

A linear chain of interacting harmonic oscillators: solutions as a Wigner Quantum System

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Abstract. We consider a quantum mechanical system consisting of a linear chain of harmonic oscillators coupled by a nearest neighbor interaction. The system configuration can be closed (periodic boundary conditions) or open (non-periodic case). We show that such systems can be considered as Wigner Quantum Systems (WQS), thus yielding extra solutions apart from the canonical solution. In particular, a class of WQS-solutions is given in terms of unitary representations of the Lie superalgebra $\mathfrak{gl}(1|n)$. In order to determine physical properties of the new solutions, one needs to solve a number of interesting but difficult representation theoretical problems. We present these problems and their solution, and show how the new results yield attractive properties for the quantum system (energy spectrum, position probabilities, spacial properties).

1. Introduction

Coupled systems describing the interaction of oscillating or scattering subsystems and the corresponding operators have been widely used in classical and quantum mechanics [1–5].

Recently, we have taken up the study of such systems as Wigner Quantum Systems (WQS) [6–8]. The particular system under consideration consists of a string of n identical harmonic oscillators, each having the same mass m and frequency ω . The position and momentum operator for the r th oscillator ($r = 1, 2, \dots, n$) are given by \hat{q}_r and \hat{p}_r respectively; more precisely, \hat{q}_r measures the displacement of the r th mass point with respect to its equilibrium position. The oscillators are coupled by some nearest neighbor coupling, represented by terms of the form $(\hat{q}_r - \hat{q}_{r+1})^2$ in the Hamiltonian. The system configuration can either be closed (periodic boundary conditions), i.e. $\hat{q}_{n+1} = \hat{q}_1$, or open (fixed wall boundary conditions), i.e. $\hat{q}_0 = \hat{q}_{n+1} = 0$.

When treating a system as a WQS, the canonical commutation relations (CCRs) are *not* required to hold. Instead, the compatibility of the Hamilton equations and the Heisenberg equations is imposed. Expressing this compatibility leads to the so-called compatibility conditions (CCs), and these need to be solved (subject to some unitarity conditions). In [9, 10], we have shown that the CCs associated with a system consisting of a coupled oscillator chain has solutions in terms of generators of the $\mathfrak{gl}(1|n)$ Lie superalgebra [11]. We are only going to work with finite-dimensional unitary irreducible representations (unirreps) of $\mathfrak{gl}(1|n)$, so in fact,

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all the solutions in this article are non canonical, and fall within the framework of Quantum Systems with a finite Hilbert space [12].

In order to determine physical properties of the system, such as the energy spectrum, position eigenvalues or position probabilities, one is led to considering problems in the representation theory of $\mathfrak{gl}(1|n)$. For the finite-dimensional unirreps of $\mathfrak{gl}(1|n)$, a Gel'fand-Zetlin basis (GZ-basis) is known, as well as the explicit action of the generators on the basis vectors of the representation [13]. These actions, especially the non-diagonal ones, on a general GZ-pattern are, however, quite involved and thus some of the tasks at hand (e.g. determining position eigenvalues and eigenvectors) are far from trivial. Nevertheless, we have been able to solve these problems quite generally [14].

In this article, we first introduce the Hamiltonians of the quantum systems. We then show briefly how to derive and manipulate the CCs such that they are in a form suitable for finding Lie superalgebraic solutions. Next it is indicated how a solution in terms of the odd generators of the Lie superalgebra $\mathfrak{gl}(1|n)$ is found. As a last part of Section 2, the unirreps of $\mathfrak{gl}(1|n)$ are introduced, focusing on one particular class of representations, namely the ladder representations $V(p)$.

Then, in Section 3, we turn to the study of the physical properties of the systems, starting with the energy spectrum, stressing the difference between the systems with open and closed boundary conditions.

Determining the energy spectrum is a relatively easy task compared to determining the spectrum of the position operators. The key point here is to introduce a second set of generators of $\mathfrak{gl}(1|n)$ and recognizing that for this second set of generators the position operators \hat{q}_r are in fact elements of a $\mathfrak{gl}(1|1)$ subalgebra. The eigenvalue problem is thus solved by considering the branching $\mathfrak{gl}(1|n) \rightarrow \mathfrak{gl}(1|1) \oplus \mathfrak{gl}(n-1)$, the multiplicity of the different eigenvalues being determined by the dimension of the various $\mathfrak{gl}(n-1)$ representations. The eigenvector problem is even harder as this involves expressing the highest weight vector w.r.t. the second set of generators in terms of the original basis vectors of the representation. For this article, we restrict ourselves completely to the ladder representations.

Finally, as an application, we give some numerical results concerning the position probabilities when the system is in a stationary state. We remark how the system behaves differently according to whether it is in the ground state or in the most excited state. Also, this behavior is in line with physical intuition about such systems. This is the subject of Section 5.

