Gel'fand-Zetlin Basis and Clebsch-Gordan Coefficients for Covariant Representations of the Lie superalgebra $\mathfrak{gl}(m|n)$

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Abstract

A Gel'fand-Zetlin basis is introduced for the irreducible covariant tensor representations of the Lie superalgebra $\mathfrak{gl}(m|n)$. Explicit expressions for the generators of the Lie superalgebra acting on this basis are determined. Furthermore, Clebsch-Gordan coefficients corresponding to the tensor product of any covariant tensor representation of $\mathfrak{gl}(m|n)$ with the natural representation $V([1,0,\ldots,0])$ of $\mathfrak{gl}(m|n)$ with highest weight (1,0,. . . ,0) are computed. Both results are steps for the explicit construction of the parastatistics Fock space.

1 Introduction

The representation theory of (basic) classical Lie (super)algebras plays a central role in many branches of mathematics and physics. The first explicit constructions of finite-dimensional irreducible representations were given by Gel'fand and Zetlin [1, 2]. They introduced a basis in any finite-dimensional irreducible $\mathfrak{gl}(n)$ module V considering the chain of subalgebras $\mathfrak{gl}(n) \supset \mathfrak{gl}(n-1) \supset \ldots \supset \mathfrak{gl}(1)$. Since each such module V is a direct sum of irreducible $\mathfrak{gl}(n-1)$ modules $V = \sum_i \oplus V_i$, where the decomposition is multiplicity free, and any irreducible $\mathfrak{gl}(1)$ module $V(1)$ is a one dimensional space, the vectors corresponding to all possible flags $V \equiv V(n) \supset V(n-1) \supset \ldots \supset V(1)$ and labeled by the highest weights of $V(k)$, constitute a basis in V. This basis is now called a Gel'fand-Zetlin (GZ) basis in the $\mathfrak{gl}(n)$ module V [1].

In a similar way one can introduce a basis in each finite-dimensional $\mathfrak{so}(n)$ module [2] considering the chain of subalgebras $\mathfrak{so}(n) \supset \mathfrak{so}(n-1) \supset \ldots \supset \mathfrak{so}(2)$. Contrary to $\mathfrak{gl}(n)$, where the basis consists of orthonormal weight vectors, the GZ-basis vectors for $\mathfrak{so}(n)$ [2] are not eigenvectors for the Cartan subalgebra (so the GZ-basis vectors are not weight vectors).

This approach does not work for the symplectic Lie algebras $\mathfrak{sp}(2n)$ since the restriction $\mathfrak{sp}(2n) \downarrow \mathfrak{sp}(2n-2)$ is not multiplicity free. Since the papers of Gel'fand and Zetlin [1, 2] were published in 1950, many different methods were developed to construct bases in the modules of the classical Lie algebras (see for instance the review paper [3]). Finally, a complete solution of the problem for the $\mathfrak{sp}(2n)$ modules was given by Molev [4] in 1999. He used finite-dimensional irreducible representations of the so called twisted Yangians. Molev applied his approach also to the orthogonal Lie algebras [5, 6]. The new basis consists of weight vectors but in turn lacks the orthogonality property. In such a way the problem to construct a natural basis for the Lie algebras $\mathfrak{so}(n)$ and $\mathfrak{sp}(2n)$, which accommodate both properties (weight vectors and orthogonality) remains an open one.

Also some steps towards a generalization of the concept of GZ-basis for basic classical Lie superalgebras have been taken (see [7–10]). Irrespective of the progress, there is still much to be done in order to complete the representation theory of the basic classical Lie superalgebras. In

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the present paper we take a step further in this respect introducing a Gel'fand-Zetlin basis in the irreducible covariant tensor representations of the general linear Lie superalgebra $\mathfrak{gl}(m|n)$ [11, 12] and writing down explicit expressions for the transformation of the basis vectors under the action of the algebra generators. In this case the Gel'fand-Zetlin basis vectors accommodate both nice properties – they are orthonormal and weight vectors. Next, using the matrix elements of the $\mathfrak{gl}(m|n)$ covariant tensor representations, we compute certain Clebsch-Gordan coefficients of the Lie superalgebra $\mathfrak{gl}(m|n)$.

The motivation for the present work comes from some physical ideas. In 1953 Green [13] introduced more general statistics than the common Fermi-Dirac and Bose-Einstein statistics, namely the parafermion and paraboson statistics. These generalizations have an algebraic formulation in terms of generators and relations. The parafermion operators $f_j^{\pm}$, satisfying

$$[[f_j^{\xi}, f_k^{\eta}], f_l^{\epsilon}] = \frac{1}{2}(\epsilon - \eta)^2 \delta_{kl} f_j^{\xi} - \frac{1}{2}(\epsilon - \xi)^2 \delta_{jl} f_k^{\eta}, \tag{1.1}$$

where $j, k, l \in \{1, 2, \ldots, m\}$ and $\eta, \epsilon, \xi \in \{+, -\}$ (to be interpreted as $+1$ and -1 in the algebraic expressions $\epsilon - \xi$ and $\epsilon - \eta$), generate the Lie algebra $\mathfrak{so}(2m+1)$ [14, 15]. Similarly, n pairs of paraboson operators $b_j^{\pm}$, $j = 1, 2, \ldots, n$, satisfying

$$[\{b_j^{\xi}, b_k^{\eta}\}, b_l^{\epsilon}] = (\epsilon - \xi)\delta_{jl} b_k^{\eta} + (\epsilon - \eta)\delta_{kl} b_j^{\xi}, \tag{1.2}$$

generate the orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2n)$ [16]. The paraboson and parafermion Fock spaces, characterized by a positive integer p, often referred to as the order of statistics, are unitary lowest weight representations of the relevant algebras with a nondegenerate lowest weight space (i.e. with a unique vacuum). Despite their importance, an explicit construction of the parafermion and paraboson Fock spaces was not known until recently. For the case of parafermions, this explicit construction was given in [17], and for parabosons in [18].

It is natural to extend these results to a system consisting of parafermions $f_j^{\pm}$ and parabosons $b_j^{\pm}$. It was proved by Palev [19] that the relative commutation relations between m parafermions (1.1) and n parabosons (1.2) can be defined in such a way that they generate the orthosymplectic Lie superalgebra $\mathfrak{osp}(2m+1|2n)$. Then the parastatistics Fock space of order p corresponds to an infinite-dimensional unitary representation of $\mathfrak{osp}(2m+1|2n)$ and it can be constructed explicitly using similar techniques as in [17, 18], namely using the branching $\mathfrak{osp}(2m+1|2n) \supset \mathfrak{gl}(m|n)$, an induced representation construction, a basis description for the covariant tensor representations of $\mathfrak{gl}(m|n)$, Clebsch-Gordan coefficients of $\mathfrak{gl}(m|n)$, and the method of reduced matrix elements. Therefore in order to construct the parastatistics Fock space first we need the covariant tensor representations of the Lie superalgebra $\mathfrak{gl}(m|n)$ in an explicit form. Since it is easy to see that the triple relations (1.1) and (1.2) imply that the set $(f_1^+, \ldots, f_m^+, b_1^+, \ldots, b_n^+)$ is a standard $\mathfrak{gl}(m|n)$ tensor of rank $(1, 0, \ldots, 0)$, for the construction of the parastatistics Fock space one needs the $\mathfrak{gl}(m|n)$ Clebsch-Gordan coefficients corresponding to the tensor product $V([\mu]^r) \otimes V([1, 0, \ldots, 0])$, where $V([\mu]^r)$ is any $\mathfrak{gl}(m|n)$ irreducible covariant tensor representation and $V([1, 0, \ldots, 0])$ is the representation of $\mathfrak{gl}(m|n)$ with highest weight $(1, 0, \ldots, 0)$. This paper deals with these two problems. In section 2 we define the Lie superalgebra $\mathfrak{gl}(m|n)$ and remind the reader of some representation theory of $\mathfrak{gl}(m|n)$, in particular of the concept of typical, atypical and covariant tensor representations. In the next section, we construct the covariant tensor representations of $\mathfrak{gl}(m|n)$ introducing a Gel'fand-Zetlin basis. We present the action of the $\mathfrak{gl}(m|n)$ generators on the basis, and give some indications of how we proved that the defining relations of the algebra are satisfied in these representations. The computation of the Clebsch-Gordan coefficients in given in section 4.

