The parastatistics Fock space and explicit infinite-dimensional representations of the Lie superalgebra $\mathfrak{osp}(2m+1|2n)$

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Abstract The defining triple relations of m pairs of parafermion operators $f_i^\pm$ and n pairs of paraboson operators $b_j^\pm$ with relative parafermion relations can be considered as defining relations for the Lie superalgebra $\mathfrak{osp}(2m+1|2n)$ in terms of $2m+2n$ generators. As a consequence of this the parastatistics Fock space of order p corresponds to an infinite-dimensional unitary irreducible representation $\mathfrak{V}(p)$ of $\mathfrak{osp}(2m+1|2n)$, with lowest weight $(-\frac{p}{2},\ldots,-\frac{p}{2}|\frac{p}{2},\ldots,\frac{p}{2})$. An explicit construction of the representations $\mathfrak{V}(p)$ is given for any m and n, as well as the computation of matrix elements of the $\mathfrak{osp}(2m+1|2n)$ generators.

1 Introduction

Standard quantum mechanics considers two types of particles, bosons $B_j^\pm$ ($[a,b]=ab-ba$)

$$[B_j^-,B_l^+]=\delta_{jl}, \qquad [B_j^-,B_l^-]=[B_j^+,B_l^+]=0, \tag{1}$$

and fermions $F_i^\pm$ ($\{a,b\}=ab+ba$)

$$\{F_i^-,F_k^+\}=\delta_{ik}, \qquad \{F_i^-,F_k^-\}=\{F_i^+,F_k^+\}=0, \tag{2}$$

and the corresponding quantum statistics, Bose-Einstein and Fermi-Dirac statistics. The n-boson Fock space with vacuum vector $|0\rangle$ satisfies

$$\langle 0|0\rangle=1, \qquad B_j^-|0\rangle=0, \quad (B_j^\pm)^\dagger=B_j^\mp \tag{3}$$

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and the other orthogonal normalized basis vectors are defined by

$$|k_1,\ldots,k_n\rangle = \frac{(B_1^+)^{k_1}\cdots(B_n^+)^{k_n}}{\sqrt{k_1!\cdots k_n!}}|0\rangle, \qquad k_1,\ldots,k_n\in\mathbb{Z}_+. \tag{4}$$

Similarly, the m-fermion Fock space is defined by

$$\langle 0|0\rangle = 1, \qquad F_i^-|0\rangle = 0, \qquad (F_i^\pm)^\dagger = F_i^\mp \qquad (i=1,\ldots,m). \tag{5}$$

and the basis vectors are as follows

$$|\theta_1,\ldots,\theta_m\rangle = (F_1^+)^{\theta_1}\cdots(F_m^+)^{\theta_m}|0\rangle, \qquad \theta_1,\ldots,\theta_m\in\{0,1\}. \tag{6}$$

Bose-Einstein and Fermi-Dirac statistics were generalized by Green [3] in 1953. He has shown that tensor fields can be quantized with creation and annihilation operators $b_j^\pm$ (parabosons), which satisfy the triple relations

$$[\{b_j^\xi, b_k^\eta\}, b_l^\varepsilon] = (\varepsilon-\xi)\delta_{jl}b_k^\eta + (\varepsilon-\eta)\delta_{kl}b_j^\xi, \tag{7}$$

whereas for spinor fields he has introduced parafermions $f_j^\pm$ postulating the commutation relations

$$[[f_j^\xi, f_k^\eta], f_l^\varepsilon] = \frac{1}{2}(\varepsilon-\eta)^2\delta_{kl}f_j^\xi - \frac{1}{2}(\varepsilon-\xi)^2\delta_{jl}f_k^\eta, \tag{8}$$

where $j,k,l\in\{1,2,\ldots\}$ and $\eta,\varepsilon,\xi\in\{+,-\}$ (or, in the algebraic expressions, $\eta,\varepsilon,\xi\in\{+1,-1\}$). The paraboson Fock space $V(p)$ is the Hilbert space with vacuum vector $|0\rangle$, defined by means of

$$\begin{aligned}&\langle 0|0\rangle = 1, \qquad b_j^-|0\rangle = 0, \qquad (b_j^\pm)^\dagger = b_j^\mp,\\ &\{b_j^-, b_k^+\}|0\rangle = p\,\delta_{jk}\,|0\rangle,\end{aligned} \tag{9}$$

and by irreducibility under the action of the algebra spanned by the elements b_j^+, b_j^-, subject to (7). In the same way, the parafermion Fock space $W(p)$ is the Hilbert space with unique vacuum vector $|0\rangle$, defined by

$$\begin{aligned}&\langle 0|0\rangle = 1, \qquad f_j^-|0\rangle = 0, \qquad (f_j^\pm)^\dagger = f_j^\mp,\\ &[f_j^-, f_k^+]|0\rangle = p\,\delta_{jk}\,|0\rangle,\end{aligned} \tag{10}$$