In Section 6 a short conclusion and outlook are given.

2. A Lie superalgebraic solution to the compatibility conditions

2.1. System with periodic boundary conditions

We start with a quick derivation of the compatibility conditions for a linear system consisting of n interacting but otherwise identical harmonic oscillators. We assume that each oscillator only interacts explicitly with its direct neighbors. The Hamiltonian of such a system is given by:

$$\hat{H}_P = \sum_{r=1}^n \left(\frac{\hat{p}_r^2}{2m} + \frac{m\omega^2}{2} \hat{q}_r^2 \right) + \sum_{r=1}^n \frac{cm}{2} (\hat{q}_r - \hat{q}_{r+1})^2, \quad (1)$$

where each oscillator has mass m and frequency ω . The position and momentum operator of each oscillator are given by \hat{q}_r and \hat{p}_r respectively. To be more precise, \hat{q}_r measures the displacement of the r th oscillator with respect to its equilibrium position. Finally, the positive constant c denotes the coupling strength. For now, we assume periodic boundary conditions, i.e. we assume that

$$\hat{q}_{n+1} \equiv \hat{q}_1. \quad (2)$$

When treating the system as a Wigner Quantum System, one imposes the compatibility between the Hamilton equations and the Heisenberg equations. It is known that some systems then also have non canonical solutions, i.e. solutions for which the canonical commutation relations between the position and momentum operators do not hold [6, 9, 15]. For the Hamiltonian under consideration, the compatibility conditions can be computed as:

$$[\hat{H}_P, \hat{q}_r] = -\frac{i\hbar}{m}\hat{p}_r (= -i\hbar\dot{\hat{q}}_r), \quad (3)$$

$$[\hat{H}_P, \hat{p}_r] = -i\hbar cm \hat{q}_{r-1} + i\hbar m(\omega^2 + 2c)\hat{q}_r - i\hbar cm \hat{q}_{r+1} (= -i\hbar\dot{\hat{p}}_r), \quad (4)$$

with $r \in \{1, 2, \dots, n\}$, $\hat{q}_{n+1} = \hat{q}_1$ and $\hat{q}_0 = \hat{q}_n$. The goal is thus to find (self-adjoint) operators \hat{q}_r and \hat{p}_r ($r = 1, \dots, n$) such that (3) and (4) are satisfied with \hat{H}_P given by (1).

In order to find solutions of the CCs (other than the canonical solution), one introduces the following discrete Fourier transforms of position and momentum operators:

$$\hat{q}_r = \sum_{j=1}^n \sqrt{\frac{\hbar}{2mn\omega_j}} \left(e^{-2\pi ijr/n} a_j^+ + e^{2\pi ijr/n} a_j^- \right), \quad (5)$$

$$\hat{p}_r = i \sum_{j=1}^n \sqrt{\frac{m\omega_j\hbar}{2n}} \left(e^{-2\pi ijr/n} a_j^+ - e^{2\pi ijr/n} a_j^- \right). \quad (6)$$

Here, the ω_j are positive numbers given by

$$\omega_j^2 = \omega^2 + 2c - 2c \cos\left(\frac{2\pi j}{n}\right) = \omega^2 + 4c \sin^2\left(\frac{\pi j}{n}\right), \quad (7)$$

while the (yet unknown) operators a_j^\pm satisfy $(a_j^\pm)^\dagger = a_j^\mp$, implying that the operators \hat{q}_r and \hat{p}_r are indeed self-adjoint. In terms of the operators a_j^\pm , the Hamiltonian has a simple expression:

$$\hat{H}_P = \sum_{j=1}^n \frac{\hbar\omega_j}{2} (a_j^- a_j^+ + a_j^+ a_j^-) = \sum_{j=1}^n \frac{\hbar\omega_j}{2} \{a_j^-, a_j^+\}. \quad (8)$$

It is now easy to check that the CCs (3)-(4) are equivalent with finding operators a_k^\pm (subject to $(a_k^\pm)^\dagger = a_k^\mp$) such that

$$\left[\sum_{j=1}^n \omega_j \{a_j^-, a_j^+\}, a_k^\pm \right] = \pm 2\omega_k a_k^\pm, \quad (k = 1, 2, \dots, n). \quad (9)$$

Note how these relations involve both commutators and anti-commutators.

2.2. System with fixed wall boundary conditions

We now consider the same system of coupled harmonic oscillators, but this time with fixed wall boundary conditions. In this case, the Hamiltonian is

$$\hat{H}_{FW} = \sum_{r=1}^n \left(\frac{\hat{p}_r^2}{2m} + \frac{m\omega^2}{2} \hat{q}_r^2 \right) + \sum_{r=0}^n \frac{cm}{2} (\hat{q}_r - \hat{q}_{r+1})^2, \quad (10)$$

with the same data as in (1) but with boundary conditions

$$\hat{q}_0 = \hat{q}_{n+1} \equiv 0 \quad (\text{and } \hat{p}_0 = \hat{p}_{n+1} \equiv 0). \quad (11)$$

Stated otherwise, the first and last oscillator (those numbered 1 and n) are assumed to be attached to a fixed wall rather than being adjacent to one another.