2 The Lie superalgebra $\mathfrak{gl}(m|n)$

The underlying vector space for the Lie superalgebra $\mathfrak{g} = \mathfrak{gl}(m|n)$ consists of the space of $(r \times r)$-matrices, with

$$r = m + n. \tag{2.1}$$

The Lie superalgebra $\mathfrak{g} = \mathfrak{gl}(m|n)$ can be defined [11, 12] through its natural matrix realization

$$\mathfrak{gl}(m|n) = \{x = \begin{pmatrix} A & B \\ C & D \end{pmatrix} | A \in M_{m\times m}, B \in M_{m\times n}, C \in M_{n\times m}, D \in M_{n\times n}\}, \tag{2.2}$$

where $M_{p\times q}$ is the space of all $p \times q$ complex matrices. The even subalgebra $\mathfrak{gl}(m|n)_{\bar{0}}$ has $B = 0$ and $C = 0$; the odd subspace $\mathfrak{gl}(m|n)_{\bar{1}}$ has $A = 0$ and $D = 0$. Note that $\mathfrak{gl}(m|n)_{\bar{0}} = \mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$. We denote by $\mathfrak{gl}(m|n)_{+1}$ the space of matrices $\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$ and by $\mathfrak{gl}(m|n)_{-1}$ the space of matrices $\begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}$. Then $\mathfrak{g} = \mathfrak{gl}(m|n)$ has a $\mathbb{Z}$-grading which is consistent with the $\mathbb{Z}_2$-grading [20], namely $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{+1}$ with $\mathfrak{g}_{\bar{0}} = \mathfrak{g}_0$ and $\mathfrak{g}_{\bar{1}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{+1}$. The Lie superalgebra is then defined by means of the bracket $[\![x, y]\!] = xy - (-1)^{\deg(x)\deg(y)} yx$, where x and y are homogeneous elements.

A basis for $\mathfrak{g} = \mathfrak{gl}(m|n)$ consists of matrices e_{ij} $(i, j = 1, 2, \ldots, r)$ with entry 1 at position (i, j) and 0 elsewhere. Alternatively, the Lie superalgebra $\mathfrak{g}$ can be defined by means of generators and relations. A Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is spanned by the elements e_{jj} $(j = 1, 2, \ldots, r)$, and a set of generators of $\mathfrak{gl}(m|n)$ is given by the Chevalley generators $h_j \equiv e_{jj}$ $(j = 1, \ldots, r)$, $e_i \equiv e_{i,i+1}$ and $f_i \equiv e_{i+1,i}$ $(i = 1, \ldots, r-1)$. Then $\mathfrak{g}$ can be defined as the free associative superalgebra over $\mathbb{C}$ and generators h_j, $(j = 1, 2, \ldots, r)$ and e_i, f_i $(i = 1, 2, \ldots, r-1)$ subject to the following relations [21–23] (unless stated otherwise, the indices below run over all possible values):

- The Cartan-Kac relations:

$$[h_i, h_j] = 0; \tag{2.3}$$
$$[h_i, e_j] = (\delta_{ij} - \delta_{i,j+1})e_j; \tag{2.4}$$
$$[h_i, f_j] = -(\delta_{ij} - \delta_{i,j+1})f_j; \tag{2.5}$$
$$[e_i, f_j] = 0 \ \text{ if } i \neq j; \tag{2.6}$$
$$[e_i, f_i] = h_i - h_{i+1} \ \text{ if } i \neq m; \tag{2.7}$$
$$\{e_m, f_m\} = h_m + h_{m+1}; \tag{2.8}$$

- The Serre relations for the e_i:

$$e_i e_j = e_j e_i \text{ if } |i - j| \neq 1; \qquad e_m^2 = 0; \tag{2.9}$$
$$e_i^2 e_{i+1} - 2e_i e_{i+1} e_i + e_{i+1} e_i^2 = 0, \ \text{for } i \in \{1, \ldots, m-1\} \cup \{m+1, \ldots, n+m-2\}; \tag{2.10}$$
$$e_{i+1}^2 e_i - 2e_{i+1} e_i e_{i+1} + e_i e_{i+1}^2 = 0, \ \text{for } i \in \{1, \ldots, m-2\} \cup \{m, \ldots, n+m-2\}; \tag{2.11}$$
$$e_m e_{m-1} e_m e_{m+1} + e_{m-1} e_m e_{m+1} e_m + e_m e_{m+1} e_m e_{m-1} + e_{m+1} e_m e_{m-1} e_m - 2e_m e_{m-1} e_{m+1} e_m = 0; \tag{2.12}$$

- The relations obtained from (2.9)–(2.12) by replacing every e_i by f_i.

The space dual to $\mathfrak{h}$ is $\mathfrak{h}^*$ and is described by the forms ϵ_i $(i = 1, \ldots, r)$ where $\epsilon_j : x \to A_{jj}$ for $1 \le j \le m$ and $\epsilon_{m+j} : x \to D_{jj}$ for $1 \le j \le n$, and where x is given as in (2.2). The components of an element $\Lambda \in \mathfrak{h}^*$ will be written as $[\mu]^r = [\mu_{1r}, \mu_{2r}, \ldots, \mu_{rr}]$ where $\Lambda = \sum_{i=1}^{r} \mu_{ir}\epsilon_i$ and μ_{ir} are complex numbers. The elements of $\mathfrak{h}^*$ are called the weights. The roots of $\mathfrak{gl}(m|n)$ take the form $\epsilon_i - \epsilon_j$ $(i \neq j)$; the positive roots are those with $1 \leq i < j \leq r$, and of importance are the mn odd positive roots

$$\beta_{ip} = \epsilon_i - \epsilon_p, \qquad \text{with } 1 \leq i \leq m \text{ and } m+1 \leq p \leq r. \tag{2.13}$$

$\Lambda \in \mathfrak{h}^*$ with components $[\mu]^r$ will be called an integral dominant weight if $\mu_{ir} - \mu_{i+1,r} \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ for all $i \neq m$ $(1 \leq i \leq r-1)$. For every integral dominant weight $\Lambda \equiv [\mu]^r$ we denote by $V^0(\Lambda)$ the simple $\mathfrak{g}_0$ module with highest weight Λ; this is simply the finite-dimensional $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$ module with $\mathfrak{gl}(m)$ labels $[\mu_{1r}, \ldots \mu_{mr}]$ and with $\mathfrak{gl}(n)$ labels $[\mu_{m+1,r}, \ldots, \mu_{rr}]$. The module $V^0(\Lambda)$ can be extended to a $\mathfrak{g}_0 \oplus \mathfrak{g}_{+1}$ module by the requirement that $\mathfrak{g}_{+1}V^0(\Lambda) = 0$. The induced $\mathfrak{g}$ module $\overline{V}([\Lambda])$, first introduced by Kac [12] and usually referred to as the Kac-module, is defined by

$$\overline{V}([\Lambda]) = \mathrm{Ind}_{\mathfrak{g}_0 \oplus \mathfrak{g}_{+1}}^{\mathfrak{g}} V^0(\Lambda) \cong U(\mathfrak{g}_{-1}) \otimes V^0(\Lambda), \tag{2.14}$$

where $U(\mathfrak{g}_{-1})$ is the universal enveloping algebra of $\mathfrak{g}_{-1}$. It follows that $\dim \overline{V}([\Lambda]) = 2^{nm} \dim V^0(\Lambda)$. By definition, $\overline{V}([\Lambda])$ is a highest weight module; unfortunately, $\overline{V}([\Lambda])$ is not always a simple $\mathfrak{g}$ module. It contains a unique maximal (proper) submodule $M[\Lambda]$, and the quotient module

$$V([\Lambda]) = \overline{V}([\Lambda])/M[\Lambda] \tag{2.15}$$

is a finite-dimensional simple module with highest weight Λ. In fact, Kac [12] proved the following:

Theorem 1 *Every finite-dimensional simple $\mathfrak{g}$ module is isomorphic to a module of type (2.15), where $\Lambda \equiv [\mu]^r \equiv [\mu_{1r}, \mu_{2,r}, \ldots, \mu_{rr}]$ is integral dominant. Moreover, every finite-dimensional simple $\mathfrak{g}$ module is uniquely characterized by its integral dominant highest weight Λ.*

An integral dominant weight $\Lambda = [\mu]^r = [\mu_{1r}, \mu_{2r}, \ldots, \mu_{rr}]$ (resp. $\overline{V}([\Lambda])$, resp. $V([\Lambda])$) is called a typical weight (resp. a typical Kac module, resp. a typical simple module) if and only if $\langle \Lambda + \rho | \beta_{ip} \rangle \neq 0$ for all odd positive roots β_{ip} of (2.13), where 2ρ is the sum of all positive roots of $\mathfrak{g}$. Otherwise Λ, $\overline{V}([\Lambda])$ and $V([\Lambda])$ are called atypical. The importance of these definitions follows from another theorem of Kac [12]:

Theorem 2 *The Kac-module $\overline{V}([\Lambda])$ is a simple $\mathfrak{g}$ module if and only if Λ is typical.*

For an integral dominant highest weight $\Lambda = [\mu]^r$ it is convenient to introduce the following labels [10]:

$$l_{ir} = \mu_{ir} - i + m + 1, \quad (1 \leq i \leq m); \qquad l_{pr} = -\mu_{pr} + p - m, \quad (m+1 \leq p \leq r). \tag{2.16}$$

In terms of these, one can deduce that $\langle \Lambda + \rho | \beta_{ip} \rangle = l_{ir} - l_{pr}$, and hence the conditions for typicality take a simple form.

Apart from the distinction between typical and atypical irreducible finite-dimensional modules of $\mathfrak{gl}(m|n)$, it is possible to distinguish between such modules on the basis of their relationship to tensor modules of various kinds. For instance, Berele and Regev [24], showed that the tensor product $V([1, 0, \ldots, 0])^{\otimes N}$ of N copies of the natural $(m+n)$-dimensional representation $V([1, 0, \ldots, 0])$ of $\mathfrak{gl}(m|n)$ is completely reducible, and the irreducible components, V_λ, can be labeled by a partition λ of N of length $l(\lambda)$ and weight $|\lambda|$, where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, with $l(\lambda) = \ell$, $|\lambda| = \lambda_1 + \lambda_2 + \ldots + \lambda_\ell = N$, and $\lambda_i \geq \lambda_{i+1} > 0$ for $i = 1, 2, \ldots, \ell - 1$, satisfying the condition $\lambda_{m+1} \leq n$. For definitions regarding partitions, see [25]. The condition $\lambda_{m+1} \leq n$ is known as the *hook* *condition*: in terms of Young diagrams, it means that the diagram of λ should be inside the (m,n)-hook [24]. The representations thus obtained are called irreducible covariant tensor representations and are necessarily finite dimensional. Then according to Theorem 1, there must exist an integral dominant weight Λ^λ such that V_λ is isomorphic to $V([\Lambda^\lambda])$. The relation between $\Lambda^\lambda \equiv [\mu]^r \equiv [\mu_{1r}, \mu_{2r}, \ldots, \mu_{rr}]$, $(r = m+n)$ and $\lambda = (\lambda_1, \lambda_2, \ldots)$ is such that [26]:

$$\mu_{ir} = \lambda_i, \quad 1 \le i \le m, \tag{2.17}$$
$$\mu_{m+i,r} = \max\{0, \lambda_i' - m\}, \quad 1 \le i \le n, \tag{2.18}$$

where λ' is the partition conjugate [25] to λ. Conversely if $\Lambda \equiv [\mu]^r \equiv [\mu_{1r}, \mu_{2r}, \ldots, \mu_{rr}]$ is integral dominant with all $\mu_{ir} \in \mathbb{Z}_+$ and

$$\mu_{mr} \ge \#\{i : \mu_{ir} > 0,\ m+1 \le i \le r\}, \tag{2.19}$$

then there exists a λ such that $V([\Lambda])$ is isomorphic to the irreducible covariant tensor module V_λ, and the components of λ are given explicitly by

$$\lambda_i = \mu_{ir}, \quad 1 \le i \le m, \tag{2.20}$$
$$\lambda_{m+i} = \#\{j : \mu_{m+j,r} \le i,\ 1 \le j \le n\}, \quad 1 \le i \le n. \tag{2.21}$$

The main feature of irreducible covariant tensor modules of $\mathfrak{gl}(m|n)$ is that their characters are known explicitly [24, 27]. Just as the characters of irreducible covariant tensor modules of $\mathfrak{gl}(m)$, which may be expressed in terms of ordinary Schur functions [28], the characters of $\mathfrak{gl}(m|n)$ can be given in terms of supersymmetric Schur functions. Following Macdonald [25], the Schur function in the variables $(\mathbf{x}) = (x_1, x_2, \ldots, x_m)$ specified by the partition σ is denoted by $s_\sigma(\mathbf{x})$. Schur functions satisfy the following product and quotient rules:

$$s_\sigma(\mathbf{x})s_\tau(\mathbf{x}) = \sum_\lambda c^\lambda_{\sigma\tau} s_\lambda(\mathbf{x}) \tag{2.22}$$
$$s_{\lambda/\tau}(\mathbf{x}) = \sum_\sigma c^\lambda_{\sigma\tau} s_\sigma(\mathbf{x}), \tag{2.23}$$

where the coefficients $c^\lambda_{\sigma\tau}$ are the famous Littlewood-Richardson coefficients, and the summations are over partitions λ and σ with $|\lambda| = |\sigma| + |\tau|$. Berele and Regev [24] proved the following:

Theorem 3 *Let $V([\Lambda^\lambda])$ be an irreducible $\mathfrak{gl}(m|n)$ covariant tensor module specified by a partition λ, and let*

$$x_i = e^{\epsilon_i}, \qquad 1 \le i \le m,$$
$$y_i = e^{\epsilon_{m+i}}, \quad 1 \le i \le n.$$

Then the character of $V([\Lambda^\lambda])$ is given by

$$\operatorname{char} V([\Lambda^\lambda]) = s_\lambda(\mathbf{x}|\mathbf{y}),$$

where $s_\lambda(\mathbf{x}|\mathbf{y})$ is the supersymmetric Schur function of $(\mathbf{x}) = (x_1, x_2, \ldots, x_m)$ and $(\mathbf{y}) = (y_1, y_2, \ldots, y_n)$ defined by

$$s_\lambda(\mathbf{x}|\mathbf{y}) = \sum_\tau s_{\lambda/\tau}(\mathbf{x}) s_{\tau'}(\mathbf{y}) = \sum_{\sigma,\tau} c^\lambda_{\sigma\tau} s_\sigma(\mathbf{x}) s_{\tau'}(\mathbf{y}),$$

with $l(\sigma) \le m$ and $l(\tau') \le n$.

