{2n}=\left(\sum_{j=-i(\neq 0)}^{i-1} m_{j,2i-1}-\sum_{j=-i+1(\neq 0)}^{i-1} m_{j,2i-2}\right)|m)^{2n},\quad 1\le i\le n; \tag{12}
$$

$$
E_{ii}|m)^{2n}=\left(\sum_{j=-i(\neq 0)}^{i} m_{j,2i}-\sum_{j=-i(\neq 0)}^{i-1} m_{j,2i-1}\right)|m)^{2n},\quad 1\le i\le n; \tag{13}
$$

$$
\begin{aligned}
E_{-i,i}|m)^{2n}&=\sum_{k=-i}^{-1}\theta_{k,2i-1}\hat{S}\Big(\sum_{s=1}^{i-1}\sum_{j=-s}^{-1}\theta_{j,2s-1}+\sum_{s=1}^{i-1}\sum_{j=1}^{s}\theta_{j,2s}+\sum_{j=-i}^{k-1}\theta_{j,2i-1}\Big)\\
&\times\left(\frac{\prod_{j=-i+1}^{-1}(l_{k,2i}-l_{j,2i-2}-1)\prod_{j=1}^{i}(l_{k,2i}-l_{j,2i})}{\prod_{j=-i(\neq k)}^{-1}(l_{k,2i}-l_{j,2i})\prod_{j=1}^{i-1}(l_{k,2i}-l_{j,2i-2}-1)}\right)^{1/2}|m)^{2n}_{+(k,2i-1)}\\
&+\sum_{k=1}^{i-1}\theta_{k,2i-2}\hat{S}\Big(\sum_{s=1}^{i-1}\sum_{j=-s}^{-1}\theta_{j,2s-1}+\sum_{s=1}^{i-2}\sum_{j=1}^{s}\theta_{j,2s}+\sum_{j=1}^{k-1}\theta_{j,2i-2}\Big)\\
&\times\left(-\frac{\prod_{j=-i+1}^{-1}(l_{j,2i-2}-l_{k,2i-1}+1)\prod_{j=1}^{i}(l_{j,2i}-l_{k,2i-1})}{\prod_{j=-i}^{-1}(l_{j,2i}-l_{k,2i-1})\prod_{j=1(\neq k)}^{i-1}(l_{j,2i-2}-l_{k,2i-1}+1)}\right)^{1/2}|m)^{2n}_{+(k,2i-1)},\\
&\qquad 1\le i\le n;
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
E_{-i-1,i}|m)^{2n}&=\sum_{k=-i}^{-1}\theta_{k,2i-1}\hat{S}\Big(\sum_{s=1}^{i-1}\sum_{j=-s}^{-1}\theta_{j,2s-1}+\sum_{s=1}^{i-1}\sum_{j=1}^{s}\theta_{j,2s}+\sum_{j=-i}^{k-1}\theta_{j,2i-1}\Big)\\
&\times\left(-\frac{\prod_{j=-i-1}^{-1}(l_{k,2i}-l_{j,2i+1})\prod_{j=1}^{i-1}(l_{k,2i-1}-l_{j,2i-1})}{\prod_{j=-i(\neq k)}^{-1}(l_{k,2i-1}-l_{j,2i-1})\prod_{j=1}^{i}(l_{k,2i-1}-l_{j,2i+1}+1)}\right)^{1/2}|m)^{2n}_{-(k,2i)}\\
&+\sum_{k=1}^{i}\theta_{k,2i}\hat{S}\Big(\sum_{s=1}^{i}\sum_{j=-s}^{-1}\theta_{j,2s-1}+\sum_{s=1}^{i-1}\sum_{j=1}^{s}\theta_{j,2s}+\sum_{j=1}^{k-1}\theta_{j,2i}\Big)\\
&\times\left(\frac{\prod_{j=-i-1}^{-1}(l_{j,2i+1}-l_{k,2i+1})\prod_{j=1}^{i-1}(l_{j,2i-1}-l_{k,2i})}{\prod_{j=-i}^{-1}(l_{j,2i-1}-l_{k,2i+1}+1)\prod_{j=1(\neq k)}^{i}(l_{j,2i+1}-l_{k,2i+1})}\right)^{1/2}|m)^{2n}_{-(k,2i)},\\
&\qquad 1\le i\le n-1;
\end{aligned}
\tag{15}
$$