and by irreducibility under the action of the algebra spanned by the elements f_j^+, f_j^-, subject to (8). In both cases the parameter p is known as the order of the corresponding para system. For $p=1$ the paraboson (parafermion) Fock space coincides with the boson (fermion) Fock space. The paraboson and parafermion Fock spaces can in principle be constructed by the so-called Green ansatz [3]. However the explicit construction of these para Fock spaces has been an open problem for many years because of the difficulties of finding a proper basis of an irreducible constituent of a p-fold tensor product [4]. In recent papers [8, 12, 13], these problems of giving complete constructions of the paraboson and parafermion Fock spaces were solved. The solutions rely on the facts that paraboson and parafermion statistics are incorporated into algebraic structures. More precisely, a finite set of parafermions $f_j^{\pm}$, $i=1,2,\dots,m$ subject to the parafermion relations (8) defines the Lie algebra $\mathfrak{so}(2m+1)$ by means of generators and relations [7, 11]. The Fock space $W(p)$ is the unitary irreducible representation of $\mathfrak{so}(2m+1)$ with lowest weight $(-\frac{p}{2},-\frac{p}{2},\dots,-\frac{p}{2})$. In a similar way, n paraboson operators $b_j^{\pm}$ subject to (7) are generating elements of the orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2n)$ [2]. The Fock space $V(p)$ is the unitary irreducible representation of $\mathfrak{osp}(1|2n)$ with lowest weight $(\frac{p}{2},\frac{p}{2},\dots,\frac{p}{2})$. If one considers an infinite number of parafermions (parabosonts) the creation and annihilation operators generate the infinite-dimentional algebra $\mathfrak{so}(\infty)$ (superalgebra $\mathfrak{osp}(1|\infty)$) [13].

In the case of a mixed system consisting of parafermions $f_j^{\pm}$ and parabosons $b_j^{\pm}$ the relative commutation relations among paraoperators were studied by Greenberg and Messiah [4]. They have shown that there can exist at most four types of relative commutation relations: straight commutation, straight anticommutation, relative paraboson, and relative parafermion relations and the most interesting case is the latter one. Palev [10] proved that m parafermions $f_j^{\pm}$ (8) and n parabosons $b_j^{\pm}$ (7) with relative parafermion relations generate the orthosymplectic Lie superalgebra $\mathfrak{osp}(2m+1|2n)$. Therefore the parastatistics Fock space corresponds to an infinite-dimensional unitary representation of $\mathfrak{osp}(2m+1|2n)$. For its explicit construction, the techniques developed in [8, 12] can be applied, namely the branching $\mathfrak{osp}(2m+1|2n)\supset\mathfrak{gl}(m|n)$, an induced module construction, a basis description for the covariant tensor representations of $\mathfrak{gl}(m|n)$ [14], Clebsch-Gordan coefficients of $\mathfrak{gl}(m|n)$ [14], and the method of reduced matrix elements.

In section 2, we define m parafermions $f_j^{\pm}$ (8) and n parabosons $b_j^{\pm}$ (7) with relative parafermion relations and the parastatistics Fock space $\mathfrak{V}(p)$. In section 3, we consider the important relation between parastatistics operators and the Lie superalgebra $\mathfrak{osp}(2m+1|2n)$, and give a description of $\mathfrak{V}(p)$ in terms of representations of $\mathfrak{osp}(2m+1|2n)$. The rest of this section is devoted to the analysis of the representations $\mathfrak{V}(p)$ for $\mathfrak{osp}(2m+1|2n)$ and to the matrix elements for any m and n. These matrix elements were recently computed [15]. We conclude the paper with some final remarks.

2 The parastatistics algebra and its Fock space $\mathfrak{V}(p)$

Consider a system of m pairs of parafermions $f_i^{\pm}\equiv c_i^{\pm}$ $,i=1,\dots,m$ and n pairs of parabosons $b_j^{\pm}\equiv c_{m+j}^{\pm}$ $,j=1,\dots,n$ with relative parafermion relations among them. The defining triple relations for such a system are given by

$$[\![[\![c_j^+,c_k^-]\!],c_l^+]\!]=2\delta_{kl}c_j^+,\quad [\![[\![c_j^+,c_k^+]\!],c_l^+]\!]=0,$$

$$[\![c_j^-,[\![c_k^+,c_l^-]\!]]\!]=2\delta_{jk}c_l^-, \quad [\![[\![c_j^-,c_k^-]\!],c_l^-]\!]=0 \tag{11}$$

or

$$[\![[\![c_j^\xi,c_k^\eta]\!],c_l^\varepsilon]\!]=-2\delta_{jl}\delta_{\varepsilon,-\xi}\varepsilon^{\langle l\rangle}(-1)^{\langle k\rangle\langle l\rangle}c_k^\eta+2\varepsilon^{\langle l\rangle}\delta_{kl}\delta_{\varepsilon,-\eta}c_j^\xi, \tag{12}$$
$$\xi,\eta,\varepsilon=\pm \text{ or } \pm 1; \quad j,k,l=1,\dots,n+m,$$

where

$$[\![a,b]\!]=ab-(-1)^{\deg(a)\deg(b)}ba \tag{13}$$

and

$$\deg(c_i^\pm)\equiv\langle i\rangle=\begin{cases}0 \text{ if } j=1,\dots,m\\ 1 \text{ if } j=m+1,\dots,n+m.\end{cases} \tag{14}$$

In the case $j,k,l=1,\dots,m$ (12) reduces to (8) and in the case $j,k,l=m+1,\dots,m+n$ (12) reduces to (7).

