ON CHARACTERS AND DIMENSION FORMULAS FOR REPRESENTATIONS OF THE LIE SUPERALGEBRA $\mathfrak{gl}(M|N)$

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We derive a new expression for the supersymmetric Schur polynomials $s_\lambda(x/y)$. The origin of this formula goes back to representation theory of the Lie superalgebra $\mathfrak{gl}(m|n)$ and gives rise to a determinantal formula for $s_\lambda(x/y)$. In the second part, we use this determinantal formula to derive new expressions for the dimension and superdimension of covariant representations V_λ of the Lie superalgebra $\mathfrak{gl}(m|n)$. In particular, we derive the t-dimension formula, giving a specialization of the character corresponding to the $\mathbb{Z}$-grading of V_λ. For a special choice of λ, the new t-dimension formula gives rise to a Hankel determinant identity.

1. Introduction

Shortly after the classification of finite-dimensional simple Lie superalgebras [1, 2], Kac considered the problem of classifying all finite-dimensional simple modules (i.e. irreducible representations) of the basic classical Lie superalgebras [3]. For a subclass of these modules, known as the 'typical' modules, Kac derived a character formula closely analogous to the Weyl character for simple modules of simple Lie algebras [3].

The problem of obtaining a character formula for the remaining 'atypical' representations has been subject of intensive investigation but is still not solved other than in various special cases. Solutions were given e.g. for covariant and contravariant tensor representations [4, 5], for so-called generic representations [6], for singly atypical representations [7, 8, 9] or for tame representations [10]. Although there is no explicit character formula, Serganova [11, 12] gave an algorithm to compute the characters of simple modules in $\mathfrak{gl}(m|n)$. This algorithm gives a method to determine the multiplicities $a_{\lambda\mu}$ of the simple modules V_μ in the composition series of a Kac module $\overline{V}_\lambda$, more specific $\operatorname{ch}\overline{V}_\lambda = \sum_\mu a_{\lambda,\mu} \operatorname{ch} V_\mu$ (where λ and μ refer here to the highest weight of the modules). In this algorithm, all weights μ are determined with nonzero $a_{\lambda,\mu}$ for a given weight λ; this turns out to be rather complicated. In [13] the authors conjectured a simple rule to find all nonzero $a_{\lambda,\mu}$ for any given weight μ. This conjecture was proved by Brundan [14].

This paper is concerned with covariant representations of $\mathfrak{gl}(m|n)$, i.e. representations for which the character is already known [4, 5, 15] to correspond to a supersymmetric Schur polynomial. In the following section, we show that these covariant modules are *tame*, leading to an alternative character formula. Comparing the two formulas, this gives rise to a new determinantal formula for supersymmetric Schur polynomials. A short overview of supersymmetric Schur functions is given in Section 3. Finally, in Section 4 we consider a specialization of the character, corresponding to the $\mathbb{Z}$-grading of the representation induced by the $\mathbb{Z}$-grading of $\mathfrak{gl}(m|n)$. This specialization, referred to as the t-dimension of the representation, is computed in two ways.

2. Covariant modules of the Lie superalgebra $\mathfrak{gl}(m|n)$.

Let $\mathfrak{g}$ be the Lie superalgebra $\mathfrak{gl}(m|n)$. Lie superalgebras are characterized by a $\mathbb{Z}_2$-grading $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$. Let $\mathfrak{h} \subset \mathfrak{g}$ be the Cartan subalgebra of $\mathfrak{g}$, and $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{+1}$ the consistent $\mathbb{Z}$-grading. The dual space $\mathfrak{h}^*$ of $\mathfrak{h}$ has a natural basis $\{\epsilon_1, \ldots, \epsilon_m, \delta_1, \ldots, \delta_n\}$, and the roots of $\mathfrak{g}$ can be expressed in terms of this basis. Let Δ be the set of all roots, Δ_0 the set of even roots, and Δ_1 the set of odd roots. One can choose a set of simple roots, but note that contrary to the case of simple Lie algebras not all such choices are equivalent. The so-called *distinguished choice* [1] is given by

$$\Pi = \{\epsilon_1 - \epsilon_2, \ldots, \epsilon_{m-1} - \epsilon_m, \epsilon_m - \delta_1, \delta_1 - \delta_2, \ldots, \delta_{n-1} - \delta_n\}. \tag{1}$$