$$E_{i,-i}|m)^{2n} = \sum_{k=-i}^{-1}(1-\theta_{k,2i-1})\hat{S}\Big(\sum_{s=1}^{i-1}\sum_{j=-s}^{-1}\theta_{j,2s-1} + \sum_{s=1}^{i-1}\sum_{j=1}^{s}\theta_{j,2s} + \sum_{j=-i}^{k-1}\theta_{j,2i-1}\Big)$$
$$\times\left(\frac{\prod_{j=-i+1}^{-1}(l_{k,2i}-l_{j,2i-2}-1)\prod_{j=1}^{i}(l_{k,2i}-l_{j,2i})}{\prod_{j=-i(\neq k)}^{-1}(l_{k,2i}-l_{j,2i})\prod_{j=1}^{i-1}(l_{k,2i}-l_{j,2i-2}-1)}\right)^{1/2} |m)^{2n}_{-(k,2i-1)}$$
$$+\sum_{k=1}^{i-1}(1-\theta_{k,2i-2})\hat{S}\Big(\sum_{s=1}^{i-1}\sum_{j=-s}^{-1}\theta_{j,2s-1} + \sum_{s=1}^{i-2}\sum_{j=1}^{s}\theta_{j,2s} + \sum_{j=1}^{k-1}\theta_{j,2i-2}\Big)$$
$$\times\left(-\frac{\prod_{j=-i+1}^{-1}(l_{j,2i-2}-l_{k,2i-1})\prod_{j=1}^{i}(l_{j,2i}-l_{k,2i-1}-1)}{\prod_{j=-i}^{-1}(l_{j,2i}-l_{k,2i-1}-1)\prod_{j=1(\neq k)}^{i-1}(l_{j,2i-2}-l_{k,2i-1})}\right)^{1/2} |m)^{2n}_{-(k,2i-1)},$$
$$1\leq i\leq n; \tag{16}$$

$$E_{i,-i-1}|m)^{2n} = \sum_{k=-i}^{-1}(1-\theta_{k,2i-1})\hat{S}\Big(\sum_{s=1}^{i-1}\sum_{j=-s}^{-1}\theta_{j,2s-1} + \sum_{s=1}^{i-1}\sum_{j=1}^{s}\theta_{j,2s} + \sum_{j=-i}^{k-1}\theta_{j,2i-1}\Big)$$
$$\times\left(-\frac{\prod_{j=-i-1}^{-1}(l_{k,2i}-l_{j,2i+1}+1)\prod_{j=1}^{i-1}(l_{k,2i-1}-l_{j,2i-1})}{\prod_{j=-i(\neq k)}^{-1}(l_{k,2i-1}-l_{j,2i-1})\prod_{j=1}^{i}(l_{k,2i-1}-l_{j,2i+1}+1)}\right)^{1/2} |m)^{2n}_{+(k,2i)}$$
$$+\sum_{k=1}^{i}(1-\theta_{k,2i})\hat{S}\Big(\sum_{s=1}^{i}\sum_{j=-s}^{-1}\theta_{j,2s-1} + \sum_{s=1}^{i-1}\sum_{j=1}^{s}\theta_{j,2s} + \sum_{j=1}^{k-1}\theta_{j,2i}\Big)$$
$$\times\left(\frac{\prod_{j=-i-1}^{-1}(l_{j,2i+1}-l_{k,2i+1})\prod_{j=1}^{i-1}(l_{j,2i-1}-l_{k,2i}+1)}{\prod_{j=-i}^{-1}(l_{j,2i-1}-l_{k,2i+1}+1)\prod_{j=1(\neq k)}^{i}(l_{j,2i+1}-l_{k,2i+1})}\right)^{1/2} |m)^{2n}_{+(k,2i)},$$
$$1\leq i\leq n-1, \tag{17}$$

where

$$\hat{S}(a)=(-1)^a;\ l_{i,2k-\varphi}=m_{i,2k-\varphi}-i\ (-k\leq i\leq -1);\ l_{p,2k-\varphi}=-m_{p,2k-\varphi}+p\ (1\leq p\leq k-\varphi),\ \varphi=0,1;\ k=1,\dots,n, \tag{18}$$

$|m)^{2n}_{\pm(i,j)}$ is the pattern obtained from $|m)^{2n}$ by replacing the entry $m_{i,j}$ by $m_{i,j}\pm 1$ and $\prod_{i=a(\neq k)}^{b}$ means the product over all i-values running from a to b, but excluding $i=k$.