The parastatistics Fock space $\mathfrak{V}(p)$ is the Hilbert space with vacuum vector $|0\rangle$, defined by means of ($j,k=1,2,\dots,m+n$)

$$\langle 0|0\rangle=1, \qquad c_j^-|0\rangle=0, \qquad (c_j^\pm)^\dagger=c_j^\mp,$$
$$[\![c_j^-,c_k^+]\!]|0\rangle=p\delta_{jk}|0\rangle, \tag{15}$$

and by irreducibility under the action of the algebra spanned by the elements c_j^+, c_j^-, $j=1,\dots,m+n$, subject to (12). The parameter p is referred to as the order of the parastatistics system.

In 1982 Palev [10] proved the following theorem.

Theorem 1 (Palev). *The Lie superalgebra generated by* $2m$ *even elements* $f_i^\pm\equiv c_i^\pm$ *(*$i=1,\dots,m$*) and* $2n$ *odd elements* $b_j^\pm\equiv c_{m+j}^\pm$ *(*$j=1,\dots,n$*) subject to the relations (12) is the orthosymplectic Lie superalgebra* $\mathfrak{osp}(2m+1|2n)$*. The Fock space* $\mathfrak{V}(p)$ *is the unitary irreducible representation of* $\mathfrak{osp}(2m+1|2n)$ *with lowest weight* $(-\frac{p}{2},\dots,-\frac{p}{2}|\frac{p}{2},\dots,\frac{p}{2})$.

Constructing a basis for the parastatistics Fock space $\mathfrak{V}(p)$ for general (integer) p-values turns out to be a difficult problem, for which we describe the solution in the rest of the paper.

3 The Lie superalgebras $\mathfrak{osp}(2m+1|2n)$ and a class of $\mathfrak{osp}(2m+1|2n)$ explicit representations

The orthosymplectic Lie superalgebra $B(m|n)\equiv\mathfrak{osp}(2m+1|2n)$ [5] consists of $(2m+2n+1\times 2m+2n+1)$ matrices of the form

$$\begin{pmatrix} a & b & u & x & x_1 \\ c & -a^t & v & y & y_1 \\ -v^t & -u^t & 0 & z & z_1 \\ y_1^t & x_1^t & z_1^t & d & e \\ -y^t & -x^t & -z^t & f & -d^t \end{pmatrix}, \tag{16}$$

with a any $(m\times m)$-matrix, b and c antisymmetric $(m\times m)$-matrices, u and v $(m\times 1)$-matrices, x,y,x_1,y_1 $(m\times n)$-matrices, z and z_1 $(1\times n)$-matrices, d any $(n\times n)$-matrix, and e and f symmetric $(n\times n)$-matrices. The even elements have $x=y=x_1=y_1=0$, $z=z_1=0$ and the odd elements are those with $a=b=c=0$, $u=v=0$, $d=e=f=0$. Denote the row and column indices running from 1 to $2m+2n+1$ and by e_{ij} the matrix with zeros everywhere except a 1 on position (i,j). The Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{osp}(2m+1|2n)$ is the subspace of diagonal matrices with basis $h_i=e_{ii}-e_{i+m,i+m}$ $(i=1,\ldots,m)$, $h_{m+i}=e_{2m+1+j,2m+1+j}-e_{2m+1+n+j,2m+1+n+j}$ $(j=1,\ldots,n)$. Denote by ε_i $(i=1,\ldots,m)$, δ_j $(j=1,\ldots,n)$ the dual basis of $\mathfrak{h}^*$.

Introducing the following multiples of the even vectors with roots $\pm\varepsilon_j$ $(j=1,\ldots,m)$

$$\begin{aligned} c_j^+ &= f_j^+ = \sqrt{2}(e_{j,2m+1}-e_{2m+1,j+m}), \\ c_j^- &= f_j^- = \sqrt{2}(e_{2m+1,j}-e_{j+m,2m+1}), \end{aligned} \tag{17}$$

and of the odd vectors with roots $\pm\delta_j$ $(j=1,\ldots,n)$

$$\begin{aligned} c_{m+j}^+ &= b_j^+ = \sqrt{2}(e_{2m+1,2m+1+n+j}+e_{2m+1+j,2m+1}), \\ c_{m+j}^- &= b_j^- = \sqrt{2}(e_{2m+1,2m+1+j}-e_{2m+1+n+j,2m+1}), \end{aligned} \tag{18}$$

it is easy to verify that these operators satisfy the triple relations (12).