For the Lie algebra $\mathfrak{gl}(m)$, the simplicity of a Gel'fand-Zetlin basis stems from the fact that the decomposition from $\mathfrak{gl}(m)$ to $\mathfrak{gl}(m-1)$ is so easy (and multiplicity free) for covariant tensor modules. Since the characters of these $\mathfrak{gl}(m)$ modules are given by Schur functions $s_\lambda(\mathbf{x})$, this decomposition is deduced from the following formula [25]:

$$s_\lambda(x_1,\ldots,x_{m-1},x_m)=\sum_\sigma s_\sigma(x_1,\ldots,x_{m-1})\cdot x_m^{|\lambda|-|\sigma|}, \tag{2.24}$$

where the sum is over all partitions σ such that

$$\lambda_1\geq\sigma_1\geq\lambda_2\geq\sigma_2\geq\cdots\geq\sigma_{m-1}\geq\lambda_m. \tag{2.25}$$

These last inequalities give rise to the so-called in-betweenness conditions in $\mathfrak{gl}(m)$ GZ-patterns. In terms of the notions introduced in [25], (2.25) means that $\lambda-\sigma$ is a *horizontal strip*.

Various interesting expressions also exist for supersymmetric Schur functions [29]. In particular, there is also a combinatorial expression in terms of supersymmetric tableaux of shape λ. From this expression (or from the one in Theorem 3), one deduces the following result:

$$s_\lambda(x_1,\ldots,x_m|y_1,\ldots,y_{n-1},y_n)=\sum_\sigma s_\sigma(x_1,\ldots,x_m|y_1,\ldots,y_{n-1})\cdot y_n^{|\lambda|-|\sigma|}, \tag{2.26}$$

where the sum is now over all partitions σ in the $(m,n-1)$-hook such that

$$\lambda'_1\geq\sigma'_1\geq\lambda'_2\geq\sigma'_2\geq\cdots\geq\sigma'_{\ell-1}\geq\lambda'_\ell\ , \tag{2.27}$$

where $\ell=\lambda_1$ is the length of λ'. In terms of the notions of [25], $\lambda-\sigma$ is a *vertical strip*. This expression will be particularly useful when decomposing the covariant tensor representation of $\mathfrak{gl}(m|n)$ characterized by λ in terms of $\mathfrak{gl}(m|n-1)$ representations.

3 Covariant tensor representations of $\mathfrak{gl}(m|n)$

Let $V([\mu]^r)$ be an irreducible covariant tensor module of $\mathfrak{gl}(m|n)$, namely the nonnegative integer r-tuple

$$[\mu]^r=[\mu_{1r},\mu_{2r},\ldots,\mu_{rr}], \tag{3.1}$$

is such that

$$\mu_{ir}-\mu_{i+1,r}\in\mathbb{Z}_+,\ \forall i\neq m,\ i=1,\ldots,r-1 \tag{3.2}$$

and

$$\mu_{mr}\geq\#\{i:\mu_{ir}>0,\ m+1\leq i\leq r\}. \tag{3.3}$$

Within a given $\mathfrak{gl}(m|n)$ module $V([\mu]^r)$ the numbers (3.1) are fixed.

For covariant tensor representations of the Lie algebra $\mathfrak{gl}(m)$, the relation between the partition characterizing the highest weight and the highest weight itself is straightforward. Moreover, the decomposition from $\mathfrak{gl}(m)$ to $\mathfrak{gl}(m-1)$ for such representations is very easy, following (2.24). That is why the GZ-basis vectors for $\mathfrak{gl}(m)$ have such a simple pattern.

For covariant tensor representations of the Lie superalgebra $\mathfrak{gl}(m|n)$ the situation is different. First of all, the relation between the partition λ characterizing the highest weight and the components of the highest weight is more involved, see (2.17)-(2.18). Therefore the conditions on the highest weight components, (3.2)-(3.3) are more complicated. Still, it is necessary to use highest weight components in the labeling of basis vectors, in order to describe the action of generators appropriately. Secondly, the decomposition from $\mathfrak{gl}(m|n)$ to $\mathfrak{gl}(m|n-1)$ for covariant tensor representations is fairly easy to describe using the partition labeling, according to (2.26). However, we need to translate this to the corresponding highest weights. This gives rise to the following propositions.

Proposition 4 *Consider the $\mathfrak{gl}(m|n)$ module $V([\mu]^r)$ as a $\mathfrak{gl}(m|n-1)$ module. Then $V([\mu]^r)$ can be represented as a direct sum of covariant simple $\mathfrak{gl}(m|n-1)$ modules,*

$$V([\mu]^r) = \sum_i \oplus V_i([\mu]^{r-1}), \tag{3.4}$$

where

I. All $V_i([\mu]^{r-1})$ carry inequivalent representations of $\mathfrak{gl}(m|n-1)$

$$[\mu]^{r-1} = [\mu_{1,r-1}, \mu_{2,r-1}, \ldots, \mu_{r-1,r-1}], \tag{3.5}$$

$$\mu_{i,r-1} - \mu_{i+1,r-1} \in \mathbb{Z}_+, \ \forall i \neq m, \ i = 1, \ldots, r-2, \tag{3.6}$$

$$\mu_{m,r-1} \geq \#\{i : \mu_{i,r-1} > 0, \ m+1 \leq i \leq r-1\}. \tag{3.7}$$

II.

$$\begin{array}{ll} 1. & \mu_{ir} - \mu_{i,r-1} = \theta_{i,r-1} \in \{0,1\}, \quad 1 \leq i \leq m, \\ 2. & \mu_{i,r} - \mu_{i,r-1} \in \mathbb{Z}_+ \ \textit{and} \ \mu_{i,r-1} - \mu_{i+1,r} \in \mathbb{Z}_+, \quad m+1 \leq i \leq r-1. \end{array} \tag{3.8}$$

Proposition 5 *Consider a covariant $\mathfrak{gl}(m|1)$ module $V([\mu]^{m+1})$ as a $\mathfrak{gl}(m)$ module. Then $V([\mu]^{m+1})$ can be represented as a direct sum of simple $\mathfrak{gl}(m)$ modules,*

$$V([\mu]^{m+1}) = \sum_i \oplus V_i([\mu]^m), \tag{3.9}$$

where

I. All $V_i([\mu]^m)$ carry inequivalent representations of $\mathfrak{gl}(m)$

$$[\mu]^m = [\mu_{1m}, \mu_{2m}, \ldots, \mu_{mm}], \ \mu_{im} - \mu_{i+1,m} \in \mathbb{Z}_+. \tag{3.10}$$

II.

$$\begin{array}{ll} 1. & \mu_{i,m+1} - \mu_{im} = \theta_{im} \in \{0,1\}, \quad 1 \leq i \leq m, \\ 2. & \textit{if} \ \mu_{m,m+1} = 0, \ \textit{then} \ \theta_{mm} = 0. \end{array} \tag{3.11}$$

Using Proposition 1, Proposition 2 and the $\mathfrak{gl}(m)$ GZ-basis we have:

Proposition 6 *The set of vectors*

$$|\mu) \equiv |\mu)^r = \left| \begin{array}{llllllll} \mu_{1r} & \cdots & \mu_{m-1,r} & \mu_{mr} & \mu_{m+1,r} & \cdots & \mu_{r-1,r} & \mu_{rr} \\ \mu_{1,r-1} & \cdots & \mu_{m-1,r-1} & \mu_{m,r-1} & \mu_{m+1,r-1} & \cdots & \mu_{r-1,r-1} & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \iddots & & \\ \mu_{1,m+1} & \cdots & \mu_{m-1,m+1} & \mu_{m,m+1} & \mu_{m+1,m+1} & & & \\ \mu_{1m} & \cdots & \mu_{m-1,m} & \mu_{mm} & & & & \\ \mu_{1,m-1} & \cdots & \mu_{m-1,m-1} & & & & & \\ \vdots & \iddots & & & & & & \\ \mu_{11} & & & & & & & \end{array} \right) = \left| \begin{array}{l} [\mu]^r \\ |\mu)^{r-1} \end{array} \right) \tag{3.12}$$

satisfying the conditions

$$\begin{array}{ll}
1. & \mu_{ir} \in \mathbb{Z}_+ \text{ are fixed and } \mu_{jr} - \mu_{j+1,r} \in \mathbb{Z}_+,\ j \neq m,\ 1 \leq j \leq r-1, \\
 & \mu_{mr} \geq \#\{i : \mu_{ir} > 0,\ m+1 \leq i \leq r\}; \\
2. & \mu_{ip} - \mu_{i,p-1} \equiv \theta_{i,p-1} \in \{0,1\}, \quad 1 \leq i \leq m;\ m+1 \leq p \leq r; \\
3. & \mu_{mp} \geq \#\{i : \mu_{ip} > 0,\ m+1 \leq i \leq p\}, \quad m+1 \leq p \leq r; \\
4. & \text{if } \mu_{m,m+1} = 0, \text{then } \theta_{mm} = 0; \\
5. & \mu_{ip} - \mu_{i+1,p} \in \mathbb{Z}_+, \quad 1 \leq i \leq m-1;\ m+1 \leq p \leq r-1; \\
6. & \mu_{i,j+1} - \mu_{ij} \in \mathbb{Z}_+ \text{ and } \mu_{i,j} - \mu_{i+1,j+1} \in \mathbb{Z}_+, \\
 & 1 \leq i \leq j \leq m-1 \text{ or } m+1 \leq i \leq j \leq r-1.
\end{array} \tag{3.13}$$

constitute a basis in $V([\mu]^r)$.

The last condition corresponds to the in-betweenness condition and ensures that the triangular pattern to the right of the $n \times m$ rectangle μ_{ip} $(1 \leq i \leq m;\ m+1 \leq p \leq r)$ in (3.12) corresponds to a classical GZ-pattern for $\mathfrak{gl}(n)$, and that the triangular pattern below this rectangle corresponds to a GZ-pattern for $\mathfrak{gl}(m)$.

We shall refer to the basis (3.12) as the GZ-basis for the covariant $\mathfrak{gl}(m|n)$ representations. The task is now to give the explicit action of the $\mathfrak{gl}(m|n)$ Chevalley generators on the basis vectors (3.12). Let $|\mu)_{\pm ij}$ be the pattern obtained from $|\mu)$ by replacing the entry μ_{ij} by $\mu_{ij}\pm 1$, and for the notations l_{ij} see formula (2.16).

The following is one of the two main results of this paper:

Theorem 7 *The transformation of the irreducible covariant tensor module* $V([\mu]^r)$ *under the action of the* $\mathfrak{gl}(m|n)$ *generators is given by:*

$$h_k|\mu) = \left(\sum_{j=1}^{k} \mu_{jk} - \sum_{j=1}^{k-1} \mu_{j,k-1}\right)|\mu), \quad 1 \leq k \leq r; \tag{3.14}$$

$$e_k|\mu) = \sum_{j=1}^{k} \left(- \frac{\prod_{i=1}^{k+1}(l_{i,k+1} - l_{jk}) \prod_{i=1}^{k-1}(l_{i,k-1} - l_{jk} - 1)}{\prod_{i\neq j=1}^{k}(l_{ik} - l_{jk})(l_{ik} - l_{jk} - 1)} \right)^{1/2} |\mu)_{jk}, \quad 1 \leq k \leq m-1; \tag{3.15}$$

$$f_k|\mu) = \sum_{j=1}^{k} \left(- \frac{\prod_{i=1}^{k+1}(l_{i,k+1} - l_{jk} + 1) \prod_{i=1}^{k-1}(l_{i,k-1} - l_{jk})}{\prod_{i\neq j=1}^{k}(l_{ik} - l_{jk} + 1)(l_{ik} - l_{jk})} \right)^{1/2} |\mu)_{-jk}, \quad 1 \leq k \leq m-1; \tag{3.16}$$

$$e_m|\mu) = \sum_{i=1}^{m} \theta_{im}(-1)^{i-1}(-1)^{\theta_{1m}+\ldots+\theta_{i-1,m}}(l_{i,m+1} - l_{m+1,m+1})^{1/2} \times \left(\frac{\prod_{k=1}^{m-1}(l_{k,m-1} - l_{i,m+1})}{\prod_{k\neq i=1}^{m}(l_{k,m+1} - l_{i,m+1})} \right)^{1/2} |\mu)_{im}; \tag{3.17}$$

$$f_m|\mu) = \sum_{i=1}^{m}(1-\theta_{im})(-1)^{i-1}(-1)^{\theta_{1m}+\ldots+\theta_{i-1,m}}(l_{i,m+1}-l_{m+1,m+1})^{1/2}$$
$$\times \left(\frac{\prod_{k=1}^{m-1}(l_{k,m-1}-l_{i,m+1})}{\prod_{k\neq i=1}^{m}(l_{k,m+1}-l_{i,m+1})}\right)^{1/2} |\mu)_{-im}; \tag{3.18}$$
$$e_p|\mu) = \sum_{i=1}^{m}\theta_{ip}(-1)^{\theta_{1p}+\ldots+\theta_{i-1,p}+\theta_{i+1,p-1}+\ldots+\theta_{m,p-1}}(1-\theta_{i,p-1})$$
$$\times \prod_{k\neq i=1}^{m}\left(\frac{(l_{i,p+1}-l_{kp})(l_{i,p+1}-l_{kp}-1)}{(l_{i,p+1}-l_{k,p+1})(l_{i,p+1}-l_{k,p-1}-1)}\right)^{1/2}$$
$$\times \left(\frac{\prod_{q=m+1}^{p-1}(l_{i,p+1}-l_{q,p-1}-1)\prod_{q=m+1}^{p+1}(l_{i,p+1}-l_{q,p+1})}{\prod_{q=m+1}^{p}(l_{i,p+1}-l_{qp}-1)(l_{i,p+1}-l_{qp})}\right)^{1/2} |\mu)_{ip}$$
$$+ \sum_{s=m+1}^{p}\left(-\frac{\prod_{q=m+1}^{p-1}(l_{q,p-1}-l_{sp}+1)\prod_{q=m+1}^{p+1}(l_{q,p+1}-l_{sp})}{\prod_{q\neq s=m+1}^{p}(l_{qp}-l_{sp})(l_{qp}-l_{sp}+1)}\right)^{1/2} \tag{3.19}$$
$$\times \prod_{k=1}^{m}\left(\frac{(l_{kp}-l_{sp})(l_{kp}-l_{sp}+1)}{(l_{k,p+1}-l_{sp})(l_{k,p-1}-l_{sp}+1)}\right)^{1/2} |\mu)_{sp}, \quad m+1\leq p\leq r-1;$$
$$f_p|\mu) = \sum_{i=1}^{m}\theta_{i,p-1}(-1)^{\theta_{1p}+\ldots+\theta_{i-1,p}+\theta_{i+1,p-1}+\ldots+\theta_{m,p-1}}(1-\theta_{ip})$$
$$\times \prod_{k\neq i=1}^{m}\left(\frac{(l_{i,p+1}-l_{kp})(l_{i,p+1}-l_{kp}-1)}{(l_{i,p+1}-l_{k,p+1})(l_{i,p+1}-l_{k,p-1}-1)}\right)^{1/2}$$
$$\times \left(\frac{\prod_{q=m+1}^{p-1}(l_{i,p+1}-l_{q,p-1}-1)\prod_{q=m+1}^{p+1}(l_{i,p+1}-l_{q,p+1})}{\prod_{q=m+1}^{p}(l_{i,p+1}-l_{qp}-1)(l_{i,p+1}-l_{qp})}\right)^{1/2} |\mu)_{-ip}$$
$$+ \sum_{s=m+1}^{p}\left(-\frac{\prod_{q=m+1}^{p-1}(l_{q,p-1}-l_{sp})\prod_{q=m+1}^{p+1}(l_{q,p+1}-l_{sp}-1)}{\prod_{q\neq s=m+1}^{p}(l_{qp}-l_{sp}-1)(l_{qp}-l_{sp})}\right)^{1/2} \tag{3.20}$$
$$\times \prod_{k=1}^{m}\left(\frac{(l_{kp}-l_{sp}-1)(l_{kp}-l_{sp})}{(l_{k,p+1}-l_{sp}-1)(l_{k,p-1}-l_{sp})}\right)^{1/2} |\mu)_{-sp}, \quad m+1\leq p\leq r-1;$$

In the above expressions, $\sum_{k\neq i=1}^{m}$ or $\prod_{k\neq i=1}^{m}$ means that k takes all values from 1 to m with $k\neq i$. If a vector from the right hand side of (3.14)-(3.20) does not belong to the module under consideration, then the corresponding term is zero even if the coefficient in front is undefined; if an equal number of factors in numerator and denominator are simultaneously equal to zero, they should be canceled out.