CLEBSCH-GORDAN COEFFICIENTS OF $\mathfrak{gl}(n|n)$

In this section we give the computed Clebsch-Gordan coefficients of $\mathfrak{gl}(n|n)$ corresponding to the tensor product $V([1,0,\dots,0])\otimes V([m]^{2n})$ of the $2n$-dimensional representation $V([1,0,\dots,0])$ of $\mathfrak{gl}(n|n)$ with any $\mathfrak{gl}(n|n)$ irreducible covariant module $V([m]^{2n})$. It is well known that:

$$V([1,0,\dots,0])\otimes V([m]^{2n}) = \bigoplus_{k=-n(\neq 0)}^{n} V([m]^{2n}_{+(k)}), \tag{19}$$

where $[m]^{2n}_{+(k)}$ is obtained from $[m]^{2n}$ by the replacement of $m_{k,2n}$ by $m_{k,2n}+1$ and on the right hand side of (19) the summands for which the conditions (8)-(9) are not fulfilled are omitted. We choose two orthonormal bases in the space (19):

$$|1_j)\otimes\left|\begin{array}{l}[m]^{2n}\\ |m)^{2n-1}\end{array}\right)\in V([1,0,\dots,0])\otimes V([m]^{2n})\quad\text{and}$$

$$\left| \begin{array}{c} [m]^{2n}_{+(k)} \\ |m')^{2n-1} \end{array} \right) \in V([m]^{2n}_{+(k)}), \quad k=-n,\ldots,-1,1,\ldots,n,$$

where the vectors $\left| \begin{array}{c} [m]^{2n} \\ |m)^{2n-1} \end{array} \right)$ and $\left| \begin{array}{c} [m]^{2n}_{+(k)} \\ |m')^{2n-1} \end{array} \right)$ satisfy the conditions (11), and $|1_j), \;\; j=1,\ldots,2n$ is a pattern which consists of $j-1$ zero rows at the bottom (denoted by $0\cdots 0 = \dot{0}$), and the first $2n-j+1$ rows are of the form $10\cdots 0$ (denoted by $1\dot{0}$). Then in general (the sum is over all possible values of j and allowed values of the GZ labels in the pattern $|m)^{2n-1}$)

$$\left| \begin{array}{c} [m]^{2n}_{+(k)} \\ |m')^{2n-1} \end{array} \right) = \sum \left(\begin{array}{c} 10\cdots 00 \\ 10\cdots 0 \\ \cdots \\ 0 \end{array} ; \begin{array}{c} [m]^{2n} \\ |m)^{2n-1} \end{array} \right| \left. \begin{array}{c} [m]^{2n}_{+(k)} \\ |m')^{2n-1} \end{array} \right) |1_j) \otimes \left| \begin{array}{c} [m]^{2n} \\ |m)^{2n-1} \end{array} \right), \tag{20}$$

where

$$\left(\begin{array}{c} 10\cdots 00 \\ 10\cdots 0 \\ \cdots \\ 0 \end{array} ; \begin{array}{c} [m]^{2n} \\ |m)^{2n-1} \end{array} \right| \left. \begin{array}{c} [m]^{2n}_{+(k)} \\ |m')^{2n-1} \end{array} \right) \equiv \left(|1_j) \; ; \begin{array}{c} [m]^{2n} \\ |m)^{2n-1} \end{array} \right| \left. \begin{array}{c} [m]^{2n}_{+(k)} \\ |m')^{2n-1} \end{array} \right)$$

are the Clebsch-Gordan coefficients (CGCs). Using the actions (12)-(13) we have for the CGCs