The operators c_j^+ are positive root vectors, and the c_j^- are negative root vectors.

We are interested in the construction of the parastatistics Fock space $\mathfrak{V}(p)$ defined by (15). It is straightforward to see that

$$[c_i^-,c_i^+]=-2h_i\ (i=1,\ldots,m),\ \text{ and }\ \{c_{m+j}^-,c_{m+j}^+\}=2h_{m+j}\ (j=1,\ldots,n). \tag{19}$$

Therefore indeed Theorem 1 holds.

In general the representations $\mathfrak{V}(p)$ can be constructed using an induced module procedure (see [15] for more details). The relevant subalgebras of $\mathfrak{osp}(2m+1|2n)$ are as follows.

Proposition 1. *A basis for the even subalgebra $\mathfrak{so}(2m+1)\oplus\mathfrak{sp}(2n)$ of $\mathfrak{osp}(2m+1|2n)$ is given by*

$$[c_i^\xi,c_k^\eta],\ c_l^\varepsilon\ (i,k,l=1,\ldots,m);\quad \{c_{m+j}^\xi,c_{m+s}^\eta\}\ (j,s=1,\ldots,n,\ \xi,\eta=\pm). \tag{20}$$

The elements

$$[\![c_j^+,c_k^-]\!] \qquad (j,k=1,\ldots,m+n) \tag{21}$$

constitute a basis for the subalgebra $\mathfrak{u}(m|n)$.

Note that with the notation $\frac{1}{2}[\![c_j^+,c_k^-]\!] \equiv E_{jk}$, the triple relations (12) imply the relations

$$[\![E_{ij},E_{kl}]\!] = \delta_{jk}E_{il} - (-1)^{deg(E_{ij})deg(E_{kl})}\delta_{li}E_{kj}. \tag{22}$$

Therefore, the elements $[\![c_j^+,c_k^-]\!]$ form, up to a factor 2, the standard basis elements of $\mathfrak{u}(m|n)$ or $\mathfrak{gl}(m|n)$.

The subalgebra $\mathfrak{u}(m|n)$ can be extended to a parabolic subalgebra $\mathscr{P}$ of $\mathfrak{osp}(2m+1|2n)$

$$\mathscr{P} = \mathrm{span}\{c_j^-, [\![c_j^+,c_k^-]\!], [\![c_j^-,c_k^-]\!] \mid j,k = 1,\ldots,m+n\}. \tag{23}$$

Because of the fact that $[\![c_j^-,c_k^+]\!]|0\rangle = p\,\delta_{jk}|0\rangle$, with $[c_i^-,c_i^+] = -2h_i$ $(i=1,\ldots,m,)$ and $\{c_{m+j}^-,c_{m+j}^+\} = 2h_{m+j}$ $(j=1,\ldots,n)$, the space spanned by $|0\rangle$ is a trivial one-dimensional $\mathfrak{u}(m|n)$ module $\mathbb{C}|0\rangle$ of weight $(-\frac{p}{2},\ldots,-\frac{p}{2}|\frac{p}{2},\ldots,\frac{p}{2})$. As $c_j^-|0\rangle = 0$, the $\mathfrak{u}(m|n)$ module $\mathbb{C}|0\rangle$ can be extended to a one-dimensional $\mathscr{P}$ module. The induced $\mathfrak{osp}(2m+1|2n)$ module $\overline{\mathfrak{V}}(p)$ is defined by

$$\overline{\mathfrak{V}}(p) = \mathrm{Ind}_{\mathscr{P}}^{\mathfrak{osp}(2m+1|2n)}\mathbb{C}|0\rangle. \tag{24}$$

This is an $\mathfrak{osp}(2m+1|2n)$ representation with lowest weight $(-\frac{p}{2},\ldots,-\frac{p}{2}|\frac{p}{2},\ldots,\frac{p}{2})$. By the Poincaré-Birkhoff-Witt theorem [6], a basis for $\overline{\mathfrak{V}}(p)$ is given by

$$\begin{aligned}&(c_1^+)^{k_1}\cdots(c_{m+n}^+)^{k_{m+n}}([\![c_1^+,c_2^+]\!])^{k_{12}}([\![c_1^+,c_3^+]\!])^{k_{13}}\cdots([\![c_{m+n-1}^+,c_{m+n}^+]\!])^{k_{m+n-1,m+n}}|0\rangle,\\ &\quad k_1,\ldots,k_{m+n},k_{12},k_{13}\ldots,k_{m-1,m},k_{m+1,m+2},k_{m+1,m+3}\ldots,k_{m+n-1,m+n}\in\mathbb{Z}_+,\\ &\quad k_{1,m+1},k_{1,m+2}\ldots,k_{1,m+n},k_{2,m+1},\ldots,k_{m,m+n}\in\{0,1\}.\end{aligned} \tag{25}$$

In general $\overline{\mathfrak{V}}(p)$ is not an irreducible representation of $\mathfrak{osp}(2m+1|2n)$. Let $M(p)$ be the maximal nontrivial submodule of $\overline{\mathfrak{V}}(p)$. Then the irreducible module, corresponding to the parastatistics Fock space, is

$$\mathfrak{V}(p) = \overline{\mathfrak{V}}(p)/M(p). \tag{26}$$

Now the aim is to determine the vectors belonging to $M(p)$, and thus find the structure of $\mathfrak{V}(p)$, and to compute the matrix elements of the algebra generators.