Then, with Δ_+ the set of positive roots, we have explicitly :

$$\begin{aligned} \Delta_{0,+} &= \{\epsilon_i - \epsilon_j | 1 \le i < j \le m\} \cup \{\delta_i - \delta_j | 1 \le i < j \le n\}, \\ \Delta_{1,+} &= \{\beta_{ij} = \epsilon_i - \delta_j | 1 \le i \le m,\ 1 \le j \le n\}. \end{aligned} \tag{2}$$

Thus in the distinguished basis there is only one simple root which is odd. As usual, we put

$$\rho_0 = \frac{1}{2}\Big(\sum_{\alpha\in\Delta_{0,+}} \alpha\Big), \qquad \rho_1 = \frac{1}{2}\Big(\sum_{\alpha\in\Delta_{1,+}} \alpha\Big), \qquad \rho = \rho_0 - \rho_1. \tag{3}$$

There is a symmetric form $(\ ,\ )$ on $\mathfrak{h}^*$ induced by the invariant symmetric form on $\mathfrak{g}$, and in the natural basis it takes the values $(\epsilon_i, \epsilon_j) = \delta_{ij}$, $(\epsilon_i, \delta_j) = 0$ and $(\delta_i, \delta_j) = -\delta_{ij}$. The odd roots are isotropic: $(\alpha, \alpha) = 0$ if $\alpha \in \Delta_1$. Let $\Lambda \in \mathfrak{h}^*$; the *atypicality* of Λ, denoted by $\mathrm{atyp}(\Lambda)$, is the maximal number of linearly independent roots β_i such that $(\beta_i, \beta_j) = 0$ and $(\Lambda, \beta_i) = 0$ for all i and j [10]. Such a set $\{\beta_i\}$ is called a Λ-maximal isotropic subset of Δ.

Given a set of positive roots Δ_+ of Δ, and a simple odd root α, one may construct a new set of positive roots [10, 16] by

$$\Delta'_+ = (\Delta_+ \cup \{-\alpha\}) \setminus \{\alpha\}. \tag{4}$$

The set Δ'_+ is called a simple reflection of Δ_+. Since we use only simple reflections with respect to simple odd roots, $\Delta_{0,+}$ remains invariant, but $\Delta_{1,+}$ will change and the new ρ is given by:

$$\rho' = \rho + \alpha. \tag{5}$$

Consider a finite-dimensional $\mathfrak{g}$-module V. Such modules are $\mathfrak{h}$-diagonalizable with weight decomposition $V = \oplus_\mu V(\mu)$, and the character is defined to be $\mathrm{ch}\, V = \sum_\mu \dim V(\mu)\, e^\mu$, where e^μ $(\mu \in \mathfrak{h}^*)$ is the formal exponential. If we fix a set of positive roots Δ_+, we may talk about the highest weight Λ of V and about the corresponding ρ. If Δ'_+ is obtained from Δ_+ by a simple α-reflection, where α is odd, and Λ' denotes the highest weight of V with respect to Δ'_+, then [10]

$$\Lambda' = \Lambda - \alpha \text{ if } (\Lambda, \alpha) \neq 0; \qquad \Lambda' = \Lambda \text{ if } (\Lambda, \alpha) = 0. \tag{6}$$

If α is a *simple* odd root from Δ_+ then $(\rho, \alpha) = \frac{1}{2}(\alpha, \alpha) = 0$ [10, p. 421], and therefore, following (5) and (6):

$$\begin{aligned} &\Lambda' + \rho' = \Lambda + \rho \ \text{ if } \ (\Lambda + \rho, \alpha) \neq 0, \\ &\Lambda' + \rho' = \Lambda + \rho + \alpha \ \text{ if } \ (\Lambda + \rho, \alpha) = 0. \end{aligned} \tag{7}$$