$$\left(\begin{array}{c} 10\cdots 00 \\ 10\cdots 0 \\ \cdots \\ 0 \end{array} ; \begin{array}{c} [m]^{2n} \\ |m)^{2n-1} \end{array} \right| \left. \begin{array}{c} [m]^{2n}_{+(k)} \\ |m')^{2n-1} \end{array} \right)$$
$$= \left(\begin{array}{c} 1\dot{0} \\ \epsilon\dot{0} \end{array} \right| \begin{array}{c} [m]^{2n} \\ [m]^{2n-1} \end{array} \left| \begin{array}{c} [m]^{2n}_{+(k)} \\ [m']^{2n-1} \end{array} \right) \times \left(\begin{array}{c} 10\cdots 00 \\ 10\cdots 0 \\ \cdots \\ 0 \end{array} ; \begin{array}{c} [m]^{2n-1} \\ |m)^{2n-2} \end{array} \right| \left. \begin{array}{c} [m']^{2n-1}_{+(k)} \\ |m')^{2n-2} \end{array} \right). \tag{21}$$

In the right hand side, the first factor is an isoscalar factor [11], and the second factor is a CGC of $\mathfrak{gl}(n|n-1)$. The middle pattern in the $\mathfrak{gl}(n|n-1)$ CGC is that of the $\mathfrak{gl}(n|n)$ CGC with the first row deleted. The middle pattern in the isoscalar factor consists of the first two rows of the middle pattern in the left hand side, so ϵ is 0 or 1. If $\epsilon=0$, then $[m']^{2n-1}=[m]^{2n-1}$. If $\epsilon=1$ then $[m']^{2n-1}=[m_{-n,2n-1},\ldots,m_{s,2n-1}+1,\ldots,m_{n-1,n-1}]=[m]^{2n-1}_{+s}$ for some s-value. Now, following the general procedure for computing CGCs we have

Theorem: The Clebsch-Gordan coefficients corresponding to the tensor product

$$V([1,0,\ldots,0]) \otimes V([m]^{2n})$$

of the natural $(2n)$-dimensional $\mathfrak{gl}(n|n)$ representation $V([1,0,\ldots,0])$ with an irreducible $\mathfrak{gl}(n|n)$ covariant tensor module $V([m]^{2n})$ are products of isoscalar factors

$$\left(|1_j) \; ; \begin{array}{c} [m]^{2n} \\ |m)^{2n-1} \end{array} \right| \left. \begin{array}{c} [m]^{2n}_{+(k)} \\ |m')^{2n-1} \end{array} \right) = \left(\begin{array}{c} 1\dot{0} \\ 1\dot{0} \end{array} \right| \begin{array}{c} [m]^{2n} \\ [m]^{2n-1} \end{array} \left| \begin{array}{c} [m]^{2n}_{+(k)} \\ [m']^{2n-1} \end{array} \right) \times \ldots$$
$$\times \left(\begin{array}{c} 1\dot{0} \\ 1\dot{0} \end{array} \right| \begin{array}{c} [m]^{j+1} \\ [m]^{j} \end{array} \left| \begin{array}{c} [m\prime]^{j+1} \\ [m']^{j} \end{array} \right) \left(\begin{array}{c} 1\dot{0} \\ 0\dot{0} \end{array} \right| \begin{array}{c} [m]^{j} \\ [m]^{j-1} \end{array} \left| \begin{array}{c} [m\prime]^{j} \\ [m]^{j-1} \end{array} \right) \times 1, \quad j=1,\ldots,2n-1. \tag{22}$$

$$\left(|1_{2n}) \; ; \begin{array}{c} [m]^{2n} \\ |m)^{2n-1} \end{array} \right| \left. \begin{array}{c} [m]^{2n}_{+(k)} \\ |m)^{2n-1} \end{array} \right) = (-1)^{\sum_{i=1}^{n}\sum_{j=-i}^{-1}\theta_{j,2i-1}+\sum_{i=1}^{n-1}\sum_{j=1}^{i}\theta_{j,2i}}$$
$$\times \left(\begin{array}{c} 1\dot{0} \\ 0\dot{0} \end{array} \right| \begin{array}{c} [m]^{2n} \\ [m]^{2n-1} \end{array} \left| \begin{array}{c} [m]^{2n}_{+(k)} \\ [m]^{2n-1} \end{array} \right) \times 1. \tag{23}$$

