Lie algebraic generalization of quantum statistics

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Abstract. Para-Fermi statistics and Fermi statistics are known to be associated with particular representations of the Lie algebra $so(2n+1) \equiv B_n$. Similarly para-Bose and Bose statistics are related with the Lie superalgebra $osp(1|2n) \equiv B(0|n)$. We develop an algebraical framework for the generalization of quantum statistics based on the Lie algebras A_n, B_n, C_n and D_n.

1. Introduction

In 1953 Green [1] introduced para-Bose and para-Fermi statistics as generalizations of Bose and Fermi statistics. Instead of the bilinear commutators $[b_i^-, b_j^+] = \delta_{ij},\ [b_i^\xi, b_j^\xi] = 0,\ \xi = \pm$ (or anti-commutators $\{f_i^-, f_j^+\} = \delta_{ij},\ \{f_i^\xi, f_j^\xi\} = 0,\ \xi = \pm$) as for Bose creation and annihilation operators (CAOs) (or Fermi CAOs), para-statistics is described by triple relations. The defining relations for n pairs of para-Fermi CAOs $F_i^\xi,\ \xi = \pm$ and $i = 1, \ldots, n$ are:

$$[[F_j^\xi, F_k^\eta], F_l^\varepsilon] = \frac{1}{2}(\varepsilon - \eta)^2 \delta_{kl} F_j^\xi - \frac{1}{2}(\varepsilon - \xi)^2 \delta_{jl} F_k^\eta,\ \xi, \eta, \varepsilon = \pm;\ j,k,l = 1, \ldots, n \quad (1)$$

and those for n pairs of para-Bose CAOs $B_i^\xi,\ \xi = \pm$ and $i = 1, \ldots, n$:

$$[\{B_j^\xi, B_k^\eta\}, B_l^\varepsilon] = (\varepsilon - \xi)\delta_{jl} B_k^\xi + (\varepsilon - \eta)\delta_{kl} B_j^\eta,\ \xi, \eta, \varepsilon = \pm;\ j,k,l = 1, \ldots, n. \quad (2)$$

It was realized by Kamefuchi and Takahashi [2], and by Ryan and Sudarshan [3], that the $2n$ operators F_i^ξ subject to the relations (1) generate the Lie algebra $so(2n+1) \equiv B_n$. Moreover, a particular representation of $so(2n+1)$ yields the relations of Fermi statistics. Similarly Ganchev and Palev [4] proved that the Lie superalgebra generated by the $2n$ operators B_i^ξ (considered as odd elements) subject to the relations (2) is the orthosymplectic Lie superalgebra $osp(1|2n) \equiv B(0|n)$ [5]. Also here there exists an $osp(1|2n)$ representation, that yields the Bose statistics. Therefore para-statistics is associated with representations of the Lie (super)algebras of class B. Motivated by these relations we introduce the concept of a generalized

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quantum statistics for a classical Lie algebra and classify all the classes of such statistics by means of their algebraic relations [6].

We should mention that examples of such statistics for each of the classical Lie algebras A_n, B_n, C_n and D_n were considered by Palev [7]-[11] (we refer to those examples as Palev's statistics), although a complete classification was never made.

2. Preliminaries, definition, method

Let G be a classical Lie algebra. A generalized quantum statistics related to G is determined by N creation x_i^+ and N annihilation operators x_i^-. Inspired by the para-Fermi, para-Bose and Palev's statistics, the operators $x_i^\pm$ should satisfy certain requirements. The $2N$ operators $x_i^\pm$ should generate the Lie algebra G, subject to triple relations. Let G_{+1} and G_{-1} be the subspaces of G spanned by the creation and annihilation operators:

$$G_{+1} = \text{span}\{x_i^+;\ i = 1\ldots,N\}, \qquad G_{-1} = \text{span}\{x_i^-;\ i = 1\ldots,N\}. \tag{3}$$