To conclude this section, we shall make some comments on the proof of this theorem. In order to prove that the explicit actions (3.14)-(3.20) give a representation of $\mathfrak{gl}(m|n)$ it is sufficient to show that (3.14)-(3.20) satisfy the relations (2.3)-(2.12) (plus the f-Serre relations). The irreducibility then follows from the fact that for any nonzero vector $x\in V([\mu]^r)$ there exists a polynomial $\mathcal{P}$ of $\mathfrak{gl}(m|n)$ generators such that $\mathcal{P}x = V([\mu]^r)$.

To show that (2.3)-(2.6) are satisfied is straightforward. The difficult Cartan-Kac relations to be verified are (2.7) and (2.8). For instance relation (2.8), with the actions (3.14)-(3.20), is valid if and only if

$$\sum_{i=1}^{m}(l_{i,m+1}-l_{m+1,m+1})\frac{\prod_{k=1}^{m-1}(l_{k,m-1}-l_{i,m+1})}{\prod_{k\neq i=1}^{m}(l_{k,m+1}-l_{i,m+1})}=\sum_{k=1}^{m-1}(l_{k,m+1}-l_{k,m-1})+l_{m,m+1}-l_{m+1,m+1}. \quad (3.21)$$

The proof of this relation is given in [30]. For the e- and f-Serre relations, the calculations are lengthy, but collecting terms with the same Gel'fand-Zetlin basis vector and taking apart the common factors, the remaining factor always reduces to a very simple algebraic expression like:

$$a(b+1)-(a+1)b=a-b, \quad \frac{1}{a(a-1)}+\frac{1}{a(a+1)}=\frac{2}{(a-1)(a+1)}, \quad (3.22)$$

from which the validity follows.

4 Clebsch-Gordan coefficients of $\mathfrak{gl}(m|n)$

In this section we compute the Clebsch-Gordan coefficients of $\mathfrak{gl}(m|n)$ corresponding to the tensor product $V([\mu]^r)\otimes V([1,0,\ldots,0])$ of any irreducible $\mathfrak{gl}(m|n)$ covariant tensor module $V([\mu]^r)$ with the natural $(m+n)$-dimensional $\mathfrak{gl}(m|n)$ representation $V([1,0,\ldots,0])$. It is well known and it is easy to see from the character formula that:

$$V([\mu]^r)\otimes V([1,0,\ldots,0])=\sum_{k=1}^{r}\oplus V([\mu]^r_{+k}), \quad (4.1)$$

where $[\mu]^r_{+k}$ is obtained from $[\mu]^r$ by the replacement of μ_{kr} by $\mu_{kr}+1$ and on the right hand side of (4.1) the summands for which the conditions (3.2)-(3.3) are not fulfilled are omitted. We choose two orthonormal bases in the space (4.1):

$$\left|\begin{array}{l}[\mu]^r\\ |\mu)^{r-1}\end{array}\right)\otimes|1_j)\in V([\mu]^r)\otimes V([1,0,\ldots,0]) \quad \text{and} \quad (4.2)$$

$$\left|\begin{array}{l}[\mu]^r_{+k}\\ |\mu')^{r-1}\end{array}\right)\in V([\mu]^r_{+k}),\quad k=1,\ldots,r, \quad (4.3)$$

where the vectors $\left|\begin{array}{l}[\mu]^r\\ |\mu)^{r-1}\end{array}\right)$ and $\left|\begin{array}{l}[\mu]^r_{+k}\\ |\mu')^{r-1}\end{array}\right)$ satisfy the conditions of Proposition 6, and $|1_j),\ j=1,\ldots,r$ is a pattern which consists of $r-j$ zero rows at the bottom (denoted by $0\cdots 0=\dot{0}$), and the first j rows are of the form $10\cdots 0$ (denoted by $1\dot{0}$). Then in general

$$\left|\begin{array}{l}[\mu]^r_{+k}\\ |\mu')^{r-1}\end{array}\right)=\sum_{|\mu)^r,|1_j)}\left(\begin{array}{l}[\mu]^r\\ |\mu)^{r-1}\end{array};\begin{array}{l}10\cdots 00\\ 10\cdots 0\\ \cdots\\ 0\end{array}\left|\begin{array}{l}[\mu]^r_{+k}\\ |\mu')^{r-1}\end{array}\right)\right.\left|\begin{array}{l}[\mu]^r\\ |\mu)^{r-1}\end{array}\right)\otimes|1_j), \quad (4.4)$$

where $\left(\begin{array}{l}[\mu]^r\\ |\mu)^{r-1}\end{array};\begin{array}{l}10\cdots 00\\ 10\cdots 0\\ \cdots\\ 0\end{array}\left|\begin{array}{l}[\mu]^r_{+k}\\ |\mu')^{r-1}\end{array}\right)\right.\equiv\left(\begin{array}{l}[\mu]^r\\ |\mu)^{r-1}\end{array};|1_j)\left|\begin{array}{l}[\mu]^r_{+k}\\ |\mu')^{r-1}\end{array}\right)\right.$ are the Clebsch-Gordan coefficients (CGCs). Acting onto both sides of relation (4.4) by the Cartan generators $h_i,\ i=$

$1, \ldots, r$ and taking into account formula (3.14) it follows that the CGC of $\mathfrak{gl}(m|n)$ vanishes if one of the relations

$$\sum_{s=1}^{p} \mu'_{sp} = \sum_{s=1}^{p} \mu_{sp}, \quad p = 1, \ldots, r-j, \tag{4.5}$$

$$\sum_{s=1}^{p} \mu'_{sp} = \sum_{s=1}^{p} \mu_{sp} + 1, \quad p = r+1-j, \ldots, r-1 \tag{4.6}$$

is not fulfilled.

Since multiple representations are absent in (4.1) we have for the CGCs

$$\left(\begin{array}{c} [\mu]^r \\ \\ |\mu)^{r-1} \end{array} ; \begin{array}{c} 10\cdots 00 \\ 10\cdots 0 \\ \cdots \\ 0 \end{array} \left| \begin{array}{c} [\mu]^r_{+k} \\ \\ |\mu')^{r-1} \end{array} \right. \right)$$

$$= \left(\begin{array}{c} [\mu]^r \\ [\mu]^{r-1} \end{array} \left| \begin{array}{c} 1\dot{0} \\ \epsilon\dot{0} \end{array} \right| \begin{array}{c} [\mu]^r_{+k} \\ [\mu']^{r-1} \end{array} \right) \times \left(\begin{array}{c} [\mu]^{r-1} \\ \\ |\mu)^{r-2} \end{array} ; \begin{array}{c} 10\cdots 00 \\ 10\cdots 0 \\ \cdots \\ 0 \end{array} \left| \begin{array}{c} [\mu']^{r-1} \\ \\ |\mu')^{r-2} \end{array} \right. \right). \tag{4.7}$$

In the right hand side, the first factor is an isoscalar factor [31], and the second factor is a CGC of $\mathfrak{gl}(m|n-1)$. The middle pattern in the $\mathfrak{gl}(m|n-1)$ CGC is that of the $\mathfrak{gl}(m|n)$ CGC with the first row deleted. The middle pattern in the isoscalar factor consists of the first two rows of the middle pattern in the left hand side, so ϵ is 0 or 1. If $\epsilon = 0$, then $[\mu']^{r-1} = [\mu]^{r-1}$. If $\epsilon = 1$ then $[\mu']^{r-1} = [\mu_{1,r-1}, \ldots, \mu_{s,r-1} + 1, \ldots, \mu_{r-1,r-1}] = [\mu]^{r-1}_{+s}$ for some s-value.