In the right hand side of (22)-(23),

$$\left(\begin{array}{c|c|c} 1\dot{0} & [m]^{t} & [m]^{t}_{+(k)} \\ \alpha\dot{0} & [m]^{t-1} & [m']^{t-1} \end{array}\right) \qquad (\alpha \in \{0,1\})$$

is a $\mathfrak{gl}(t|t) \supset \mathfrak{gl}(t|t-1)$ $(1 \leq t \leq n)$ or a $\mathfrak{gl}(t|t-1) \supset \mathfrak{gl}(t-1|t-1)$ $(2 \leq t \leq n)$ isoscalar factor following the chain of subalgebras

$$\mathfrak{gl}(n|n) \supset \mathfrak{gl}(n|n-1) \supset \mathfrak{gl}(n-1|n-1) \supset \mathfrak{gl}(n-1|n-2) \supset \mathfrak{gl}(n-2|n-2) \supset \cdots \supset \mathfrak{gl}(1|1) \supset \mathfrak{gl}(1|0) \equiv \mathfrak{gl}(1) \qquad (24)$$

For the $\mathfrak{gl}(t|t) \supset \mathfrak{gl}(t|t-1)$ isoscalar factors there are six different expressions, depending on the position of the pattern changes in the right hand side. These six expressions are given by:

$$\left(\begin{array}{c|c|c} 1\dot{0} & [m]^{2t} & [m]^{2t}_{+(k)} \\ 0\dot{0} & [m]^{2t-1} & [m]^{2t-1} \end{array}\right) = (-1)^{t+k}(1-\theta_{k,2t-1})(-1)^{\sum_{i=k}^{-1}\theta_{i,2t-1}}$$
$$\times\left(\prod_{i=-t(\neq k)}^{-1}\left(\frac{l_{k,2t}-l_{i,2t}+1}{l_{k,2t}-l_{i,2t-1}}\right)\frac{\prod_{s=1}^{t-1}(l_{k,2t}-l_{s,2t-1})}{\prod_{s=1}^{t}(l_{k,2t}-l_{s,2t}+1)}\right)^{1/2} \qquad (1 \leq t \leq n;\ -t \leq k \leq -1); \qquad (25)$$

$$\left(\begin{array}{c|c|c} 1\dot{0} & [m]^{2t} & [m]^{2t}_{+(k)} \\ 0\dot{0} & [m]^{2t-1} & [m]^{2t-1} \end{array}\right) = \left(\prod_{i=-t}^{-1}\left(\frac{l_{i,2t}-l_{k,2t}}{l_{i,2t-1}-l_{k,2t}+1}\right)\frac{\prod_{s=1}^{t-1}(l_{s,2t-1}-l_{k,2t}+1)}{\prod_{s=1(\neq k)}^{t}(l_{s,2t}-l_{k,2t})}\right)^{1/2}$$
$$(2 \leq t \leq n;\ 1 \leq k \leq t); \qquad (26)$$

$$\left(\begin{array}{c|c|c} 10 & [m]^{2} & [m]^{2}_{+(1)} \\ 0 & [m]^{1} & [m]^{1} \end{array}\right) = (-1)^{\theta_{11}}\left(\frac{l_{-1,2}-l_{1,2}}{l_{-1,1}-l_{1,2}+1}\right)^{1/2}$$

$$\left(\begin{array}{c|c|c} 1\dot{0} & [m]^{2t} & [m]^{2t}_{+(k)} \\ 1\dot{0} & [m]^{2t-1} & [m]^{2t-1}_{+(q)} \end{array}\right) = (-1)^{k+q}(-1)^{\sum_{i=\min(k+1,q+1)}^{\max(k-1,q-1)}\theta_{i,2t-1}}(\delta_{kq}+(1-\delta_{kq})\theta_{q,2t-1}(1-\theta_{k,2t-1}))$$
$$\times\left(\prod_{i=-t(\neq k,q)}^{-1}\frac{(l_{i,2t-1}-l_{k,2t-1}-1-\delta_{kq}+2\theta_{i,2t-1})(l_{i,2t-1}-l_{q,2t-1})}{(l_{i,2t}-l_{k,2t})(l_{i,2t}-l_{q,2t})}\right)^{\theta_{q,2t-1}/2}$$
$$\times\frac{1}{(l_{k,2t}-l_{q,2t})^{1-\delta_{kq}}}\left(\prod_{s=1}^{t}\left(\frac{l_{q,2t}-l_{s,2t}}{l_{k,2t}-l_{s,2t}+1}\right)\prod_{s=1}^{t-1}\left(\frac{l_{k,2t}-l_{s,2t-1}}{l_{q,2t-1}-l_{s,2t-1}}\right)\right)^{\theta_{q,2t-1}/2} \qquad (-t \leq k,q \leq -1); \qquad (27)$$

