# DEFORMED JACOBSON GENERATORS OF THE ALGEBRA $\mathbf{U_q[sl(n+1)]}$ AND THEIR FOCK REPRESENTATIONS


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A description of the quantum algebra $U_q[sl(n+1)]$ via a new set of generators, called deformed Jacobson generators, is given. It provides an alternative to the canonical description of $U_q[sl(n+1)]$ in terms of Chevalley generators. The Jacobson generators satisfy trilinear commutation relations and define $U_q[sl(n+1)]$ as a deformed Lie triple system. Fock representations are constructed and the action of the Jacobson generators on the Fock basis is written down. Finally, Dyson and Holstein-Primakoff realizations are given.


## 1 Deformed Jacobson generators of $U_q[sl(n+1)]$

We describe the quantum algebra $U_q[sl(n+1)]$ in terms of a new set of generators [1], which are different from the known Chevalley generators. The deformation however is the usual Hopf algebra deformation [2]. In the classical case, these new generators describe the Lie algebra $sl(n+1)$ using relations, which were introduced for the first time by Jacobson[3] in the context of Lie triple systems. For this reason we call the new generators for $U_q[sl(n+1)]$ deformed Jacobson generators (JGs).

The definition of deformed JGs can be best presented in the framework of a set of Cartan-Weyl elements $e_{ij}$, $i,j=0,\ldots,n$ of $U_q[gl(n+1)]$. These Cartan-Weyl elements can be expressed in terms of the usual Chevalley gener-

ators of $U_q[gl(n+1)]$ [4,5]. Recall that a construction of Cartan-Weyl elements is necessary in order to describe a complete basis (PBW-basis) of $U_q[gl(n+1)]$. We refer to $e_{ij}$ as a positive root vector (resp. negative root vector) if $i<j$ (resp. $i>j$). Fix a total order for the set of elements $e_{ij}$ :
for the positive root vectors

$$e_{ij} < e_{kl}, \text{ if } i<k \text{ or } i=k \text{ and } j<l; \tag{1}$$

for the negative root vectors $e_{ij}$ one takes the same rule (1), and total order is fixed by choosing

$$\text{positive root vectors} < \text{negative root vectors} < e_{ii}. \tag{2}$$

The difference between $U_q[sl(n+1)]$ and $U_q[gl(n+1)]$ is in the elements of the Cartan subalgebra. For $U_q[gl(n+1)]$ the Cartan subalgebra is generated by $e_{ii}$ $(i=0,\ldots,n)$ and for $U_q[sl(n+1)]$ by the elements $H_i = e_{00} - e_{ii}$ $(i = 1,\ldots,n)$. We use also the elements $L_i = q^{H_i}$ and $\bar{L}_i = q^{-H_i}$. Then a complete set of relations between these Cartan-Weyl elements is given by

$$[H_i, H_j] = 0; \quad [H_i, e_{jk}] = (\delta_{0j} - \delta_{0k} - \delta_{ij} + \delta_{ik})e_{jk}; \tag{3}$$

for two positive root vectors $e_{ij} < e_{kl}$ :

$$[e_{ij}, e_{kl}]_{q^{\delta_{jl}-\delta_{jk}+\delta_{ik}}} = \delta_{jk}e_{il} + (q-q^{-1})\theta(l>j>k>i)e_{kj}e_{il}; \tag{4}$$

for two negative root vectors $e_{ij} > e_{kl}$ :

$$[e_{ij}, e_{kl}]_{q^{-\delta_{jl}+\delta_{jk}-\delta_{ik}}} = \delta_{jk}e_{il} - (q-q^{-1})\theta(i>k>j>l)e_{kj}e_{il}; \tag{5}$$

and finally for a positive root vector $e_{ij}$ and a negative root vector $e_{kl}$ :

$$\begin{aligned}
[e_{ij}, e_{kl}] = {} & \frac{\delta_{il}\delta_{jk}}{q-q^{-1}}\left(L_j\bar{L}_i - \bar{L}_jL_i\right) + \\
& \Big((q-q^{-1})\theta(j>k>i>l)e_{kj}e_{il} - \delta_{il}\theta(j>k)e_{kj} + \delta_{jk}\theta(i>l)e_{il}\Big)L_i\bar{L}_k + \\
& L_j\bar{L}_l\Big(-(q-q^{-1})\theta(k>j>l>i)e_{il}e_{kj} - \delta_{il}\theta(k>j)e_{kj} + \delta_{jk}\theta(l>i)e_{il}\Big),
\end{aligned} \tag{6}$$

where

$$[a,b]_x = ab - xba; \quad \theta(i_1 > i_2 > \ldots > i_r) = \begin{cases} 1, \text{ if } i_1 > i_2 > \ldots > i_r, \\ 0, \text{ otherwise.} \end{cases} \tag{7}$$