The treatment of this system as a Wigner Quantum system is very similar to that of the system with periodic boundary conditions. In fact, the CCs are identical to (3) and (4) but with boundary conditions given by (11) (and with \hat{H}_P replaced by \hat{H}_{FW}). These new boundary conditions, however, imply the use of a different kind of transform (instead of the discrete Fourier transform). In this case, the following transformations are introduced:

$$\hat{q}_r = \sum_{j=1}^n \sqrt{\frac{\hbar}{m(n+1)\tilde{\omega}_j}} \sin\left(\frac{rj\pi}{n+1}\right) (a_j^+ + a_j^-), \quad (12)$$

$$\hat{p}_r = i \sum_{j=1}^n \sqrt{\frac{m\tilde{\omega}_j\hbar}{n+1}} \sin\left(\frac{rj\pi}{n+1}\right) (a_j^+ - a_j^-), \quad (13)$$

where the $\tilde{\omega}_j$ are positive numbers given by

$$\tilde{\omega}_j^2 = \omega^2 + 2c - 2c \cos\left(\frac{j\pi}{n+1}\right) = \omega^2 + 4c \sin^2\left(\frac{j\pi}{2(n+1)}\right). \quad (14)$$

As before, the operators a_j^\pm satisfy the adjointness conditions $(a_j^\pm)^\dagger = a_j^\mp$. The Hamiltonian \hat{H}_{FW} , when expressed in terms of the operators a_j^\pm is given by

$$\hat{H}_{FW} = \sum_{j=1}^n \frac{\hbar\tilde{\omega}_j}{2} \{a_j^-, a_j^+\}, \quad (15)$$

which is identical to (8) but with ω_j replaced by $\tilde{\omega}_j$. Also the CCs are identical to (9) up to the same replacement. This implies that the algebraic solutions will be similar, even though the conclusions about physical properties will be different due to the different numerical values of the numbers ω_j and $\tilde{\omega}_j$ (and due to the different transformations used).

2.3. Solution of the CCs in terms of $\mathfrak{gl}(1|n)$ generators

The CCs (9) involve both commutators and anti-commutators. Moreover, they resemble certain algebraic relations satisfied by the odd $\mathfrak{gl}(1|n)$ generators. The Lie superalgebra $\mathfrak{gl}(1|n)$ [11] has standard basis elements e_{jk} ($j, k = 0, 1, \dots, n$), where e_{k0} and e_{0k} ($k = 1, \dots, n$) are the odd elements (having degree one) while the other elements are the even elements (having degree zero). The bracket is given by:

$$[[e_{ij}, e_{kl}]] = \delta_{jk} e_{il} - (-1)^{\deg(e_{ij})\deg(e_{kl})} \delta_{il} e_{kj}, \quad (16)$$

and we impose the star condition $e_{ij}^\dagger = e_{ji}$. Then, one can give a solution of (9) in terms of the odd elements of this algebra. More explicitly, one has

$$a_j^- = \sqrt{\frac{2\beta_j}{\omega_j}} e_{j0}, \quad a_j^+ = \sqrt{\frac{2\beta_j}{\omega_j}} e_{0j} \quad (j = 1, \dots, n) \quad (17)$$

where

$$\beta_j = -\omega_j + \frac{1}{n-1} \sum_{k=1}^n \omega_k, \quad (j = 1, \dots, n). \quad (18)$$

Clearly, all these numbers β_j should be nonnegative. It can be proved [9] that (for $n \geq 4$) this implies that the coupling constant is bounded above by some positive critical value c_0 (dependent upon n). As was to be expected, the same solution is valid for the case of fixed wall boundary conditions but with the constants ω_j replaced by $\tilde{\omega}_j$ (the β_j are then denoted naturally by $\tilde{\beta}_j$). Also in the fixed wall boundary conditions case, the coupling constant c is bounded above by a critical value \tilde{c}_0 (different but similar to c_0) [10].

2.4. Unitary irreducible representations of $\mathfrak{gl}(1|n)$

In order to study physical properties of the systems, one has to work with specific unitary irreducible representations of $\mathfrak{gl}(1|n)$. These unitary representations $W = W([m]_{n+1})$ of $\mathfrak{gl}(1|n)$ are well known [16]: they are labeled by some $(n+1)$ -tuple $[m]_{n+1}$ subject to certain conditions. Even more: for such representations, a Gel'fand-Zetlin basis has been constructed and the explicit action of the $\mathfrak{gl}(1|n)$ generators on the basis vectors of the representation is also known [13]. Explicit actions of generators on a Gel'fand-Zetlin basis are usually quite involved, and this is also the case for $\mathfrak{gl}(1|n)$. In particular, the action of the odd generators e_{0j} and e_{j0} on a GZ-basis vector is very complicated, see [13, Eqs. (2.25) and (2.26)].