From this, one deduces that for the $\mathfrak{g}$-module V, $\mathrm{atyp}(\Lambda+\rho)$ is independent of the choice of Δ_+; then $\mathrm{atyp}(\Lambda + \rho)$ is referred to as the atypicality of V (if $\mathrm{atyp}(\Lambda+\rho) = 0$, V is typical, otherwise it is atypical). If one can choose a $(\Lambda+\rho)$-maximal isotropic subset S_Λ in Δ_+ such that $S_\Lambda \subset \Pi \subset \Delta_+$ (Π is the set of simple roots with respect to Δ_+), then the $\mathfrak{g}$-module V is called *tame*, and a character formula is known due to Kac and Wakimoto [10]. It reads:

$$\mathrm{ch}\, V = j_\Lambda^{-1} e^{-\rho} R^{-1} \sum_{w \in W} \varepsilon(w) w \Big(e^{\Lambda+\rho} \prod_{\beta \in S_\Lambda} (1 + e^{-\beta})^{-1} \Big), \tag{8}$$

where

$$R = \prod_{\alpha\in\Delta_{0,+}} (1-e^{-\alpha}) \Big/ \prod_{\alpha\in\Delta_{1,+}} (1+e^{-\alpha}) \tag{9}$$

and j_Λ is a normalization coefficient to make sure that the coefficient of e^Λ on the right hand side of (8) is 1.

Covariant modules, a particular class of finite-dimensional simple $\mathfrak{gl}(m|n)$-modules, are characterized by a partition λ inside the (m,n)-hook, i.e. such that $\lambda_{m+1} \le n$. The representation characterized by the partition λ is denoted by V_λ. In the distinguished basis fixed by (1), the highest weight Λ_λ of V_λ in the standard ϵ-δ-basis is given by [17]

$$\Lambda_\lambda = \lambda_1\epsilon_1 + \cdots + \lambda_m\epsilon_m + \nu'_1\delta_1 + \cdots + \nu'_n\delta_n, \tag{10}$$

where $\nu'_j = \max\{0, \lambda'_j - m\}$ for $1 \le j \le n$ (and λ' is the partition conjugate to λ). Let us consider the atypicality of V_λ, in the distinguished basis. It is sufficient to compute the numbers $(\Lambda_\lambda + \rho, \beta_{ij})$, with $\beta_{ij} = \epsilon_i - \delta_j$, for $1 \le i \le m$ and $1 \le j \le n$, and count the number of zeros. It is convenient to put these numbers $(\Lambda_\lambda + \rho, \beta_{ij})$ in a $(m \times n)$-matrix (the atypicality matrix [8, 17]).

From the combinatorics of atypicality matrices [8], the following proposition holds.

Proposition 2.1. *The atypicality matrix of a partition λ has its zeros on row i and in column $\lambda_i + m - i + 1$ ($i = m, m-1, \ldots$) as long as these column indices are not exceeding n.*

With a partition λ we can associate the (m,n)-index k:

Definition 2.2. For λ a partition inside the (m,n)-hook, the (m,n)-index of λ is the number

$$k = \min\{i | \lambda_i + m + 1 - i \le n\}, \qquad (1 \le k \le m+1). \tag{11}$$