$$\left(\begin{array}{c|c|c} 1\dot{0} & [m]^{2t} & [m]^{2t}_{+(k)} \\ 1\dot{0} & [m]^{2t-1} & [m]^{2t-1}_{+(q)} \end{array}\right) = (-1)^{k+t+1}(-1)^{\sum_{i=-t}^{k-1}\theta_{i,2t-1}}(1-\theta_{k,2t-1})\left(\frac{1}{l_{k,2t}-l_{q,2t-1}}\right)^{1/2}$$
$$\times\left(\prod_{i=-t(\neq k)}^{-1}\left(\frac{(l_{i,2t-1}-l_{k,2t-1}-1+2\theta_{i,2t-1})(l_{i,2t-1}-l_{q,2t-1}+1)}{(l_{i,2t}-l_{k,2t})(l_{i,2t}-l_{q,2t-1})}\right)\right)^{1/2}$$
$$\times\left(\prod_{s=1}^{t}\left(\frac{|l_{s,2t}-l_{q,2t-1}|}{(l_{k,2t}-l_{s,2t}+1)}\right)\prod_{s=1(\neq q)}^{t-1}\left(\frac{l_{k,2t}-l_{s,2t-1}}{|l_{s,2t-1}-l_{q,2t-1}+1|}\right)\right)^{1/2}$$
$$(-t \leq k \leq -1, \quad 1 \leq q \leq t-1); \qquad (28)$$

$$\left(\begin{array}{c|c|c} 1\dot{0} & [m]^{2t} & [m]^{2t}_{+(k)} \\ 1\dot{0} & [m]^{2t-1} & [m]^{2t-1}_{+(q)} \end{array}\right) = (-1)^{q+t+1}(-1)^{\sum_{i=q+1}^{-1}\theta_{i,2t-1}}\theta_{q,2t-1}\left(\frac{1}{l_{q,2t}-l_{k,2t}+1}\right)^{1/2}$$
$$\times\left(\prod_{i=-t}^{-1}\left(\frac{l_{i,2t}-l_{k,2t}}{l_{i,2t-1}-l_{k,2t}+1}\right)\prod_{i=-t(\neq q)}^{-1}\left|\frac{l_{q,2t-1}-l_{i,2t-1}}{l_{q,2t}-l_{i,2t}}\right|\prod_{s=1(\neq k)}^{t}\left|\frac{l_{q,2t}-l_{s,2t}}{l_{s,2t}-l_{k,2t}}\right|\prod_{s=1}^{t-1}\left|\frac{l_{s,2t-1}-l_{k,2t}+1}{l_{q,2t}-l_{s,2t-1}-1}\right|\right)^{1/2}$$
$$(1 \leq k \leq t, \quad -t \leq q \leq -1); \qquad (29)$$

$$\left(\begin{array}{c|c|c} 1\dot{0} & [m]^{2t} & [m]^{2t}_{+(k)} \\ 1\dot{0} & [m]^{2t-1} & [m]^{2t-1}_{+(q)} \end{array}\right) = S(k,q)(-1)^{\sum_{i=-t}^{-1}\theta_{i,2t-1}} \left(\prod_{i=-t}^{-1} \left(\frac{(l_{i,2t}-l_{k,2t})(l_{i,2t-1}-l_{q,2t-1}+1)}{(l_{i,2t-1}-l_{k,2t}+1)(l_{i,2t}-l_{q,2t-1})} \right) \right)^{1/2}$$

$$\times \left(\prod_{s=1(\neq k)}^{t} \left| \frac{l_{s,2t}-l_{q,2t-1}}{l_{s,2t}-l_{k,2t}} \right| \prod_{s=1(\neq q)}^{t-1} \left| \frac{l_{s,2t-1}-l_{k,2t}+1}{l_{s,2t-1}-l_{q,2t-1}+1} \right| \right)^{1/2} \quad (1 \le k \le t, \quad 1 \le q \le t-1). \tag{30}$$