As a special case, we consider a simple, yet interesting and quite rich, class of finite-dimensional representations $W([1, p-1, 0, \dots, 0]) \equiv V(p)$, characterized by a positive integer p . A notation more convenient than the general GZ-patterns for the basis vectors of $V(p)$ is:

$$w(\theta; s) \equiv w(\theta; s_1, s_2, \dots, s_n), \quad \theta \in \{0, 1\}, \quad s_i \in \{0, 1, 2, \dots\}, \quad \text{and } \theta + s_1 + \dots + s_n = p. \quad (19)$$

The action of the $\mathfrak{gl}(1|n)$ generators on the basis (19) is given by ($1 \leq k \leq n$) [13, 14]:

$$e_{00}w(\theta; s) = \theta w(\theta; s), \quad (20)$$

$$e_{kk}w(\theta; s) = s_k w(\theta; s), \quad (21)$$

$$e_{k0}w(\theta; s) = \theta \sqrt{s_k + 1} w(1 - \theta; s_1, \dots, s_k + 1, \dots, s_n), \quad (22)$$

$$e_{0k}w(\theta; s) = (1 - \theta) \sqrt{s_k} w(1 - \theta; s_1, \dots, s_k - 1, \dots, s_n). \quad (23)$$

From these one deduces the action of other elements e_{kl} . The basis $w(\theta; s)$ of $V(p)$ is orthonormal, i.e. $\langle w(\theta; s), w(\theta'; s') \rangle = \delta_{\theta, \theta'} \delta_{s, s'}$, and with respect to this inner product the action of the generating elements satisfies the conjugacy relations $e_{k0}^\dagger = e_{0k}$ and $e_{0k}^\dagger = e_{k0}$.

3. Energy spectrum of the systems

As can be seen from (8) and (17), the Hamiltonian for both systems is a linear combination of elements e_{kk} ($k = 0, 1, \dots, n$). Since such elements act diagonally on the basis vectors of any representation $W([m]_{n+1})$, [13], the basis vectors of such a representation will be eigenvectors of the Hamiltonian. Stated otherwise, the basis vectors of the representation are the stationary states of the system.

We will now restrict our attention to the ladder representations $V(p)$ introduced in the previous section. In particular, for fixed wall boundary conditions we have:

$$\hat{H}_{FW} = \hbar(\tilde{\beta} e_{00} + \sum_{k=1}^n \tilde{\beta}_k e_{kk}),$$

with $\tilde{\beta} = \sum_{k=1}^n \tilde{\beta}_k = \sum_{k=1}^n \tilde{\omega}_k$. From (20) and (21) it follows immediately that:

$$\hat{H}_{FW} w(\theta; s) = \hbar(\tilde{\beta}\theta + \sum_{k=1}^n \tilde{\beta}_k s_k) w(\theta; s) = \hbar \tilde{E}_{\theta, s} w(\theta; s).$$

When $c = 0$, one has that $\tilde{\beta}_k = \omega/(n-1)$ and $\tilde{\beta} = \omega n/(n-1)$, so in this case there are only two eigenvalues namely

$$\tilde{E}_{0,s} = \frac{\omega p}{n-1} \quad \text{and} \quad \tilde{E}_{1,s} = \frac{\omega p}{n-1} + \omega$$

with multiplicities

$$\binom{p+n-1}{n-1} \quad \text{and} \quad \binom{p+n-2}{n-1}$$

respectively. For $c > 0$, these two energy levels split into

$$\binom{p+n-1}{n-1} + \binom{p+n-2}{n-1}$$

energy levels, each with multiplicity one. Figure 1(a) gives an example.

In the case of periodic boundary conditions, however, the system has much greater symmetry and hence one expects fewer different energy levels with higher degeneracies. Indeed, in [10] it is shown that the number of energy levels is in general given by

$$\binom{p + \lfloor n/2 \rfloor}{p} + \binom{p + \lfloor n/2 \rfloor - 1}{p-1}.$$

Figure 1(b) illustrated this. Also, the degeneracy of an energy level $E_{\theta,s}$ can in general be calculated from θ and s .