In the rest of this paper, k will denote the (m,n)-index of λ. By Proposition 2.1, $m-k+1$ is the atypicality of V_λ. If $k = m+1$, then $S_{\Lambda_\lambda} = \emptyset$ and V_λ is typical and trivially tame. Thus in the following, we shall assume that $k \le m$. In the initial situation, Δ_+ corresponds to the distinguished choice, and Π is the distinguished set of simple roots (1). The highest weight of V_λ is given by Λ_λ. Denote $\Lambda^{(1)} = \Lambda_\lambda$, $\rho^{(1)} = \rho$ and $\Pi^{(1)} = \Pi$. Now we perform a sequence of simple odd $\alpha^{(i)}$-reflections; each of these reflections preserve $\Delta_{0,+}$ but may change $\Lambda^{(i)}+\rho^{(i)}$ and $\Pi^{(i)}$. Denote the sequence of reflections by:

$$\Lambda^{(1)}+\rho^{(1)},\,\Pi^{(1)} \quad \stackrel{\alpha^{(1)}}{\longrightarrow} \quad \Lambda^{(2)}+\rho^{(2)},\,\Pi^{(2)} \quad \stackrel{\alpha^{(2)}}{\longrightarrow} \quad \cdots \quad \stackrel{\alpha^{(f)}}{\longrightarrow} \quad \Lambda'+\rho',\,\Pi'$$

where, at each stage, $\alpha^{(i)}$ is an odd root from $\Pi^{(i)}$. For given λ, consider the following sequence of odd roots (with positions on row m, row $m-1$, …, row k):

$$\begin{array}{ll} \text{row } m: & \beta_{m,1},\beta_{m,2},\ldots,\beta_{m,\lambda_k-k+m} \\ \text{row } m-1: & \beta_{m-1,1},\beta_{m-1,2},\ldots,\beta_{m-1,\lambda_k-k+m-1} \\ \vdots & \vdots \\ \text{row } k: & \beta_{k,1},\beta_{k,2},\ldots,\beta_{k,\lambda_k} \end{array} \tag{12}$$

in this particular order (i.e. starting with $\beta_{m,1}$ and ending with β_{k,λ_k}). Then we have [18]:

Lemma 2.3. *The sequence (12) is a proper sequence of simple odd reflections for Λ_λ, i.e. $\alpha^{(i)}$ is a simple odd root from $\Pi^{(i)}$. At the end of the sequence, one finds:*

$$\begin{aligned}\Pi' = \{&\epsilon_1-\epsilon_2,\epsilon_2-\epsilon_3,\ldots,\epsilon_{k-2}-\epsilon_{k-1},\epsilon_{k-1}-\delta_1,\delta_1-\delta_2,\delta_2-\delta_3,\ldots,\\ &\delta_{\lambda_k-1}-\delta_{\lambda_k},\delta_{\lambda_k}-\epsilon_k,\epsilon_k-\delta_{\lambda_k+1},\delta_{\lambda_k+1}-\epsilon_{k+1},\epsilon_{k+1}-\delta_{\lambda_k+2},\ldots,\\ &\delta_{\lambda_k+m-k}-\epsilon_m,\epsilon_m-\delta_{\lambda_k+m+1-k},\delta_{\lambda_k+m+1-k}-\delta_{\lambda_k+m+2-k},\ldots,\delta_{n-1}-\delta_n\}.\end{aligned}$$

Furthermore,

$$\Lambda'+\rho' = \Lambda_\lambda+\rho+\sum_{i=k+1}^{m}\sum_{j=\lambda_i+1}^{\lambda_k-k+i}\beta_{i,j}. \tag{13}$$

The set Π' in Lemma 2.3 contains a maximal isotropic subset

$$S_{\Lambda'} = \{\epsilon_k-\delta_{\lambda_k+1},\epsilon_{k+1}-\delta_{\lambda_k+2},\ldots,\epsilon_m-\delta_{\lambda_k+m+1-k}\}.$$

So we have:

Corollary 2.4. *Every covariant module V_λ is tame.*

Thus, the character formula of Kac and Wakimoto becomes:

$$\operatorname{ch} V_\lambda = j_{\Lambda'}^{-1}e^{-\rho'}R'^{-1}\sum_{w\in W}\varepsilon(w)w\Big(e^{\Lambda'+\rho'}\prod_{\beta\in S_{\Lambda'}}(1+e^{-\beta})^{-1}\Big).$$