Define the Jacobson generators of $U_q[sl(n+1)]$ to be the following Cartan-Weyl vectors :

$$a_i^- = e_{0i}, \qquad a_i^+ = e_{i0}, \qquad H_i, \qquad i = 1,\ldots,n. \tag{8}$$

Then from (6) one obtains :

$$e_{ji} = -\bar{L}_i[a_i^-, a_j^+], \quad (i < j); \qquad e_{ji} = -[a_i^-, a_j^+]L_j, \quad (i > j). \tag{9}$$

In order to be able to order all Cartan-Weyl generators in an arbitrary expression in the right order we need a complete set of relations in terms of the deformed JG's. The result follows :

*Theorem 1.* A set of Cartan-Weyl elements of $U_q[sl(n+1)]$ is given by $H_i$, $a_i^\pm$, $[a_i^+, a_j^-]$ $(i \neq j = 1, \ldots, n)$. A complete set of supercommutation relations between these elements is given by :

$$[H_i, H_j] = 0; \qquad [H_i, a_j^\pm] = \mp(1 + \delta_{ij})a_j^\pm; \tag{10}$$

$$[a_i^-, a_i^+] = \frac{L_i - \bar{L}_i}{q - q^{-1}}; \qquad [a_i^\eta, a_j^\eta]_q = 0 \quad (i < j); \tag{11}$$

$$\begin{aligned} [[a_i^\eta, a_j^{-\eta}], a_k^\eta]_{q^{\xi(1+\delta_{ik})}} &= \eta\delta_{jk}L_k^{-\xi\eta}a_i^\eta + \epsilon(j,k,i)(q - \bar{q})[a_k^\eta, a_j^{-\eta}]a_i^\eta \\ &= \eta\delta_{jk}L_k^{-\xi\eta}a_i^\eta + \epsilon(j,k,i)q^\xi(q - \bar{q})a_i^\eta]a_k^\eta, a_j^{-\eta}], \end{aligned} \tag{12}$$

where $(j - i)\xi > 0$, $\xi$, $\eta = \pm$; $\quad \bar{q} = q^{-1}$;

and $\epsilon(j,k,i) = \begin{cases} 1, & \text{if } j > k > i; \\ -1, & \text{if } j < k < i; \\ 0, & \text{otherwise.} \end{cases}$

This description of $U_q[sl(n+1)]$ in terms of deformed JGs is referred to as a deformed Lie triple system.

## 2 Fock representations

We construct Fock modules using the induced module procedure. $G = U_q[sl(n+1)]$, with Cartan-Weyl elements $H_i$, $a_i^\pm$ and $[a_i^+, a_j^-]$ $(i \neq j = 1, \ldots, n)$, has a subalgebra $A = U_q[gl(n)]$ with Cartan-Weyl elements $H_i$ and $[a_i^+, a_j^-]$ $(i \neq j = 1, \ldots, n)$. Define a trivial one-dimensional $A$ module as follows :

$$[a_i^-, a_j^+]|0\rangle = 0, \quad i \neq j = 1, \ldots, n; \quad H_i|0\rangle = p|0\rangle, \tag{13}$$

where $p$ is any complex number. Let $P$ be the (associative) subalgebra of $G = U_q[sl(n+1)]$ generated by the elements of $A$ and $\{a_i^-; i = 1, \ldots, n\}$. The one-dimensional $A$ module $\mathbf{C}|0\rangle$ can be extended to a one-dimensional $P$ module by requiring $a_i^-|0\rangle = 0$, $i = 1, \ldots, n$. Now the $G$ module $\bar{W}_p$ is defined as $\bar{W}_p = \mathrm{Ind}_P^G\, \mathbf{C}|0\rangle$. Clearly $\bar{W}_p$ is freely generated by the generators $a_i^+$ $(i = 1, \ldots, n)$ acting on $|0\rangle$. Therefore a basis for $\bar{W}_p$ is given by

$$|p; l_1, l_2, \ldots, l_n\rangle \equiv (a_1^+)^{l_1}(a_2^+)^{l_2}\ldots(a_n^+)^{l_n}|0\rangle, \quad l_i \in \mathbf{Z}_+, \quad i = 1, \ldots, n. \tag{14}$$