4. On the eigenvalues and eigenvectors for the position operators

Under the solution (17), the position operators \hat{q}_r are quite arbitrary odd elements of the Lie superalgebra $\mathfrak{gl}(1|n)$ as can be seen from (12). Since the action of the odd generators on a GZ-pattern is very complicated, determining the spectrum of the position operators is an involved task. However, in [14] a method was developed to determine the eigenvalues (and eigenvectors) of such an odd operator for a general unirrep $W([m]_{n+1})$ of $\mathfrak{gl}(1|n)$. This method goes as follows: in a first step, a new set of generators $E_{0j}^{(r)}$ and $E_{j0}^{(r)}$ (and $E_{00}^{(r)}$) for $\mathfrak{gl}(1|n)$ is introduced such that the operator \hat{q}_r has an easy form in terms of these new generators. More in particular, the operator \hat{q}_r will be an element of a $\mathfrak{gl}(1|1)$ subalgebra generated by $E_{00}^{(r)}, E_{nn}^{(r)}, E_{0n}^{(r)}$ and $E_{n0}^{(r)}$:

$$\hat{q}_r \sim E_{0n}^{(r)} + E_{n0}^{(r)}.$$

Hence, it is clear that the decomposition of the representation $W([m]_{n+1})$ as a $\mathfrak{gl}(1|1) \oplus \mathfrak{gl}(n-1)$ representation solves the eigenvalue problem. The eigenvalues are given by solving the $\mathfrak{gl}(1|1)$ eigenvalue problem for the various $\mathfrak{gl}(1|1)$ representations occurring, and their multiplicities by the dimension of the corresponding $\mathfrak{gl}(n-1)$ representation (for which there is a known dimension formula). The eigenvector problem is much harder, and involves expressing the new GZ-patterns in terms of the old GZ-patterns.

Let us now go into a little more detail. First, the position operator \hat{q}_r is written as:

$$\hat{q}_r = \sqrt{\frac{\hbar}{mn}} \sum_{j=1}^n \left(\gamma_j e^{2\pi i j r/n} e_{j0} + \gamma_j e^{-2\pi i j r/n} e_{0j} \right) \quad (24)$$

$$= \sqrt{\frac{\hbar \gamma}{mn}} (E_{n0}^{(r)} + E_{0n}^{(r)}), \quad (25)$$

where we use the notation

$$\gamma_j = \sqrt{\beta_j/\omega_j} \quad (j = 1, \dots, n) \quad \text{and} \quad \gamma = \gamma_1^2 + \dots + \gamma_n^2. \quad (26)$$

(a) Fixed Wall Boundary Conditions

(b) Periodic Boundary Conditions

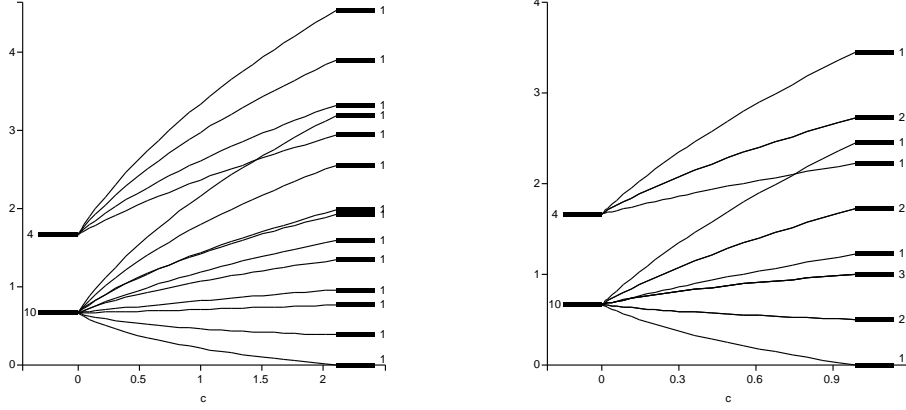


Figure 1. (a) The energy levels of the quantum system with fixed wall boundary conditions for $n = 4$ in the representation $V(p)$ with $p = 2$ and $\hbar = \omega = 1$; c ranges from 0 to \tilde{c}_0 . The vertical axis gives the energy values and the numbers next to the levels refer to their multiplicity. When $c = 0$, there are only two energy levels with multiplicities 10 and 4 respectively. When $0 < c < \tilde{c}_0$, however, there are (in general) 14 energy levels each with multiplicity 1. (b) The energy levels of the quantum system with periodic boundary conditions for $n = 4$ in the representation $V(p)$ with $p = 2$ and $\hbar = \omega = 1$; c ranges from 0 to c_0 . For $c = 0$, one always has the same system irrespective of the boundary conditions in place, so again we find the same two energy levels. When $c > 0$, however, there are in general only 9 different energy levels, some non-degenerate others two-fold and one three-fold degenerate.

This defines the operators $E_{n0}^{(r)}$ and $E_{0n}^{(r)}$. Then, this is extended to $E_{j0}^{(r)}$ (and $E_{0j}^{(r)}$) with $1 \leq j \leq n-1$ by using a unitary (Hessenberg) matrix. The operators $E_{j0}^{(r)}$ and $E_{0j}^{(r)}$ ($j = 1, \dots, n$) then also generate $\mathfrak{sl}(1|n)$ as they also satisfy the defining relations of $\mathfrak{sl}(1|n)$ [14, 17]. (The operator $E_{00}^{(r)}$ is then defined in the appropriate way.)