For this purpose, let us first consider the character of $\overline{\mathfrak{V}}(p)$: this is a formal infinite series of terms $\nu x_1^{j_1}x_2^{j_2}\ldots x_m^{j_m}\ y_1^{j_{m+1}}y_2^{j_{m+2}}\ldots y_n^{j_{m+n}}$, where the exponents carry a weight $(j_1,\ldots,j_m|j_{m+1},\ldots,j_{m+n})$ of $\overline{\mathfrak{V}}(p)$ and ν is the dimension of this weight space. The vacuum vector $|0\rangle$ of $\overline{\mathfrak{V}}(p)$, of weight $(-\frac{p}{2},\ldots,-\frac{p}{2}|\frac{p}{2},\ldots,\frac{p}{2})$, yields a term $x_1^{-\frac{p}{2}}\ldots x_m^{-\frac{p}{2}}y_1^{\frac{p}{2}}\ldots y_n^{\frac{p}{2}}$ in the character $\mathrm{char}\,\overline{\mathfrak{V}}(p)$ and from the basis vectors (25) it follows that

$$\mathrm{char}\,\overline{\mathfrak{V}}(p) = \frac{(x_1)^{-p/2}\cdots(x_m)^{-p/2}(y_1)^{p/2}\cdots(y_n)^{p/2}\prod_{i,j}(1+x_iy_j)}{\prod_i(1-x_i)\prod_{i<k}(1-x_ix_k)\prod_j(1-y_j)\prod_{j<l}(1-y_jy_l)}. \tag{27}$$

Such expressions can be expanded in terms of supersymmetric Schur functions, valid for general m and n.

Proposition 2 (Cummins and King). *Consider two sets of variables*

$$(\mathbf{x}) = (x_1, x_2, \ldots, x_m), \qquad (\mathbf{y}) = (y_1, y_2, \ldots, y_n).$$

Then [1]

$$\frac{\prod_{i,j}(1+x_iy_j)}{\prod_i(1-x_i)\prod_{i<k}(1-x_ix_k)\prod_j(1-y_j)\prod_{j<l}(1-y_jy_l)} \\ = \sum_{\lambda\in\mathscr{H}} s_\lambda(x_1,\ldots,x_m|y_1,\ldots,y_n) = \sum_{\lambda\in\mathscr{H}} s_\lambda(\mathbf{x}|\mathbf{y}). \tag{28}$$

In the right hand side, the sum is over all partitions λ satisfying the so called hook condition $\lambda_{m+1} \le n$ ($\lambda \in \mathscr{H}$), and $s_\lambda(\mathbf{x}|\mathbf{y})$ is the supersymmetric Schur function [9] defined by

$$s_\lambda(\mathbf{x}|\mathbf{y}) = \sum_\tau s_{\lambda/\tau}(\mathbf{x}) s_{\tau'}(\mathbf{y}) = \sum_{\sigma,\tau} c^\lambda_{\sigma\tau} s_\sigma(\mathbf{x}) s_{\tau'}(\mathbf{y}),$$

where $\ell(\sigma) \le m$, $\ell(\tau') \le n$ and $|\lambda| = |\sigma| + |\tau|$. Herein, some standard notation [9] is used: for a partition λ, $\ell(\lambda)$ is the length of λ and $|\lambda|$ its weight; τ' is the partition conjugate to τ; $c^\lambda_{\sigma\tau}$ are the Littlewood-Richardson coefficients; and $s_\nu(\mathbf{x})$ is the ordinary Schur function.

Now it is well known that the characters of the irreducible covariant $\mathfrak{u}(m|n)$ tensor representations $V([\Lambda^\lambda])$ are given by such supersymmetric Schur functions $s_\lambda(x|y)$ ($\lambda \in \mathscr{H}$). The relation between the partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$, $\lambda_{m+1} \le n$ and the highest weights $\Lambda^\lambda \equiv [\mu]^r \equiv [\mu_{1r}, \ldots, \mu_{mr}|\mu_{m+1,r} \ldots, \mu_{rr}]$ ($r = m+n$) of the irreducible covariant $\mathfrak{u}(m|n)$ tensor representations is known [16]:

$$\begin{aligned} &\mu_{ir} = \lambda_i, \quad 1 \le i \le m, \\ &\mu_{m+i,r} = \max\{0, \lambda'_i - m\}, \quad 1 \le i \le n, \end{aligned} \tag{29}$$