Putting, as usual, $e^{\epsilon_i} = x_i$ and $e^{\delta_i} = y_i$, we have shown [18] that this expression can be rewritten in a determinantal form:

$$\operatorname{ch} V_\lambda = \pm D^{-1} \det \begin{pmatrix} \left(\frac{1}{x_i+y_j}\right)_{\substack{1\le i\le m\\ 1\le j\le n}} & \left(x_i^{\lambda_j+m-n-j}\right)_{\substack{1\le i\le m\\ 1\le j\le k-1}} \\ \left(y_j^{\lambda'_i+n-m-i}\right)_{\substack{1\le i\le n-m+k-1\\ 1\le j\le n}} & 0 \end{pmatrix} \tag{14}$$

with

$$D = \frac{\prod_{i<j}(x_i - x_j)\prod_{i<j}(y_i - y_j)}{\prod_{i,j}(x_i + y_j)}.$$

3. Supersymmetric Schur polynomials.

Let us now recall some notions of *supersymmetric S-functions* [4, 19, 20]. The ring of doubly symmetric polynomials in $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_n)$ is $S(x, y) = S(x) \otimes_{\mathbb{Z}} S(y)$. An element $p \in S(x, y)$ has the *cancellation property* if it satisfies the following: when the substitution $x_1 = t$, $y_1 = -t$ is made in p, the resulting polynomial is independent of t. We denote $S(x/y)$ the subring of $S(x, y)$ consisting of the elements satisfying the cancellation property. The elements of $S(x/y)$ are the supersymmetric polynomials [20].

The complete supersymmetric functions $h_r(x/y)$ belong to $S(x/y)$, and are defined by

$$h_r(x/y) = \sum_{j=0}^{r} h_{r-j}(x)e_j(y), \tag{15}$$

where $h_i(x)$ resp. $e_j(y)$ are the usual complete resp. elementary symmetric functions [21]. We can use $h_r(x/y)$ to give a first formula for the supersymmetric S-functions:

$$s_\lambda(x/y) = \det\Big(h_{\lambda_i-i+j}(x/y)\Big)_{1\le i,j\le \ell(\lambda)}, \tag{16}$$

where $\ell(\lambda)$ denotes the length of λ. The polynomials $s_\lambda(x/y)$ are identically zero when $\lambda_{m+1} > n$.

Berele and Regev [4] showed that the character of a covariant module is a supersymmetric Schur function: $\operatorname{ch} V_\lambda = s_\lambda(x/y)$. Thus, our formula (14) gives an alternative expression for $s_\lambda(x/y)$. In fact, (14) is the supersymmetric equivalent of the determinantal formula for the symmetric Schur polynomials as a quotient of two alternants.

In [21, §I.3, exercise 23], Macdonald shows that the supersymmetric S-polynomials satisfy four properties (see also [22]) which also characterize these polynomials. [a] We have shown that the polynomials defined by means of the right hand side of (14) do indeed satisfy these four properties. This provides us with an independent proof of the determinantal expression (14).

4. The t-dimension of a covariant module.

Consider a finite-dimensional irreducible representation V of $\mathfrak{g}$ with weight decomposition $V = \oplus_\mu V(\mu)$. Recall that the character is $\operatorname{ch} V = \sum_\mu \dim V(\mu)\, e^\mu$. We shall consider a specialization of the character determined by

$$\begin{aligned} &F(e^{\epsilon_i}) = 1 \ (i = 1, \ldots, m) \\ &F(e^{\delta_j}) = t \ (j = 1, \ldots, n). \end{aligned} \tag{17}$$