A unirrep of $\mathfrak{gl}(1|1)$ is characterized by two real numbers a and b , such that either $a + b > 0$ or $a + b = 0$. When $a + b > 0$, the representation $W([a, b])$ is two-dimensional with weights (a, b) and $(a - 1, b + 1)$. Furthermore, calling the basis vectors of the representation v and w one has:

$$e_{01}v = 0, \quad e_{10}v = \sqrt{a+b}w, \quad e_{01}w = \sqrt{a+b}v, \quad e_{10}w = 0.$$

In this case, it is clear that $(e_{01} + e_{10})(v \pm w) = \pm\sqrt{a+b}(v \pm w)$. When $a + b = 0$, the representation $W([a, -a])$ is one-dimensional and one has that $(e_{01} + e_{10})(v) = 0$. In this case the weight is given by $(a, -a)$.

In the case of the ladder representation $V(p)$ the decomposition $\mathfrak{gl}(1|n) \rightarrow \mathfrak{gl}(1|1) \oplus \mathfrak{gl}(n-1)$ is given by

$$V(p) \rightarrow W([0, 0]) \times V([p, 0, \dots, 0]) \oplus \bigoplus_{K=0}^{p-1} W([1, p-1-K]) \times V([K, 0, \dots, 0]), \quad (27)$$

where the $\mathfrak{gl}(n-1)$ representation $V([K, 0, \dots, 0])$ has dimension $\binom{n-2+K}{n-2}$. So \hat{q}_r has $2p + 1$

eigenvalues in all, namely $\pm x_K = \pm \sqrt{\frac{h\gamma}{mn}(p-K)}$, where $0 \leq K \leq p-1$, with multiplicities $\binom{n-2+K}{n-2}$, and $x_p = 0$ with multiplicity $\binom{n-2+p}{n-2}$.

One can also explicitly compute the eigenvectors of \hat{q}_r in the ladder representation as follows. First, any basis vector $w(\theta; s_1, \dots, s_n)$ can be obtained from the highest weight vector $w(1; p-1, 0, \dots, 0)$ by acting on it with negative root vectors. Explicitly, one has:

$$w(\theta; s_1, \dots, s_n) = \frac{e_{n,n-1}^{p-\theta-\sum_{j=1}^{n-1} s_j} e_{n-1,n-2}^{p-\theta-\sum_{j=1}^{n-2} s_j} \dots e_{32}^{p-\theta-\sum_{j=1}^2 s_j} e_{21}^{p-\theta-\sum_{j=1}^1 s_j} e_{10}^{1-\theta}}{\sqrt{p^{1-\theta} \prod_{k=1}^{n-1} (p-\theta-\sum_{j=1}^k s_j)! (s_k+1)_{p-\theta-\sum_{j=1}^k s_j}}} w(1; p-1, 0, \dots, 0),$$

where $(a)_j = a(a+1)\dots(a+j-1)$ is the Pochhammer symbol or rising factorial.

We now define a basis $v(\phi; t_1, \dots, t_n)$ relative to the new generators E_{jk} as follows (for brevity, the superscripts r are temporarily dropped):

$$v(\phi; t_1, \dots, t_n) = \frac{E_{n,n-1}^{p-\phi-\sum_{j=1}^{n-1} t_j} E_{n-1,n-2}^{p-\phi-\sum_{j=1}^{n-2} t_j} \dots E_{32}^{p-\phi-\sum_{j=1}^2 t_j} E_{21}^{p-\phi-\sum_{j=1}^1 t_j} E_{10}^{1-\phi}}{\sqrt{p^{1-\phi} \prod_{k=1}^{n-1} (p-\phi-\sum_{j=1}^k t_j)! (t_k+1)_{p-\phi-\sum_{j=1}^k t_j}}} v(1; p-1, 0, \dots, 0). \quad (28)$$

Of course, in order to make this definition work, we have to express the highest weight vector $v(1; p-1, 0, \dots, 0)$ in terms of the ‘‘old’’ basis vectors $w(\theta; s)$. So, one has to look for the unique vector that is annihilated by all positive root vectors $E_{jk}^{(r)}$ with $k > j$. For the ladder representation $V(p)$ one has that

$$v(1; p-1, 0, \dots, 0) = \frac{1}{(\gamma_1^2 + \gamma_2^2)^{(p-1)/2}} \sum_{u=0}^{p-1} (-1)^u e^{-2\pi i r u/n} \sqrt{\binom{p-1}{u}} \times \gamma_1^{p-1-u} \gamma_2^u w(1; u, p-1-u, 0, \dots, 0). \quad (29)$$