where λ' is the partition conjugate [9] to λ. Therefore the formula (28) gives the branching to $\mathfrak{u}(m|n)$ of the $\mathfrak{osp}(2m+1|2n)$ representation $\overline{\mathfrak{V}}(p)$. This also gives a possibility to label the basis vectors of $\overline{\mathfrak{V}}(p)$. For each irreducible covariant $\mathfrak{u}(m|n)$ tensor representations one can use the Gelfand-Zetlin basis (GZ) [14] and the union of all these GZ basis is then the basis for $\overline{\mathfrak{V}}(p)$. In such a way the new basis of $\overline{\mathfrak{V}}(p)$ consists of vectors of the form

$$|\mu) \equiv |\mu)^r = \left|\begin{array}{llllllll} \mu_{1r} & \cdots & \mu_{m-1,r} & \mu_{mr} & \mu_{m+1,r} & \cdots & \mu_{r-1,r} & \mu_{rr} \\ \mu_{1,r-1} & \cdots & \mu_{m-1,r-1} & \mu_{m,r-1} & \mu_{m+1,r-1} & \cdots & \mu_{r-1,r-1} & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \iddots & & \\ \mu_{1,m+1} & \cdots & \mu_{m-1,m+1} & \mu_{m,m+1} & \mu_{m+1,m+1} & & & \\ \mu_{1m} & \cdots & \mu_{m-1,m} & \mu_{mm} & & & & \\ \mu_{1,m-1} & \cdots & \mu_{m-1,m-1} & & & & & \\ \vdots & \iddots & & & & & & \\ \mu_{11} & & & & & & & \end{array}\right), \quad (30)$$

which satisfy the conditions

$$\begin{array}{l} 1.\ \mu_{ir} \in \mathbb{Z}_+ \text{ are fixed and } \mu_{jr} - \mu_{j+1,r} \in \mathbb{Z}_+,\ j \neq m,\ 1 \leq j \leq r-1, \\ \quad \mu_{mr} \geq \#\{i : \mu_{ir} > 0,\ m+1 \leq i \leq r\}; \\ 2.\ \mu_{ip} - \mu_{i,p-1} \equiv \theta_{i,p-1} \in \{0,1\}, \quad 1 \leq i \leq m;\ m+1 \leq p \leq r; \\ 3.\ \mu_{mp} \geq \#\{i : \mu_{ip} > 0,\ m+1 \leq i \leq p\}, \quad m+1 \leq p \leq r; \\ 4.\ \text{if } \mu_{m,m+1} = 0, \text{then } \theta_{mm} = 0; \\ 5.\ \mu_{ip} - \mu_{i+1,p} \in \mathbb{Z}_+, \quad 1 \leq i \leq m-1;\ m+1 \leq p \leq r-1; \\ 6.\ \mu_{i,j+1} - \mu_{ij} \in \mathbb{Z}_+ \text{ and } \mu_{i,j} - \mu_{i+1,j+1} \in \mathbb{Z}_+, \\ \quad 1 \leq i \leq j \leq m-1 \text{ or } m+1 \leq i \leq j \leq r-1. \end{array} \quad (31)$$

Note that the last m lines of the triangular GZ-array correspond to a GZ-pattern of $\mathfrak{gl}(m)$, whereas the last n columns correspond to a GZ-pattern for $\mathfrak{gl}(n)$. The conditions above follow from the correspondence between a highest weight in partition notation and its coordinates, see (29), and from the fact that for covariant representations, the decomposition from $\mathfrak{u}(m|n)$ to $\mathfrak{u}(m|n-1)$ is governed by

$$s_\lambda(\mathbf{x}|\mathbf{y}) = \sum_\sigma s_\sigma(\mathbf{x}|y_1,\ldots,y_{n-1})\, y_n^{|\lambda|-|\sigma|}. \quad (32)$$

In this last expression, the sum is over all partitions σ such that $\lambda - \sigma$ is a vertical strip [9]. That actually explains why the $\theta_{i,p}$'s in (31) take values in $\{0,1\}$.

Now the task is to give the explicit action of the generating elements $c_i^\pm$ (12) of $\mathfrak{osp}(2m+1|2n)$. For this purpose, we introduce the following notations:

$$|\mu) \equiv |\mu)^r = \left|\begin{array}{l} [\mu]^r \\ |\mu)^{r-1} \end{array}\right),$$

$([\mu]^r) = (\mu_{1r}, \mu_{2r}, \ldots, \mu_{rr})$ and $([\mu]^r_{\pm k}) = (\mu_{1r}, \ldots, \mu_{kr} \pm 1, \ldots, \mu_{rr})$. Then