In addition to all this we followed the general procedure for computing Clebsch-Gordan coefficients. First, the highest weight vector of the irreducible module $V([\mu]^r_{+1})$ is equal to the tensor product of the two highest weight vectors of the components of the tensor product $V([\mu]^r) \otimes V([1, 0, \ldots, 0])$. Then any other vector in the same irreducible module $V([\mu]^r_{+1})$ is obtained by acting with polynomials of negative root vectors on this vector. The highest weight vector in the irreducible module $V([\mu]^r_{+2})$ is (up to a phase) fixed by the requirement that it is orthogonal to the unique vector in $V([\mu]^r_{+1})$ with the same weight, namely $[\mu_{1r}, \mu_{2r}+1, \mu_{3r}, \mu_{4r}, \ldots, \mu_{rr}]$ as of the highest weight vector in this second space $V([\mu]^r_{+2})$. Then again, all vectors in the irreducible module $V([\mu]^r_{+2})$ are found by the actions of polynomials of negative root vectors of the algebra to the corresponding highest weight vector of $V([\mu]^r_{+2})$. Next the highest weight vector of $V([\mu]^r_{+3})$ has to be orthogonal to all vectors in $V([\mu]^r_{+1})$ and $V([\mu]^r_{+2})$ with weight $(\mu_{1r}, \mu_{2r}, \mu_{3r}+1, \mu_{4r}, \ldots, \mu_{rr})$ (the highest weight of $V([\mu]^r_{+3})$). Note that following this general procedure for computing CGCs one should have in mind two other important facts from representation theory of Lie superalgebras. First: Lie superalgebra representation spaces are $\mathbb{Z}_2$-graded spaces and for the considered irreducible $\mathfrak{gl}(m|n)$ covariant tensor modules $V([\mu]^r) = V_{\bar{0}}([\mu]^r) \oplus V_{\bar{1}}([\mu]^r)$ there are two possibilities for the $\mathbb{Z}_2$-grading, namely $|\mu) \in V_{\bar{0}}([\mu]^r)$ (resp. $|\mu) \in V_{\bar{1}}([\mu]^r)$) if $\sum_{i=1}^{m} \sum_{p=m+1}^{r} \theta_{i,p-1} = \sum_{i=1}^{m} \sum_{p=m+1}^{r} (\mu_{i,p} - \mu_{i,p-1})$ is even (resp. odd). The first grading will be referred to as the natural grading, and the other one as the opposite grading. Second: The action of a Lie superalgebra generator g in a tensor product of two $\mathfrak{g}$-modules V and W is given by

$$g(x \otimes y) = gx \otimes y + (-1)^{\deg(g)\deg(x)} x \otimes gy, \quad x \in V, \quad y \in W,$$

and in the computations only the grading of the first module V plays role. Because of this we fix that in (4.1) $V([1, 0, \ldots, 0])$ has the natural grading. As a consequence, the vectors $|1_j)$ of $V([1, 0, \ldots, 0])$ have the following degree:

$$\deg|1_j) = \bar{1} \text{ if } 1 \le j \le n, \qquad \deg|1_j) = \bar{0} \text{ if } n+1 \le j \le n+m. \tag{4.8}$$

The degree of the vectors is important since in general the vector $|\mu)^{r-1}$ from the $\mathfrak{gl}(m|n-1)$ module does not necessarily have the same grading as the vector $|\mu)^{r}$ from the $\mathfrak{gl}(m|n)$ module.

Iterating the procedure for computing Clebsch-Gordan coefficients and the corresponding isoscalar factors, it is clear that there are two distinct cases. First, when $1 \leq j \leq n$, one will finally reach a trivial CGC of $\mathfrak{gl}(m|n-j)$ which is equal to 1 because the middle pattern consist of zeros only; in this case the computed $\mathfrak{gl}(m|n)$ CGC is a product of isoscalar factors only. Secondly, when $n+1 \leq j \leq n+m$, the iteration leads to a product of isoscalar factors times a CGC of $\mathfrak{gl}(m)$. For these simple $\mathfrak{gl}(m)$ CGCs, there exist closed form expressions, see e.g. [18, 31]. Thus we reach to the following:

Theorem 8 *The Clebsch-Gordan coefficients corresponding to the tensor product*

$$V([\mu]^r) \otimes V([1,0,\ldots,0])$$

of an irreducible $\mathfrak{gl}(m|n)$ covariant tensor module $V([\mu]^r)$ with the natural $(m+n)$-dimensional $\mathfrak{gl}(m|n)$ representation $V([1,0,\ldots,0])$ are

- *products of isoscalar factors (for $j = 1,\ldots,n$)*

$$\left(\begin{array}{c}[\mu]^r\\ \\ |\mu)^{r-1}\end{array};\ |1_j)\ \left|\ \begin{array}{c}[\mu]^r_{+k}\\ \\ |\mu')^{r-1}\end{array}\right.\right) = \xi(-1)^{\sum_{q=1}^{j}\sum_{i=1}^{m}\theta_{i,r-q}}\left(\begin{array}{c}[\mu]^r\\ [\mu]^{r-1}\end{array}\left|\begin{array}{c}\dot{10}\\ \dot{10}\end{array}\right|\begin{array}{c}[\mu]^r_{+k}\\ [\mu']^{r-1}\end{array}\right)\times\cdots$$
$$\times\left(\begin{array}{c}[\mu]^{r+2-j}\\ [\mu]^{r+1-j}\end{array}\left|\begin{array}{c}\dot{10}\\ \dot{10}\end{array}\right|\begin{array}{c}[\mu']^{r+2-j}\\ [\mu']^{r+1-j}\end{array}\right)\left(\begin{array}{c}[\mu]^{r+1-j}\\ [\mu]^{r-j}\end{array}\left|\begin{array}{c}\dot{10}\\ \dot{00}\end{array}\right|\begin{array}{c}[\mu']^{r+1-j}\\ [\mu']^{r-j}\end{array}\right)\times 1, \tag{4.9}$$

  where $\xi = (-1)^{\deg([\mu]^r)}$, $\deg([\mu]^r)$ being the degree of the highest weight vector of $V([\mu]^r)$;