Since the defining relations should be triple relations, this implies that it is natural to make the following requirements:

$$[[x_i^+,x_j^+],x_k^+] = 0, \quad [[x_i^-,x_j^-],x_k^-] = 0,$$
$$[[x_i^+,x_j^+],x_k^-] = \text{a lineair combination of } x_l^+,\ [[x_i^+,x_j^-],x_k^+] = \text{a lineair combination of } x_l^+,$$
$$[[x_i^+,x_j^-],x_k^-] = \text{a lineair combination of } x_l^-,\ [[x_i^-,x_j^-],x_k^+] = \text{a lineair combination of } x_l^-.$$

Let $G_{\pm 2} = [G_{\pm 1}, G_{\pm 1}]$ and $G_0 = [G_{+1}, G_{-1}]$, then we require $G_{-2} \oplus G_{-1} \oplus G_0 \oplus G_{+1} \oplus G_{+2}$ to be a **Z**-grading of G. Let ω be the standard anti-involution of the Lie algebra G (characterized by $\omega(x) = x^\dagger$ in the standard defining representation of G, where $x^\dagger$ denotes the Hermitian conjugate of the matrix x in this representation) then we shall assume $\omega(x_i^+) = x_i^-$. And finally, we shall require that the CAOs $x_i^\pm$ are root vectors of the Lie algebra G.

Definition. *Let G be a classical Lie algebra, with anti-involution ω. A set of $2N$ root vectors $x_i^\pm$ $(i = 1,\ldots,N)$ is called a set of creation and annihilation operators for G if:*

$\omega(x_i^\pm) = x_i^\mp$,

$G = G_{-2} \oplus G_{-1} \oplus G_0 \oplus G_{+1} \oplus G_{+2}$ *is a* **Z***-grading of G, with* $G_{\pm 1} = span\{x_i^\pm,\ i = 1\ldots,N\}$ *and* $G_{j+k} = [G_j, G_k]$.

The algebraic relations $\mathscr{R}$ satisfied by the operators $x_i^\pm$ are the relations of a generalized quantum statistics (GQS) associated with G.

A consequence of this definition is that G_0 is a subalgebra of G spanned by root vectors of G, i.e. G_0 is a regular subalgebra of G. By the adjoint action, the remaining G_i's are G_0-modules. Thus the following technique can be used in order to classify all GQS associated with G:

1. Determine all regular subalgebras G_0 of G [12].

2. For each regular subalgebra G_0, determine the decomposition of G into simple G_0-modules g_k $(k = 1,2,\ldots)$.

3. Investigate whether there exists a **Z**-grading of G of the form $G = G_{-2} \oplus G_{-1} \oplus G_0 \oplus G_{+1} \oplus G_{+2}$, where each G_i is either directly a module g_k or else a sum of such modules $g_1 \oplus g_2 \oplus \cdots$, such that $\omega(G_{+i}) = G_{-i}$.

A summary of the classification process for the classical Lie algebras A_n, B_n, C_n and D_n follows.

3. Classification

3.1. The Lie algebra $A_n = sl(n+1)$

Let G be the special linear Lie algebra $sl(n+1)$, consisting of traceless $(n+1)\times(n+1)$ matrices. The root vectors of G are the elements e_{jk} ($j \neq k = 1,\ldots,n+1$), where e_{jk} is a matrix with zeros everywhere except a 1 on the intersection of row j and column k. The corresponding root is $\varepsilon_j - \varepsilon_k$, in the usual basis. The anti-involution is such that $\omega(e_{jk}) = e_{kj}$. In order to find regular subalgebras of $G = A_n$, one should delete nodes from the Dynkin diagram of G or from its extended Dynkin diagram.

Step 1. Delete node i from the Dynkin diagram. Then $sl(n+1) = G_{-1}\oplus G_0 \oplus G_{+1}$, with $G_0 = H + sl(i)\oplus sl(n-i+1)$, $\quad G_{-1} = \mathrm{span}\{e_{kl};\ k=1,\ldots,i,\ l=i+1,\ldots,n+1\}$ and $N = i(n-i+1)$. For $i=1$, $N=n$, the rank of A_n. Putting $a_j^- = e_{1,j+1}, \quad a_j^+ = e_{j+1,1},\ j=1,\ldots,n$, the corresponding relations $\mathscr{R}$ read ($j,k,l=1,\ldots,n$):

$$[a_j^+,a_k^+] = [a_j^-,a_k^-] = 0,\ [[a_j^+,a_k^-],a_l^+] = \delta_{jk}a_l^+ + \delta_{kl}a_j^+, \quad [[a_j^+,a_k^-],a_l^-] = -\delta_{jk}a_l^- - \delta_{jl}a_k^-.$$