*Theorem 2.* The transformation of the basis (14) of $\bar{W}_p$ under the action of the JGs reads $(i = 1, \ldots, n)$, denoting $l_1 + l_2 + \ldots + l_n$ by $L$ :

$$H_i|p; l_1, \ldots, l_n\rangle = (p - l_i - L)|p; l_1, \ldots, l_n\rangle, \tag{15}$$

$$a_i^+|p; l_1, \ldots, l_n\rangle = \bar{q}^{l_1+\ldots+l_{i-1}}|p; l_1, \ldots, l_{i-1}, l_i + 1, l_{i+1}, \ldots, l_n\rangle, \tag{16}$$

$$\begin{aligned} a_i^-|p; l_1, \ldots, l_n\rangle = q^{l_1+\ldots+l_{i-1}}[l_i][p - L + 1] \\ \times|p; l_1, \ldots, l_{i-1}, l_i - 1, l_{i+1}, \ldots, l_n\rangle, \end{aligned} \tag{17}$$

where $[x] = (q^x - q^{-x})/(q - q^{-1})$.

The action of the elements $H_i$ and $a_i^\pm$ on the basis vectors of $\bar{W}_p$, determined in *Theorem 2* imply that $\bar{W}_p$ has an invariant submodule when $p$ is a nonnegative integer. From now on we shall assume that $p \in \mathbf{Z}_+$. Then we have

*Corollary.* The $U_q[sl(n+1)]$ module $\bar{W}_p$ has an invariant submodule $V_p$ with basis vectors $|p; l_1, l_2, \ldots, l_n\rangle$, with $L = \sum_{i=1}^n l_i > p$.

The quotient module $W_p = \bar{W}_p/V_p$ carries an irreducible representation for $U_q[sl(n+1)]$ and is referred to as a Fock module. The basis vectors of $W_p$ are given by (the representatives of)

$$|p; l_1, l_2, \ldots, l_n\rangle, \text{ with } L = \sum_{i=1}^n l_i \leq p. \tag{18}$$

Now we consider those Fock modules relevant for physical applications. A representation of $U_q[sl(n+1)]$ is unitary if the representation space is a Hilbert space and the representatives of $a_i^\pm$ and $H_i$ satisfy

$$(a_i^+)^\dagger = a_i^-, \qquad (a_i^-)^\dagger = a_i^+, \qquad (H_i)^\dagger = H_i. \tag{19}$$

In each Fock module $W_p$ define a scalar product ( , ) by postulating :

$$(|0\rangle\ ,\ |0\rangle) = \langle 0|0\rangle = 1, \quad (a_i^\pm v, w) = (v, a_i^\mp w), \quad \forall v, w \in W_p. \tag{20}$$

Then any two vectors $|p; l_1, l_2, \ldots, l_n\rangle$ and $|p; l_1', l_2', \ldots, l_n'\rangle$ with $(l_1, l_2, \ldots, l_n) \neq (l_1', l_2', \ldots, l_n')$ are orthogonal and

$$(|p; l_1, l_2, \ldots, l_n\rangle, |p; l_1, l_2, \ldots, l_n\rangle) = \frac{[p]!}{[p-L]!}\prod_{i=1}^n [l_i]!\ . \tag{21}$$

A careful analysis shows

*Theorem 3.* The irreducible Fock module $W_p$ $(p \geq 2)$ is unitary if and only if $q$ is a phase, i.e. $q = e^{i\phi}$, with $-\frac{\pi}{p} < \phi < \frac{\pi}{p}$.

Under these conditions, define an orthonormal basis of $W_p$ as follows :

$$|p;l_1,l_2,\ldots,l_n) = \sqrt{\frac{[p-L]!}{[p]![l_1]!\ldots[l_n]!}}|p;l_1,l_2,\ldots,l_n\rangle, \quad 0\leq L\leq p. \tag{22}$$

In the new basis (22) the transformations (15)-(16) read $(i=1,\ldots,n)$ :

$$H_i|p;l_1,\ldots,l_n) = (p-l_i-L)|p;l_1,\ldots,l_n), \tag{23}$$

$$\begin{aligned} a_i^-|p;l_1,\ldots,l_n) = \\ \times q^{l_1+\ldots+l_{i-1}}\sqrt{[l_i][p-L+1]}\ |p;l_1,\ldots l_{i-1},l_i-1,l_{i+1},\ldots,l_n), \end{aligned} \tag{24}$$

$$\begin{aligned} a_i^+|p;l_1,\ldots,l_n) = \\ \times \bar{q}^{l_1+\ldots+l_{i-1}}\sqrt{[l_i+1][p-L]}\ |p;l_1,\ldots l_{i-1},l_i+1,l_{i+1},\ldots,l_n). \end{aligned} \tag{25}$$