This specialization is consistent with the $\mathbb{Z}$-grading of $\mathfrak{g}$, and the corresponding $\mathbb{Z}$-grading of V. The specialization of the character of V under F is referred to as the t-dimension of V and denoted by $\dim_t(V)$:

$$\dim_t(V) = F(\operatorname{ch} V) = \sum_\mu \dim V(\mu) F(e^\mu). \tag{18}$$

The t-dimension of V stands for the polynomial

$$F(e^\Lambda) \sum_{j\in\mathbb{Z}_+} (\dim V_{-j})\ t^j, \tag{19}$$

where $V = V_0 \oplus V_{-1} \oplus V_{-2} \oplus \cdots$ is the $\mathbb{Z}$-grading of V. Note that for the $\mathbb{Z}_2$-grading $V = V_{\bar 0} \oplus V_{\bar 1}$ we have $V_{\bar 0} = V_0 \oplus V_{-2} \oplus \cdots$ and $V_{\bar 1} = V_{-1} \oplus V_{-3} \oplus \cdots$. Therefore, the ordinary dimension of V is found by putting $t = 1$ in the expression for the t-dimension, whereas the superdimension of V is found by putting $t = -1$. So the t-dimension can also be seen as an extension of the notion of dimension and superdimension.

In order to compute the t-dimension there are two useful formulas, namely (16) and (14). Since $x_i = e^{\epsilon_i}$ and $y_j = e^{\delta_j}$, the specialization (17) corresponds to putting each $x_i = 1$ and $y_j = t$ in $s_\lambda(x/y)$. For the elementary and complete symmetric functions, such specializations are well-

[a] Our convention for the $s_\lambda(x/y)$ is slightly different from that of Macdonald's $s_\lambda^{\mathrm{Mac}}(x/y)$: $s_\lambda(x/y) = s_\lambda^{\mathrm{Mac}}(x/-y)$.

known:

$$h_r(x_1,\dots,x_m)\Big|_{x_i=1} = \binom{m+r-1}{r} = \binom{m+r-1}{m-1}, \tag{20}$$

$$e_r(x_1,\dots,x_m)\Big|_{x_i=1} = \binom{m}{r}. \tag{21}$$

Thus it follows from (15) and (16) that

Proposition 4.1. *The t-dimension of V_λ is given by the determinant*

$$\dim_t(V_\lambda) = \det_{1\le i,j\le \ell(\lambda)} \left(\sum_{l=0}^{\lambda_i-i+j} \binom{m+\lambda_i-i+j-l-1}{\lambda_i-i+j-l}\binom{n}{l} t^l \right). \tag{22}$$

Although this formula is simple to derive, it should be observed that in general the matrix elements in the right hand side of (22) do not have a "closed form" expression [23]: they remain polynomials of type ${}_2F_1(-t)$ in t. Even for $t=1$, the expression $\sum_{l=1}^{r}\binom{m+r-l-1}{r-l}\binom{n}{l}$ cannot be simplified in general. Only for $t=-1$, i.e. the superdimension formula $\operatorname{sdim} V_\lambda$, the expression (22) can be simplified:

$$\operatorname{sdim} V_\lambda = \det_{1\le i,j\le \ell(\lambda)} \left(\binom{m-n-1+\lambda_i-i+j}{\lambda_i-i+j} \right) \tag{23}$$

$$= \frac{\prod_{i<j}(\lambda_i-i-\lambda_j+j)}{\prod_i(\lambda_i-i+l(\lambda))} \prod_i (m-n+1-i)_{\lambda_i}. \tag{24}$$

Herein, $(a)_n = a(a+1)\cdots(a+n-1)$ is the Pochhammer symbol [24], and the determinant in (23) can be written in closed form using [25, (3.11)]. So in general the superdimension has a closed form expression (24), whereas the dimension has not.