These are the relations of A-statistics [7]-[8], [11], [13]-[14]. For $i=2$, $N=2(n-1)$, let

$$a_{-j}^- = e_{1,j+2}, \quad a_{+j}^- = e_{2,j+2}, \quad a_{-j}^+ = e_{j+2,1}, \quad a_{+j}^+ = e_{j+2,2}, \quad j=1,\ldots,n-1. \tag{4}$$

Now the corresponding relations are ($\xi,\eta,\varepsilon=\pm;\ j,k,l=1,\ldots,n-1$):

$$\begin{aligned}
&[a_{\xi j}^+,a_{\eta k}^+] = [a_{\xi j}^-,a_{\eta k}^-] = 0, \quad [a_{\xi j}^+,a_{-\xi k}^-] = 0,\ j\neq k,\\
&[a_{-j}^+,a_{-k}^-] = [a_{+j}^+,a_{+k}^-],\ j\neq k,\ [a_{+j}^+,a_{-j}^-] = [a_{+k}^+,a_{-k}^-],\ [a_{-j}^+,a_{+j}^-] = [a_{-k}^+,a_{+k}^-],\\
&[[a_{\xi j}^+,a_{\eta k}^-],a_{\varepsilon l}^+] = \delta_{\eta\varepsilon}\delta_{jk}a_{\xi l}^+ + \delta_{\xi\eta}\delta_{kl}a_{\varepsilon j}^+,\ [[a_{\xi j}^+,a_{\eta k}^-],a_{\varepsilon l}^-] = -\delta_{\xi\varepsilon}\delta_{jk}a_{\eta l}^- - \delta_{\xi\eta}\delta_{jl}a_{\varepsilon k}^-.
\end{aligned}$$

Step 2. Delete node i and j from the Dynkin diagram. Then $sl(n+1) = G_{-2}\oplus G_{-1}\oplus G_0 \oplus G_{+1}\oplus G_{+2}$, with $G_0 = H + sl(i)\oplus sl(j-i)\oplus sl(n+1-j)$. There are six simple G_0-modules and three different ways in which these G_0-modules can be combined. To characterize these three cases, it is sufficient to give only G_{-1}:

$$\begin{aligned}
&G_{-1} = \mathrm{span}\{e_{kl},e_{lp};\ k=1,\ldots,i,\ l=i+1,\ldots,j,\ p=j+1,\ldots,n+1\},\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad N=(j-i)(n+1-j+i);\\
&G_{-1} = \mathrm{span}\{e_{kl},e_{pk};\ k=1,\ldots,i,\ l=i+1,\ldots,j,\ p=j+1,\ldots,n+1\},\ N=i(n+1-i);\\
&G_{-1} = \mathrm{span}\{e_{kl},e_{lp};\ k=1,\ldots,i,\ p=i+1,\ldots,j,\ l=j+1,\ldots,n+1\},\ N=j(n+1-j).
\end{aligned}$$

For $j-i=1$ one can label the CAOs as follows:

$$a_k^- = e_{k,i+1},\ a_k^+ = e_{i+1,k},\ k=1,\ldots,i; \quad a_k^- = e_{i+1,k+1},\ a_k^+ = e_{k+1,i+1},\ k=i+1,\ldots,n.$$