## 3 Dyson and Holstein-Primakoff realizations of $U_q[sl(n+1)]$

Denote by $W(n)$ the algebra of all polynomials of $n$ pairs of Bose operators $b_i^\pm$, namely $[b_i^-,b_j^+]=\delta_{ij},\ [b_i^+,b_j^+]=[b_i^-,b_j^-]=0$. Let $N_i=b_i^+b_i^-,\ \ N=\sum_{j=1}^n N_j$.

*Theorem 4.* (Dyson realization). The linear map $\rho : U_q[sl(n+1)] \to W(n)$, defined on the Jacobson generators as

$$\begin{aligned} \rho(H_i) &= p - b_i^+b_i^- - \sum_{j=1}^n b_j^+b_j^- = p - N_i - N, \\ \rho(a_i^-) &= q^{N_1+\ldots+N_{i-1}}\frac{[N_i+1]}{N_i+1}[p-N]b_i^-, \\ \rho(a_i^+) &= \bar{q}^{N_1+\ldots+N_{i-1}}b_i^+, \end{aligned} \tag{26}$$

is a homomorphism of $U_q[sl(n+1)]$ into $W(n)$ for any $p\in \mathbf{C}$.

The Dyson realization defines an infinite-dimensional representation of $U_q[sl(n+1)]$ in the Fock space $\mathcal{F}(n)$ with basis :

$$(b_1^+)^{r_1}\ldots(b_n^+)^{r_n}|0\rangle \equiv |r_1,r_2,\ldots,r_n\rangle,\ r_i\in \mathbf{Z}_+,\ i=1,\ldots,n. \tag{27}$$

If $p$ is a positive integer, $p\in\mathbf{N}$, the representation is indecomposable. If $p\notin\mathbf{N}$ the representation is irreducible. In both cases however it is impossible to introduce a scalar product in $\mathcal{F}(n)$ for which the operators (26) satisfy (19). This is a disadvantage of the Dyson realization.

*Theorem 5.* (Holstein-Primakoff realization). The linear map $\varrho : U_q[sl(n+1)] \to W(n)$, defined on the Jacobson generators as

$$\varrho(H_i) = p - b_i^+b_i^- - \sum_{j=1}^n b_j^+b_j^- = p - N_i - N,$$

$$\varrho(a_i^-) = q^{N_1+\ldots+N_{i-1}} \sqrt{\frac{[N_i+1]}{N_i+1}[p-N]} b_i^-, \tag{28}$$

$$\varrho(a_i^+) = \bar{q}^{N_1+\ldots+N_{i-1}} \sqrt{\frac{[N_i]}{N_i}[p-N+1]} b_i^+,$$

is a homomorphism of $U_q[sl(n+1)]$ into $W(n)$.

If $p \in \mathbf{N}$, then $\mathcal{F}_0(n)$ with basis :

$$\frac{(b_1^+)^{r_1}(b_2^+)^{r_2}\ldots(b_n^+)^{r_n}}{\sqrt{r_1!r_2!\ldots r_n!}}|0\rangle \equiv |r_1, r_2, \ldots, r_n), \quad r_1+\ldots+r_n \leq p \tag{29}$$

is a finite-dimensional irreducible module; the hermicity condition (19) holds if and only if $q = e^{i\phi}$, $-\frac{\pi}{p} < \phi < \frac{\pi}{p}$. From (9) and (28) one obtains the Holstein-Primakoff realization of the remaining Cartan-Weyl elements of $U_q[sl(n+1)]$ :

$$\varrho(e_{ji}) = q^{N_{j+1}+N_{j+2}+\ldots+N_{i-1}} \sqrt{\frac{[N_j][N_i+1]}{N_j(N_i+1)}} b_j^+ b_i^-, \; j < i,$$

$$\varrho(e_{ji}) = \bar{q}^{N_{i+1}+N_{i+2}+\ldots+N_{j-1}} \sqrt{\frac{[N_j][N_i+1]}{N_j(N_i+1)}} b_j^+ b_i^-, \; j > i. \tag{30}$$

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