Also for the $\mathfrak{gl}(t|t-1) \supset \mathfrak{gl}(t-1|t-1)$ isoscalar factors there are six different expressions, depending on the position of the pattern changes in the right hand side. These six expressions are given by:

$$\left(\begin{array}{c|c|c} 1\dot{0} & [m]^{2t-1} & [m]^{2t-1}_{+(k)} \\ \dot{0} & [m]^{2t-2} & [m]^{2t-2} \end{array}\right)$$

$$= (-1)^{t+k} \left(\frac{\prod_{s=-t+1}^{-1}(l_{k,2t-1}-l_{s,2t-2})}{\prod_{s=-t(\neq k)}^{-1}(l_{k,2t-1}-l_{s,2t-1})} \prod_{i=1}^{t-1} \left(\frac{l_{k,2t-1}-l_{i,2t-1}}{l_{k,2t-1}-l_{i,2t-2}} \right) \right)^{1/2} \quad (-t \le k \le -1); \tag{31}$$

$$\left(\begin{array}{c|c|c} 1\dot{0} & [m]^{2t-1} & [m]^{2t-1}_{+(k)} \\ \dot{0} & [m]^{2t-2} & [m]^{2t-2} \end{array}\right)$$

$$= \theta_{k,2t-2}(-1)^{k}(-1)^{\sum_{i=k+1}^{t-1}\theta_{i,2t-2}}$$

$$\times \left(\frac{\prod_{s=-t+1}^{-1}(l_{s,2t-2}-l_{k,2t-1}+1)}{\prod_{s=-t}^{-1}(l_{s,2t-1}-l_{k,2t-1}+1)} \prod_{i=1(\neq k)}^{t-1} \left(\frac{l_{i,2t-1}-l_{k,2t-1}+1}{l_{i,2t-2}-l_{k,2t-1}+1} \right) \right)^{1/2} \quad (1 \le k \le t-1); \tag{32}$$

$$\left(\begin{array}{c|c|c} 10\dot{0} & [m]^{2t-1} & [m]^{2t-1}_{+(k)} \\ 1\dot{0} & [m]^{2t-2} & [m]^{2t-2}_{+(q)} \end{array}\right)$$

$$= (-1)^{k+q} S(-k,-q) \left(\prod_{i=-t(\neq k)}^{-1} \left(\frac{l_{q,2t-2}-l_{i,2t-1}+1}{l_{k,2t-1}-l_{i,2t-1}} \right) \prod_{i=-t+1(\neq q)}^{-1} \left(\frac{l_{k,2t-1}-l_{i,2t-2}}{l_{q,2t-2}-l_{i,2t-2}+1} \right) \right)^{1/2}$$

$$\times \left(\prod_{s=1}^{t-1} \frac{(l_{k,2t-1}-l_{s,2t-1})(l_{q,2t-2}-l_{s,2t-2}+1)}{(l_{k,2t-1}-l_{s,2t-2})(l_{q,2t-2}-l_{s,2t-1}+1)} \right)^{1/2} \quad (-t \le k \le -1,\ -t+1 \le q \le -1); \tag{33}$$

$$\left(\begin{array}{c|c|c} 10\dot{0} & [m]^{2t-1} & [m]^{2t-1}_{+(k)} \\ 1\dot{0} & [m]^{2t-2} & [m]^{2t-2}_{+(q)} \end{array}\right) = (-1)^{k+t}(-1)^{\sum_{i=1}^{q-1}\theta_{i,2t-2}}(1-\theta_{q,2t-2}) \left(\frac{1}{l_{k,2t-1}-l_{q,2t-2}+1} \right)^{1/2}$$