Starting from the new determinantal formula (14) and using properties of symmetric polynomials and row and column operations, we can deduce a new t-dimension formula (see [26]):

Theorem 4.2. *The t-dimension of V_λ is* $\dim_t(V_\lambda) = \pm(1+t)^{mn}\det(R(\lambda))$, *with $R(\lambda)$ given by*

$$\begin{pmatrix} \left(\frac{(-1)^{i+j}}{(1+t)^{i+j-1}} \binom{i+j-2}{j-1} \right)_{\substack{1\le i\le m\\ 1\le j\le n}} & \left(\binom{\lambda_j+m-n-j}{i-1} \right)_{\substack{1\le i\le m\\ 1\le j\le k-1}} \\ \left(t^{\lambda'_i+n-m-i-j+1} \binom{\lambda'_i+n-m-i}{j-1} \right)_{\substack{1\le i\le n-m+k-1\\ 1\le j\le n}} & 0 \end{pmatrix}. \tag{25}$$

Compared to (22), this formula has the advantage that each matrix element is a simple binomial coefficient multiplied by a power of t or $(1+t)$, and no longer a finite series of type ${}_2F_1(-t)$. The order of the matrix in (22) is the length of the partition where as the order of the matrix in (25) is always less than or equal to $m+n$. So in general the second formula is easier to compute, see Example 4.3.

Example 4.3. In $\mathfrak{gl}(2,2)$, with $\lambda = (4,1,1,1)$, (22) resp. (25) yield:

$$\begin{aligned}
&\dim_t\big(V_{(4,1,1,1)}\big) = \\
&= \det\begin{pmatrix} 5+8t+3t^2 & 6+10t+4t^2 & 7+12t+5t^2 & 8+14t+6t^2 \\ 1 & 2+2t & 3+4t+t^2 & 4+6t+2t^2 \\ 0 & 1 & 2+2t & 3+4t+t^2 \\ 0 & 0 & 1 & 2+2t \end{pmatrix} \\
&= (1+t)^4 \det\begin{pmatrix} \frac{1}{1+t} & \frac{-1}{(1+t)^2} & 1 \\ \frac{-1}{(1+t)^2} & \frac{2}{(1+t)^3} & 3 \\ t^3 & 3t^2 & 0 \end{pmatrix} = 2t^2(6t^2+13t+6)(1+t).
\end{aligned}$$

From the equality of the two formulas (22) and (25) we can deduce certain poperties of the t-dimension for particular V_λ. An interesting application follows for the special case of $\lambda = \Big((n-a)^{(m-a)}\Big)$, where $a = 0,1,\ldots,\min(m,n)$. For such partitions, the determinant given in (25) can be reduced and simplifies to:

$$\dim{}_t(V_\lambda) = (1+t)^{(m-a)(n-a)} \det_{1\le i,j\le a}\left(\binom{i+j+m+n-2a-2}{j-1} \right).$$

Taking out common factors in rows and columns in this determinant, it becomes

$$\prod_{i=1}^{a} \frac{(i+m+n-2a-1)!}{(i+n-a-1)!(i+m-a-1)!} \det_{1\le i,j\le a}\Big((m+n-2a+i)_{j-1}\Big).$$

This remaining determinant can easily be computed (see [26]). So we finally obtain

$$\begin{aligned}
\dim{}_t(V_\lambda) &= (1+t)^{(m-a)(n-a)} \det_{1\le i,j\le a}\left(\binom{i+j+m+n-2a-2}{j-1} \right) \\
&= (1+t)^{(m-a)(n-a)} \prod_{i=0}^{a-1} \frac{\binom{m+n-2a+i}{n-a+i}}{\binom{m-a+i}{i}}.
\end{aligned}$$

Using this, we can prove the following result:

Corollary 4.4. *Let r and s be positive integers with $s \leq r$, u and v arbitrary variables, and*

$$A_i = \sum_{l=0}^{r} \binom{u+i-l}{r-l}\binom{u}{l} v^l = \binom{u+i}{r}\, {}_2F_1\left({-r, -u \atop -u-i}; -v\right).$$

Then the Hankel determinant is given by

$$\det_{0\leq i,j\leq s}(A_{i+j}) = (-1)^{s(s+1)/2}(1+v)^{(s+1)(r-s)} \prod_{i=1}^{r-s} \frac{\binom{2s+u-r+i}{s+1}}{\binom{s+i}{s+1}}.$$

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