Proposition 3. *The explicit actions of the Lie superalgebra generators $c_j^\pm$ on a basis of $\overline{\mathfrak{V}}(p)$ are as follows:*

$$c_j^+|\mu) = \sum_{k,\mu'} \left(\begin{array}{l} [\mu]^r \\ |\mu)^{r-1} \end{array} ; \begin{array}{l} 10\cdots00 \\ 10\cdots0 \\ \cdots \\ 0 \end{array} \left| \begin{array}{l} [\mu]^r_{+k} \\ |\mu')^{r-1} \end{array} \right.\right) \times G_k([\mu]^r) \left|\begin{array}{l} [\mu]^r_{+k} \\ |\mu')^{r-1} \end{array}\right), \quad (33)$$

$$c_j^-|\mu) = \sum_{k,\mu'} \left(\begin{matrix} [\mu]^r_{-k} \\ |\mu')^{r-1} \end{matrix} ; \begin{matrix} 10\cdots 00 \\ 10\cdots 0 \\ \cdots \\ 0 \end{matrix} \left| \begin{matrix} [\mu]^r \\ |\mu)^{r-1} \end{matrix} \right. \right) \times G_k([\mu]^r_{-k}) \left| \begin{matrix} [\mu]^r_{-k} \\ |\mu')^{r-1} \end{matrix} \right). \tag{34}$$

The first factor in the right hand sides of (33)-(34) is a $\mathfrak{u}(m|n)$ Clebsch-Gordan coefficient (CGC) given by formulae (4.9)-(4.17) in [14], and the second factor is a reduced matrix element. *The reduced matrix elements G_k ($k = 1,\ldots,m+n = r$) are given by:*

$$\begin{aligned} & G_k(\mu_{1r},\mu_{2r},\ldots,\mu_{rr}) = \\ & \left(-\frac{(\mathscr{E}_m(\mu_{kr}+m-n-k)+1)\prod_{j\neq k=1}^m(\mu_{kr}-\mu_{jr}-k+j)}{\prod_{j\neq \frac{k}{2}=1}^{\lfloor m/2\rfloor}(\mu_{kr}-\mu_{2j,r}-k+2j)(\mu_{kr}-\mu_{2j,r}-k+2j+1)} \right)^{1/2} \\ & \times \prod_{j=1}^n \left(\frac{\mu_{kr}+\mu_{m+j,r}+m-j-k+2}{\mu_{kr}+\mu_{m+j,r}+m-j-k+2-\mathscr{E}_{m+\mu_{m+j,r}}} \right)^{1/2} \end{aligned} \tag{35}$$

for $k \leq m$ and k even;

$$\begin{aligned} & G_k(\mu_{1r},\mu_{2r},\ldots,\mu_{rr}) = \\ & \left(\frac{(p-\mu_{kr}+k-1)(\mathscr{O}_m(\mu_{kr}+m-n-k)+1)\prod_{j\neq k=1}^m(\mu_{kr}-\mu_{jr}-k+j)}{\prod_{j\neq \frac{k+1}{2}=1}^{\lceil m/2\rceil}(\mu_{kr}-\mu_{2j-1,r}-k+2j-1)(\mu_{kr}-\mu_{2j-1,r}-k+2j)} \right)^{1/2} \\ & \times \prod_{j=1}^n \left(\frac{\mu_{kr}+\mu_{m+j,r}+m-j-k+2}{\mu_{kr}+\mu_{m+j,r}+m-j-k+2-\mathscr{O}_{m+\mu_{m+j,r}}} \right)^{1/2} \end{aligned} \tag{36}$$

for $k \leq m$ and k odd. The remaining expressions for $k = 1,2,\ldots,n$ are

$$\begin{aligned} & G_{m+k}(\mu_{1r},\mu_{2r},\ldots,\mu_{rr}) = (-1)^{\mu_{m+k+1,r}+\mu_{m+k+2,r}+\ldots+\mu_{rr}} \\ & \times \Big((\mathscr{O}_{\mu_{m+k,r}}(\mu_{m+k,r}-k+n)+1)(\mathscr{E}_{m+\mu_{m+k,r}}(p+\mu_{m+k,r}+m-k)+1) \Big)^{1/2} \\ & \times \left(\frac{\prod_{j=1}^{\lfloor m/2\rfloor}(\mathscr{E}_{m+\mu_{m+k,r}}(\mu_{2j,r}+\mu_{m+k,r}-2j-k+m+1)+1)}{\prod_{j=1}^{\lceil m/2\rceil}(\mathscr{E}_{m+\mu_{m+k,r}}(\mu_{2j-1,r}+\mu_{m+k,r}-2j-k+m+1)+1)} \right)^{1/2} \\ & \times \left(\frac{\prod_{j=1}^{\lceil m/2\rceil}(\mathscr{O}_{m+\mu_{m+k,r}}(\mu_{2j-1,r}+\mu_{m+k,r}-2j-k+m+2)+1)}{\prod_{j=1}^{\lfloor m/2\rfloor}(\mathscr{O}_{m+\mu_{m+k,r}}(\mu_{2j,r}+\mu_{kr}-2j-k+m)+1)} \right)^{1/2} \\ & \times \prod_{j\neq k=1}^n \left(\frac{\mu_{m+j,r}-\mu_{m+k,r}-j+k}{\mu_{m+j,r}-\mu_{m+k,r}-j+k-\mathscr{O}_{\mu_{m+j,r}-\mu_{m+k,r}}} \right)^{1/2}. \end{aligned} \tag{37}$$