$$\times \left(\prod_{i=-t(\neq k)}^{-1} \left(\frac{l_{i,2t-1}-l_{q,2t-2}}{l_{k,2t-1}-l_{i,2t-1}} \right) \prod_{i=-t+1}^{-1} \left(\frac{l_{k,2t-1}-l_{i,2t-2}}{l_{i,2t-2}-l_{q,2t-2}} \right) \right)^{1/2}$$

$$\times \left(\prod_{s=1(\neq q)}^{t-1} \left(\frac{(l_{k,2t-1}-l_{s,2t-1})(l_{s,2t-2}-l_{q,2t-2})}{(l_{k,2t-1}-l_{s,2t-2})(l_{s,2t-1}-l_{q,2t-2})} \right) \right)^{1/2}$$

$$(-t \le k \le -1, \quad 1 \le q \le t-1); \tag{34}$$

$$\left(\begin{array}{c|c|c} 10\dot{0} & [m]^{2t-1} & [m]^{2t-1}_{+(k)} \\ 1\dot{0} & [m]^{2t-2} & [m]^{2t-2}_{+(q)} \end{array}\right) = (-1)^{k+q+t}(-1)^{\sum_{i=k+1}^{t-1}\theta_{i,2t-2}}\theta_{k,2t-2} \left(\frac{1}{l_{q,2t-2}-l_{k,2t-1}+1} \right)^{1/2}$$

$$\times \left(\prod_{i=-t}^{-1} \left(\frac{|l_{i,2t-1}-l_{q,2t-2}-1|}{l_{i,2t-1}-l_{k,2t-1}+1} \right) \prod_{i=-t+1(\neq q)}^{-1} \left(\frac{l_{i,2t-2}-l_{k,2t-1}+1}{|l_{i,2t-2}-l_{q,2t-2}-1|} \right) \right)^{1/2}$$

$$\times \left(\prod_{s=1(\neq k)}^{t-1} \left(\frac{(l_{q,2t-2}-l_{s,2t-2}+1)(l_{s,2t-1}-l_{k,2t-1}+1)}{(l_{q,2t-2}-l_{s,2t-1}+1)(l_{s,2t-2}-l_{k,2t-1}+1)} \right) \right)^{1/2} \quad (1 \le k \le t-1, \quad -t+1 \le q \le -1); \tag{35}$$

$$
\left(\begin{array}{c} 10\dot{0} \\ 1\dot{0} \end{array} \middle| \begin{array}{c} [m]^{2t-1} \\ [m]^{2t-2} \end{array} \middle| \begin{array}{c} [m]^{2t-1}_{+(k)} \\ [m]^{2t-2}_{+(q)} \end{array} \right) = (-1)^{k-1}(-1)^{\sum_{i=1}^{t-1}\theta_{i,2t-2}}(\delta_{kq} + (1-\delta_{kq})\theta_{k,2t-2}(1-\theta_{q,2t-2}))
$$

$$
\times \frac{1}{(l_{k,2t-1} - l_{q,2t-1})^{1-\delta_{kq}}} \left(\prod_{i=-t}^{-1} \left(\frac{l_{i,2t-1} - l_{q,2t-2}}{l_{i,2t-1} - l_{k,2t-1} + 1} \right) \prod_{i=-t+1}^{-1} \left(\frac{l_{i,2t-2} - l_{k,2t-1} + 1}{l_{i,2t-2} - l_{q,2t-2}} \right) \right)^{(1-\theta_{q,2t-2})/2}
$$

$$
\times \left(\prod_{s=1(\neq k,q)}^{t-1} \frac{(l_{s,2t-2} - l_{q,2t-2} + 1 - \delta_{kq} + 2\theta_{s,2t-2})(l_{s,2t-2} - l_{q,2t-2})}{(l_{s,2t-1} - l_{k,2t-1})(l_{s,2t-1} - l_{q,2t-2})} \right)^{(1-\theta_{q,2t-2})/2} \quad (1 \leq k, q \leq t-1); \qquad (36)
$$

In all of the formulas above, $\dot{0}$ stands for an appropriate sequence of zeros in the pattern, and we use

$$
S(k,q) = \begin{cases} 1 & \text{for} \quad k \leq q \\ -1 & \text{for} \quad k > q. \end{cases} . \qquad (37)
$$

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