Using

$$\langle k\rangle = \begin{cases} 0 & \text{if} \quad k=1,\ldots,i \\ 1 & \text{if} \quad k=i+1,\ldots,n \end{cases} \tag{5}$$

the algebraic relations read ($\xi=\pm;\ k,l,m=1,\ldots,n$):

$$\begin{aligned}
&[a_k^+,a_l^+] = [a_k^-,a_l^-] = 0, \qquad k,l=1,\ldots,i \text{ or } k,l=i+1,\ldots,n,\\
&[a_k^-,a_l^+] = [a_k^+,a_l^-] = 0, \qquad k=1,\ldots,i,\ l=i+1,\ldots,n,\\
&[[a_k^+,a_l^-],a_m^+] = (-1)^{\langle l\rangle+\langle m\rangle}\delta_{kl}a_m^+ + (-1)^{\langle l\rangle+\langle m\rangle}\delta_{lm}a_k^+,\ k,l=1,\ldots,i \text{ or } k,l=i+1,\ldots,n,\\
&[[a_k^+,a_l^-],a_m^-] = -(-1)^{\langle l\rangle+\langle m\rangle}\delta_{kl}a_m^- - (-1)^{\langle l\rangle+\langle m\rangle}\delta_{km}a_l^-,\ k,l=1,\ldots,i \text{ or } k,l=i+1,\ldots,n,\\
&[[a_k^\xi,a_l^\xi],a_m^{-\xi}] = -\delta_{km}a_l^\xi + \delta_{lm}a_k^\xi,\ k=1,\ldots,i,\ l=i+1,\ldots,n, \qquad [[a_k^\xi,a_l^\xi],a_m^\xi]=0.
\end{aligned} \tag{6}$$

The relations (6) with $n=2m$ and $i=m$ are the commutation relations of the (Palev's) causal A-statistics investigated in [10].

Step 3. If we delete 3 or more nodes from the Dynkin diagram, the corresponding **Z**-grading of $sl(n+1)$ has no longer the required properties.

Step 4. If we delete node i from the extended Dynkin diagram, the remaining diagram is again of type A_n, so $G_0=G$, and there are no CAOs.

Step 5. If we delete 2 (3) nodes from the extended Dynkin diagram we go to Step 1 (Step 2).

Step 6. If we delete 4 or more nodes from the extended Dynkin diagram, the corresponding **Z**-grading of $sl(n+1)$ has no longer the required properties.

Following the same procedure we give only the most interesting cases for the algebras B_n-D_n.

3.2. The Lie algebra $B_n=so(2n+1)$

Delete node 1 from the Dynkin diagram. Then $so(2n+1)=G_{-1}\oplus G_0\oplus G_{+1}$ with $G_0=H+so(2n-1)$, $G_{-1}=\text{span}\{e_{1,2n+1}-e_{2n+1,n+1},\ e_{1,k+n}-e_{k,n+1},\ e_{1k}-e_{k+n,n+1};\ k=2,\ldots,n\}$ and $N=2n-1$. Let us denote the CAOs by:

$$\begin{aligned}
&b^-_{00}=e_{1,2n+1}-e_{2n+1,n+1},\ b^+_{00}=e_{2n+1,1}-e_{n+1,2n+1},\\
&b^-_{-k}=e_{1,n+k+1}-e_{k+1,n+1},\ b^+_{-k}=e_{n+k+1,1}-e_{n+1,k+1}, \qquad k=1,\ldots,n-1,\\
&b^-_{+k}=e_{1,k+1}-e_{n+k+1,n+1},\ b^+_{+k}=e_{k+1,1}-e_{n+1,n+k+1}, \qquad k=1,\ldots,n-1.
\end{aligned}\tag{7}$$

The corresponding relations $\mathscr{R}$ are given by ($\xi,\eta,\varepsilon=0,\pm;\ i,j,k=1,\ldots,n-1$):

$$\begin{aligned}
&[b^+_{\xi i},b^+_{\eta j}]=[b^-_{\xi i},b^-_{\eta j}]=0,\quad [b^+_{-i},b^-_{-j}]=[b^-_{+i},b^+_{+j}], \qquad i\neq j,\\
&[b^+_{00},b^-_{-j}]=[b^-_{00},b^+_{+j}],\quad [b^+_{00},b^-_{+j}]=[b^-_{00},b^+_{-j}],\\
&[[b^+_{\xi i},b^-_{\eta j}],b^+_{\varepsilon k}]=\delta_{ij}\delta_{\xi\eta}b^+_{\varepsilon k}+\delta_{jk}\delta_{\eta\varepsilon}b^+_{\xi i}-\delta_{ik}\delta_{\xi,-\varepsilon}b^+_{-\eta j},\\
&[[b^+_{\xi i},b^-_{\eta j}],b^-_{\varepsilon k}]=-\delta_{ij}\delta_{\xi\eta}b^-_{\varepsilon k}-\delta_{ik}\delta_{\xi\varepsilon}b^-_{\eta j}+\delta_{jk}\delta_{\eta,-\varepsilon}b^-_{-\xi i}.
\end{aligned}$$