We remark that Proposition 4 of [14] solves this particular problem for arbitrary unirreps $W([m]_{n+1})$. Using the actions (22) and (23), it is now easy to check that the orthonormal eigenvectors for $\pm x_K \neq 0$ are:

$$\psi_{r,\pm x_K,t} = \frac{1}{\sqrt{2}} v(1; t_1, \dots, t_{n-1}, p-1-K) \pm \frac{1}{\sqrt{2}} v(0; t_1, \dots, t_{n-1}, p-K), \quad (30)$$

where $t_1 + \dots + t_{n-1} = K$. For the eigenvalue 0, the eigenvectors read

$$\psi_{r,0,t} = v(0; t_1, \dots, t_{n-1}, 0), \quad t_1 + \dots + t_{n-1} = p. \quad (31)$$

Note how the multiplicity labels t are in accordance with the previously stated multiplicities of the eigenvalues x_K .

5. Application: position probabilities

For the ladder representations $V(p)$, the expressions (30) (or (31)) together with (29) and (28) give the eigenvectors of the position operators \hat{q}_r . Although, admittedly, these expressions do not yield an analytical expression of the eigenvectors in terms of the basis vectors $w(\theta; s)$, they do allow to compute them efficiently using only matrix multiplications. Stated otherwise, the coefficients $C_{r,\pm x_K,t}^{\theta,s}$ in the expansion

$$\psi_{r,\pm x_K,t} = \sum_{\theta,s} C_{r,\pm x_K,t}^{\theta,s} w(\theta; s),$$

may be efficiently computed (numerically).

It is a well known fact (postulate) of quantum mechanics that when measuring an observable, the measurement always yields an eigenvalue of the (self-adjoint) operator associated with that observable. The probability of measuring a certain eigenvalue when the system is in a particular state is determined by the expansion of that state in terms of (orthonormal) eigenvectors of the operator at hand.

So, when the quantum system is in the fixed stationary state $w(\theta, s)$, the probability of measuring for \hat{q}_r the eigenvalue $\pm x_K$ is given by:

$$P(\theta, s, r, \pm x_K) = \sum_{t_1 + \dots + t_{n-1} = K} \left| C_{r, \pm x_K, t}^{\theta, s} \right|^2. \quad (32)$$

Also, from (30) it is immediately clear that

$$P(\theta, s, r, x_K) = P(\theta, s, r, -x_K). \quad (33)$$

For now, we will restrict our attention to the case of periodic boundary conditions, the case of fixed wall boundary conditions being elaborated upon in [10]. For the case of periodic boundary conditions, it is clear that the position probabilities are independent of the number of the oscillator r , as all oscillators are equivalent. So, one can take $r = 1$. As an explicit example, we choose $n = 4$ (so there are four coupled oscillators) and $p = 10$. This means that each position operator has 21 distinct eigenvalues. We will plot the position probability distributions for the ground state (this is the state $w(0; 0, p, 0, 0) = w(0; 0, 10, 0, 0)$) and for the most excited state (this is the state $w(1; 0, 0, 0, p - 1) = w(1; 0, 0, 0, 9)$). These distributions are given in Figure 2, for some values of the coupling constant c .

Let us make a number of observations on these distributions. When the system is in the ground state, the probability distribution function of each position operator is symmetric around its equilibrium position and unimodal. Of course it is also discrete (as we are working in finite-dimensional representations). As the coupling constant c increases, the ‘‘peak’’ around the equilibrium position is sharper. In other words, as the coupling constant becomes larger, the oscillators are more likely to be close to their equilibrium position when the system is in its ground state.

When the system is in its most excited state, the position probabilities are quite different. The probability of finding the oscillator in its equilibrium position is zero. On the other hand, there are certain peaks away from the equilibrium position. As c increases, these peaks are further away from the equilibrium position. In other words, as the coupling constant becomes larger, the oscillators are more likely to be further away from their equilibrium position when the system is in its most excited state.

Note that in this figure one also observes the fact that the range of the spectrum of the position operators depends on the coupling constant c [10].

6. Conclusion

In this article, we have treated a linear chain of harmonic oscillators as a Wigner Quantum System. By suitably manipulating the compatibility conditions, we showed how a solution in terms of odd elements of the Lie superalgebra $\mathfrak{gl}(1|n)$ can be found. The physical properties of the system are then determined by the $\mathfrak{gl}(1|n)$ unitary representation used. Investigating these physical properties leads to quite interesting and challenging problems in the representation theory of $\mathfrak{gl}(1|n)$, and a solution to these problems was presented. The specific examples and figures given used a simple yet interesting class of representations $V(p)$ with p an integer. The properties obtained for such a system, e.g. the position probabilities are in line with physical intuition.