Herein $\mathscr{E}$ and $\mathscr{O}$ are the even and odd functions defined by

$$\begin{aligned} &\mathscr{E}_j = 1 \text{ if } j \text{ is even and 0 otherwise,} \\ &\mathscr{O}_j = 1 \text{ if } j \text{ is odd and 0 otherwise;} \end{aligned} \tag{38}$$

where obviously $\mathscr{O}_j = 1 - \mathscr{E}_j$, but it is still convenient to use both notations. Also, note that products such as $\prod_{j\neq k=1}^{s}$ means "the product over all j-values running from 1 to s, but excluding $j = k$". The notation $\lfloor a \rfloor$ (resp. $\lceil a \rceil$) refers to the *floor* (resp. *ceiling*) of a, i.e. the largest integer not exceeding a (resp. the smallest integer greater than or equal to a).

Now, taking into account the general conditions (31), the only factor in the right hand sides of (35)-(37) that may become zero appears in (36) and is

$$p - \mu_{kr} + k - 1 \qquad (k \leq m \text{ and } k \text{ odd}).$$

For $k = 1$ this factor is $(p - \mu_{1r})$, and μ_{1r} is the largest integer in the first row of the GZ-pattern (30) (which is also the first part of the partition λ, see (29)). Starting from the vacuum vector, with a GZ-pattern consisting of all zeros, one can raise the entries in the GZ-pattern by applying the operators c_j^+. However, when μ_{1r} has reached the value p it can no longer be increased. As a consequence, all vectors $|\mu)$ with $\mu_{1r} > p$ belong to the submodule $M(p)$. This gives the structure of $\mathfrak{V}(p)$.

Theorem 2. *An orthonormal basis for the space $\mathfrak{V}(p)$ is given by the vectors $|\mu)$, see (30)-(31), with $\mu_{1r} \leq p$. The action of the Cartan algebra elements of $\mathfrak{osp}(2m+1|2n)$ is:*

$$\begin{aligned} h_k|\mu) &= \left(-\frac{p}{2} + \sum_{j=1}^{k} \mu_{jk} - \sum_{j=1}^{k-1} \mu_{j,k-1}\right)|\mu), \quad k = 1,\ldots,m; \\ h_k|\mu) &= \left(\frac{p}{2} + \sum_{j=1}^{k} \mu_{jk} - \sum_{j=1}^{k-1} \mu_{j,k-1}\right)|\mu), \quad k = m+1,\ldots,r. \end{aligned} \tag{39}$$

The action of the operators $c_j^\pm$, $j = 1,\ldots,r$ is given by (33)-(34), where the CGCs are found in [14] (see formulae (4.9)-(4.17)) and the reduced matrix elements are given by (35)-(37).

4 Summary and conclusion

In the present paper we have constructed the Fock spaces $\mathfrak{V}(p)$ of m parafermions and n parabosons with relative parafermion relations among them, which are the unitary irreducible representations of $\mathfrak{osp}(2m+1|2n)$ with lowest weight of the form $(-\frac{p}{2},\ldots,-\frac{p}{2}|\frac{p}{2},\ldots,\frac{p}{2})$. The subalgebra $\mathfrak{u}(m|n)$ of $\mathfrak{osp}(2m+1|2n)$, generated by all supercommutators of the parafermions and parabosons, and its covariant tensor representations play a crucial role in the analysis. For each irreducible covariant $\mathfrak{u}(m|n)$ tensor representation the known Gelfand-Zetlin basis follows the decomposition $\mathfrak{u}(m|n) \supset \mathfrak{u}(m|n-1) \supset \ldots \supset \mathfrak{u}(m|1) \supset \mathfrak{u}(m) \supset \mathfrak{u}(m-1) \supset \ldots \supset \mathfrak{u}(1)$.

The real interest is in such quantum systems (mixed systems of parafermions and parabosons) with infinite degrees of freedom ($m \to \infty$ and $n \to \infty$). It is clear that the GZ-basis used here cannot be used for such a purpose: as $m \to \infty$ in (30), there is no longer control over n. In order to investigate such systems one should construct the irreducible covariant tensor representations of $\mathfrak{u}(n|n)$ in another Gelfand-Zetlin basis, namely following the decomposition $\mathfrak{u}(n|n) \supset \mathfrak{u}(n|n-1) \supset \mathfrak{u}(n-1|n-1) \ldots \supset \mathfrak{u}(2|2) \supset \mathfrak{u}(2|1) \supset \mathfrak{u}(1|1) \supset \mathfrak{u}(1)$. We hope to report this result soon.

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