Delete node i ($i=2,\ldots,n$) from the Dynkin diagram; then $so(2n+1)=G_{-2}\oplus G_{-1}\oplus G_0\oplus G_{+1}\oplus G_{+2}$ with $G_0=H+sl(i)\oplus so(2(n-i)+1)$, $G_{-1}=\text{span}\{e_{j,2n+1}-e_{2n+1,n+j},\ e_{j,k+n}-e_{k,n+j},\ e_{jk}-e_{k+n,n+j};\ j=1,\ldots,i,\ k=i+1,\ldots,n\}$, and $N=2i(n-i)+i$. The case with $i=n$ is the para-Fermi case presented in the Introduction.

3.3. The Lie algebra $C_n=sp(2n)$

Delete node i ($i=1,\ldots,n-1$) from the Dynkin diagram. Then $sp(2n)=G_{-2}\oplus G_{-1}\oplus G_0\oplus G_{+1}\oplus G_{+2}$ with $G_0=H+sl(i)\oplus sp(2(n-i))$, $G_{-1}=\text{span}\{e_{k,n+l}+e_{l,n+k},\ e_{kl}-e_{n+l,n+k};\ k=1,\ldots,i,\ l=i+1,\ldots,n\}$ and $N=2i(n-i)$. For $i=1$, let us denote the CAOs by

$$\begin{aligned}
&c^-_{-j}=e_{1,n+j+1}+e_{j+1,n+1},\ c^-_{+j}=e_{1,j+1}-e_{n+j+1,n+1}, \quad j=1,\ldots,n-1,\\
&c^+_{-j}=e_{n+j+1,1}+e_{n+1,j+1},\ c^+_{+j}=e_{j+1,1}-e_{n+1,n+j+1}, \quad j=1,\ldots,n-1.
\end{aligned}\tag{8}$$

Then the corresponding relations $\mathscr{R}$ read, with $\xi,\eta,\varepsilon,\gamma=\pm$ or ±1, and $j,k,l=1,\ldots,n-1$:

$$\begin{aligned}
&[c^\eta_{\xi j},c^\eta_{\xi k}]=0,\quad [[c^\gamma_{\xi j},c^\gamma_{\eta k}],c^\gamma_{\varepsilon l}]=0,\quad [[c^\xi_{-j},c^\xi_{+k}],c^{-\xi}_{\eta l}]=2\eta\,\delta_{jk}c^\xi_{-\eta l},\\
&[c^+_{-j},c^-_{-k}]=[c^-_{+j},c^+_{+k}],\quad [c^-_{-j},c^-_{+k}]=[c^+_{-j},c^+_{+k}]=0, \qquad j\neq k,\\
&[[c^+_{\xi j},c^-_{\eta k}],c^+_{\varepsilon l}]=\delta_{\xi\eta}\delta_{jk}c^+_{\varepsilon l}+\delta_{\eta\varepsilon}\delta_{kl}c^+_{\xi j}+(-1)^{\eta\varepsilon}\delta_{\xi,-\varepsilon}\delta_{jl}c^+_{-\eta k},\\
&[[c^+_{\xi j},c^-_{\eta k}],c^-_{\varepsilon l}]=-\delta_{\xi\eta}\delta_{jk}c^-_{\varepsilon l}-\delta_{\xi\varepsilon}\delta_{jl}c^-_{\eta k}+(-1)^{\xi\eta}\delta_{\eta,-\varepsilon}\delta_{kl}c^-_{-\xi j}.
\end{aligned}\tag{9}$$