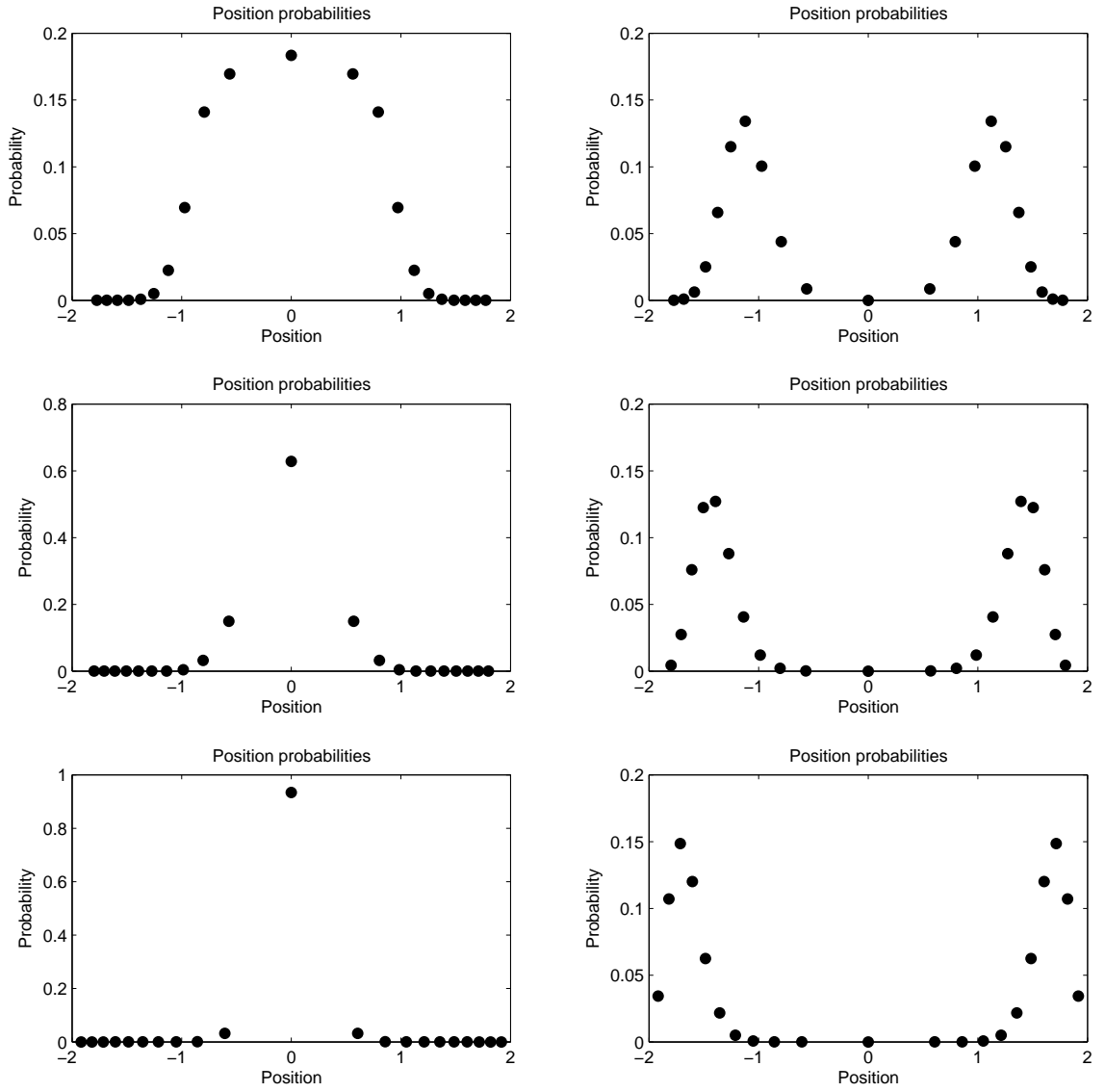


Figure 2. Position probability distribution function for the position operator \hat{q}_1 , in the periodic boundary case, when $n = 4$ and $p = 10$. In the three rows, $c = 0.1$, $c = 0.4$, $c = 0.8$. The plotted value is $P(\theta, s, 1, \pm x_K)$ for each of the 21 eigenvalues $\pm x_K$ ($K = 0, 1, \dots, 10$) of \hat{q}_1 . This is given for the case when the system is in the stationary state $w(\theta; s)$ corresponding to the ground state (minimum energy) in the left column and in the stationary state $w(\theta; s)$ corresponding to the most excited state (maximum energy).

Although we restricted ourselves here to the case of $\mathfrak{gl}(1|n)$ solutions, other types of solutions exist, more specifically in terms of $\mathfrak{osp}(1|2n)$ elements. We expect to further investigate this system (and others, such as a multi-dimensional anisotropic oscillator) using a newly constructed class of infinite dimensional unirreps of $\mathfrak{osp}(1|2n)$ [18]. As this class of solutions includes the canonical solution, such a study will be most interesting, although we expect it to be computationally difficult.

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