For $i=n-1$, let us also denote the CAOs by $c^{\pm}_{j}$:

$$\begin{aligned} c^-_{-j} &= e_{j,2n}+e_{n,n+j},\ c^-_{+j}=e_{jn}-e_{2n,n+j}, \quad j=1,\ldots,n-1,\\ c^+_{-j} &= e_{2n,j}+e_{n+j,n},\ c^+_{+j}=e_{nj}-e_{n+j,2n}, \quad j=1,\ldots,n-1. \end{aligned} \tag{10}$$

Now, the corresponding relations read, with $\xi,\eta,\varepsilon,\gamma=\pm$ or ±1, $j,k,l=1,\ldots,n-1$:

$$\begin{aligned} &[c^{\eta}_{\xi j},c^{\eta}_{\xi k}]=0, \quad [[c^{\gamma}_{\xi j},c^{\gamma}_{\eta k}],c^{\gamma}_{\varepsilon l}]=0,\\ &[c^+_{+j},c^-_{-k}]=[c^-_{+j},c^+_{-k}]=0, \qquad j\neq k,\\ &[[c^{\varepsilon}_{\xi j},c^{-\varepsilon}_{\xi k}],c^{\varepsilon}_{\eta l}]=\xi\eta\delta_{jk}c^{\varepsilon}_{\eta l}+\delta_{kl}c^{\varepsilon}_{\eta j}, \quad [[c^{\varepsilon}_{+j},c^{-\varepsilon}_{-k}],c^{\xi}_{\eta l}]=(\varepsilon\xi-\eta)\delta_{jk}c^{\xi}_{-\eta l},\\ &[[c^{\varepsilon}_{+j},c^{\varepsilon}_{-k}],c^{-\varepsilon}_{\xi l}]=-\xi\delta_{jl}c^{\varepsilon}_{-\xi k}-\xi\delta_{kl}c^{\varepsilon}_{-\xi j}. \end{aligned} \tag{11}$$

This set of CAOs, together with their relations (11), was constructed earlier in [7].

When node n is deleted from the Dynkin diagram of C_n, then $sp(2n)=G_{-1}\oplus G_0\oplus G_{+1}$ with $G_0=H+sl(n)$ and $G_{-1}=\{e_{j,n+k}+e_{k,n+j};\ 1\leq j\leq k\leq n\}$. There are $N=\frac{n(n+1)}{2}$ commuting annihilation operators, and the relations $\mathscr{R}$ will not be given explicitly.

3.4. The Lie algebra $D_n=so(2n)$

When node 1 is deleted from the Dynkin diagram of D_n, then $so(2n)=G_{-1}\oplus G_0\oplus G_{+1}$ with $G_0=H+D_{n-1}$, $G_{-1}=\mathrm{span}\{e_{1i}-e_{n+i,n+1},\ e_{1,n+i}-e_{i,n+1};\ i=2,\ldots,n\}$ and $N=2(n-1)$ Denoting the CAOs by

$$\begin{aligned} d^-_{-i} &= e_{1,n+i+1}-e_{i+1,n+1},\ d^-_{+i}=e_{1,i+1}-e_{n+i+1,n+1}, \qquad i=1,\ldots,n-1,\\ d^+_{-i} &= e_{n+i+1,1}-e_{n+1,i+1},\ d^+_{+i}=e_{i+1,1}-e_{n+1,n+i+1}, \qquad i=1,\ldots,n-1, \end{aligned} \tag{12}$$

then, for $\xi,\eta,\varepsilon=\pm$ and $i,j,k=1,\ldots,n-1$, the relations $\mathscr{R}$ are given by:

$$\begin{aligned} &[d^{\varepsilon}_{\xi i},d^{\varepsilon}_{\eta j}]=0, \qquad [d^+_{-i},d^-_{+i}]=[d^+_{+i},d^-_{-i}]=0,\\ &[[d^+_{\xi i},d^-_{\eta j}],d^-_{\varepsilon k}]=-\delta_{\xi\eta}\delta_{ij}d^-_{\varepsilon k}-\delta_{\xi\varepsilon}\delta_{ik}d^-_{\eta j}+\delta_{\eta,-\varepsilon}\delta_{jk}d^-_{-\xi,i},\\ &[[d^+_{\xi i},d^-_{\eta j}],d^+_{\varepsilon k}]=\delta_{\xi\eta}\delta_{ij}d^+_{\varepsilon k}+\delta_{\eta\varepsilon}\delta_{jk}d^+_{\xi i}-\delta_{\xi,-\varepsilon}\delta_{ik}d^+_{-\eta,j}. \end{aligned} \tag{13}$$

Although the relations (13) are new, the existence of the set of CAOs (12) was pointed out in [7].

When node i $(i=2,\ldots,n-2)$ is deleted from the Dynkin diagram of D_n, then $so(2n)=G_{-2}\oplus G_{-1}\oplus G_0\oplus G_{+1}\oplus G_{+2}$ with $G_0=sl(i)\oplus so(2(n-i))$, $G_{-1}=\mathrm{span}\{e_{kl}-e_{n+l,n+k},\ e_{k,n+l}-e_{l,n+k};\ k=1,\ldots,i,\ l=i+1,\ldots,n\}$ and $N=2i(n-i)$.

Delete node n from the Dynkin diagram, then $so(2n)=G_{-1}\oplus G_0\oplus G_{+1}$ with $G_0=H+sl(n)$, $G_{-1}=\mathrm{span}\{e_{j,k+n}-e_{k,j+n};\ 1\leq j<k\leq n\}$ and $N=\frac{n(n-1)}{2}$.

Delete nodes $n-1$ and n from the Dynkin diagram. Then $so(2n)=G_{-2}\oplus G_{-1}\oplus G_0\oplus G_{+1}\oplus G_{+2}$ with $G_0=H+sl(n-1)$. There are six G_0-modules and three ways in which these G_0-modules can be combined, namely with:

$$G_{-1} = \mathrm{span}\{e_{jn}-e_{2n,n+j},\ e_{j,2n}-e_{n,n+j};\ j=1,\ldots,n-1\}, \tag{14}$$

$$G_{-1} = \mathrm{span}\{e_{jn}-e_{2n,n+j},\ j=1,\ldots,n-1;\ e_{n+j,k}-e_{n+k,j},\ 1\leq j<k\leq n-1\}, \tag{15}$$

$$G_{-1} = \mathrm{span}\{e_{j+n,n}-e_{2n,j},\ j=1,\ldots,n-1;\ e_{j,k+n}-e_{k,j+n},\ 1\leq j<k\leq n-1\}. \tag{16}$$

For (14), we have $N=2(n-1)$; for (15) and (16), we have $N=\frac{n(n-1)}{2}$. It turns out that (15) and (16) are isomorphic to each other. Denote the CAOs of (14) by

$$\begin{array}{ll} d^-_{-i}=e_{i,2n}-e_{n,n+i},\ d^-_{+i}=e_{in}-e_{2n,n+i}, & i=1,\ldots,n-1, \\ d^+_{-i}=e_{2n,i}-e_{n+i,n},\ d^+_{+i}=e_{ni}-e_{n+i,2n}, & i=1,\ldots,n-1. \end{array} \tag{17}$$

Then, with $\xi,\eta,\varepsilon,\gamma=\pm$ or ±1 and $i,j,k=1,\ldots,n-1$, the relations are explicitly given by:

$$\begin{array}{l} [d^\eta_{\xi i},d^\eta_{\xi j}]=0, \quad [[d^\gamma_{\xi i},d^\gamma_{\eta j}],d^\gamma_{\varepsilon k}]=0, \\ [d^+_{-i},d^-_{+j}]=[d^+_{+i},d^-_{-j}]=0, \quad [d^-_{+i},d^-_{-i}]=[d^+_{+i},d^+_{-i}]=0, \\ [[d^\xi_{+i},d^\xi_{-j}],d^{-\xi}_{\varepsilon k}]=-\delta_{ik}d^\xi_{-\varepsilon j}+\delta_{jk}d^\xi_{-\varepsilon i}, \quad [[d^\eta_{\xi i},d^{-\eta}_{\xi j}],d^\eta_{\varepsilon k}]=\xi\varepsilon\delta_{ij}d^\eta_{\varepsilon k}+\delta_{jk}d^\eta_{\varepsilon i}. \end{array} \tag{18}$$

The set of CAOs (17) with relations (18) is the example that was considered earlier in [7] and [9].

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