- *products of isoscalar factors and a $\mathfrak{gl}(m)$ CGC [18, 31] (for $j = n+1,\ldots,r$)*

$$\left(\begin{array}{c}[\mu]^r\\ \\ |\mu)^{r-1}\end{array};\ |1_j)\ \left|\ \begin{array}{c}[\mu]^r_{+k}\\ \\ |\mu')^{r-1}\end{array}\right.\right) = \left(\begin{array}{c}[\mu]^r\\ [\mu]^{r-1}\end{array}\left|\begin{array}{c}\dot{10}\\ \dot{10}\end{array}\right|\begin{array}{c}[\mu]^r_{+k}\\ [\mu']^{r-1}\end{array}\right)\times\cdots$$
$$\times\left(\begin{array}{c}[\mu]^{m+1}\\ [\mu]^{m}\end{array}\left|\begin{array}{c}\dot{10}\\ \dot{10}\end{array}\right|\begin{array}{c}[\mu']^{m+1}\\ [\mu']^{m}\end{array}\right)\left(\begin{array}{c}[\mu]^m\\ \\ |\mu)^{m-1}\end{array};\ |1_{j-n})\ \left|\ \begin{array}{c}[\mu']^m\\ \\ |\mu')^{m-1}\end{array}\right.\right); \tag{4.10}$$

and the isoscalar factors are given by:

$$\left(\begin{array}{c}[\mu]^r\\ [\mu]^{r-1}\end{array}\left|\begin{array}{c}\dot{10}\\ \dot{00}\end{array}\right|\begin{array}{c}[\mu]^r_{+k}\\ [\mu]^{r-1}\end{array}\right)$$
$$= (-1)^{k-1}(-1)^{\sum_{i=k}^{m}\theta_{i,r-1}}\left(\prod_{i\neq k=1}^{m}\left(\frac{l_{kr}-l_{ir}+1}{l_{kr}-l_{i,r-1}}\right)\frac{\prod_{p=m+1}^{r-1}(l_{kr}-l_{p,r-1})}{\prod_{p=m+1}^{r}(l_{kr}-l_{pr}+1)}\right)^{1/2}\quad 1\leq k\leq m; \tag{4.11}$$

$$\left(\begin{array}{c} [\mu]^r \\ [\mu]^{r-1} \end{array} \middle| \begin{array}{c} 1\dot{0} \\ 0\dot{0} \end{array} \middle| \begin{array}{c} [\mu]^r_{+k} \\ [\mu]^{r-1} \end{array} \right) = \left(\prod_{i=1}^{m} \left(\frac{l_{ir} - l_{kr}}{l_{i,r-1} - l_{kr} + 1} \right) \frac{\prod_{p=m+1}^{r-1} (l_{p,r-1} - l_{kr} + 1)}{\prod_{p \neq k = m+1}^{r} (l_{pr} - l_{kr})} \right)^{1/2}$$

$$m+1 \leq k \leq r; \tag{4.12}$$

$$\left(\begin{array}{c} [\mu]^r \\ [\mu]^{r-1} \end{array} \middle| \begin{array}{c} 1\dot{0} \\ 1\dot{0} \end{array} \middle| \begin{array}{c} [\mu]^r_{+k} \\ [\mu]^{r-1}_{+q} \end{array} \right) = (-1)^{k+q} (-1)^{\sum_{i=\min(k+1,q+1)}^{\max(k-1,q-1)} \theta_{i,r-1}} S(k,q)$$

$$\times \left(\prod_{i \neq k,q=1}^{m} \frac{(l_{i,r-1} - l_{k,r-1} - 1 - \delta_{kq} + 2\theta_{i,r-1})(l_{i,r-1} - l_{q,r-1})}{(l_{ir} - l_{kr})(l_{ir} - l_{qr})} \right)^{\frac{\theta_{q,r-1}}{2}}$$

$$\times \frac{1}{(l_{kr} - l_{qr})^{1-\delta_{kq}}} \left(\prod_{p=m+1}^{r} \left(\frac{l_{qr} - l_{pr}}{l_{kr} - l_{pr} + 1} \right) \prod_{p=m+1}^{r-1} \left(\frac{l_{kr} - l_{q,r-1}}{l_{q,r-1} - l_{p,r-1}} \right) \right)^{\frac{\theta_{q,r-1}}{2}} \quad 1 \leq k, q \leq m; \tag{4.13}$$

$$\left(\begin{array}{c} [\mu]^r \\ [\mu]^{r-1} \end{array} \middle| \begin{array}{c} 1\dot{0} \\ 1\dot{0} \end{array} \middle| \begin{array}{c} [\mu]^r_{+k} \\ [\mu]^{r-1}_{+q} \end{array} \right) = (-1)^{k} (-1)^{\sum_{i=1}^{k-1} \theta_{i,r-1}} \left(\frac{1}{l_{kr} - l_{q,r-1}} \right)^{1/2}$$

$$\times \left(\prod_{i \neq k=1}^{m} \left(\frac{(l_{i,r-1} - l_{k,r-1} - 1 + 2\theta_{i,r-1})(l_{i,r-1} - l_{q,r-1} + 1)}{(l_{ir} - l_{kr})(l_{ir} - l_{q,r-1})} \right) \right)^{1/2}$$

$$\times \left(\prod_{p=m+1}^{r} \left(\frac{|l_{pr} - l_{q,r-1}|}{(l_{kr} - l_{pr} + 1)} \right) \prod_{p \neq q = m+1}^{r-1} \left(\frac{l_{kr} - l_{p,r-1}}{|l_{p,r-1} - l_{q,r-1} + 1|} \right) \right)^{1/2}$$

$$1 \leq k \leq m, \quad m+1 \leq q \leq r-1; \tag{4.14}$$

$$\left(\begin{array}{c} [\mu]^r \\ [\mu]^{r-1} \end{array} \middle| \begin{array}{c} 1\dot{0} \\ 1\dot{0} \end{array} \middle| \begin{array}{c} [\mu]^r_{+k} \\ [\mu]^{r-1}_{+q} \end{array} \right) = (-1)^{q} (-1)^{\sum_{i=q+1}^{m} \theta_{i,r-1}} \left(\frac{1}{l_{qr} - l_{kr} + 1} \right)^{1/2}$$

$$\times \left(\prod_{i=1}^{m} \left(\frac{l_{ir} - l_{kr}}{l_{i,r-1} - l_{kr} + 1} \right) \prod_{i \neq q=1}^{m} \left| \frac{l_{q,r-1} - l_{i,r-1}}{l_{qr} - l_{ir}} \right| \prod_{p \neq k=m+1}^{r} \left| \frac{l_{qr} - l_{pr}}{l_{pr} - l_{kr}} \right| \prod_{p=m+1}^{r-1} \left| \frac{l_{p,r-1} - l_{kr} + 1}{l_{qr} - l_{p,r-1} - 1} \right| \right)^{1/2}$$

$$m+1 \leq k \leq r, \quad 1 \leq q \leq m; \tag{4.15}$$

$$\left(\begin{array}{c} [\mu]^r \\ [\mu]^{r-1} \end{array} \middle| \begin{array}{c} 1\dot{0} \\ 1\dot{0} \end{array} \middle| \begin{array}{c} [\mu]^r_{+k} \\ [\mu]^{r-1}_{+q} \end{array} \right) = S(k,q) \left(\prod_{i=1}^{m} \left(\frac{(l_{ir} - l_{kr})(l_{i,r-1} - l_{q,r-1} + 1)}{(l_{i,r-1} - l_{kr} + 1)(l_{ir} - l_{q,r-1})} \right) \right)^{1/2}$$

$$\times \left(\prod_{p \neq k=m+1}^{r} \left| \frac{l_{pr} - l_{q,r-1}}{l_{pr} - l_{kr}} \right| \prod_{p \neq q=m+1}^{r-1} \left| \frac{l_{p,r-1} - l_{kr} + 1}{l_{p,r-1} - l_{q,r-1} + 1} \right| \right)^{1/2} \quad m+1 \leq k \leq r, \quad m+1 \leq q \leq r-1; \tag{4.16}$$

$$S(k,q) = \left\{ \begin{array}{rl} 1 & for \quad k \leq q \\ -1 & for \quad k > q. \end{array} \right. \tag{4.17}$$

The expressions in this Theorem look fairly complicated, however they are easy to use in practice. Let us give an example, and apply Theorem 8 to the Lie superalgebra $\mathfrak{gl}(2|3)$, both for the case $j \leq n$ and $j > n$.

$$
\begin{aligned}
&\left(\begin{array}{lllll} \mu_{15} & \mu_{25} & \mu_{35} & \mu_{45} & \mu_{55} \\ \mu_{15}-1 & \mu_{25}-1 & \mu_{34} & \mu_{44} & \\ \mu_{15}-2 & \mu_{25}-1 & \mu_{33} & & \\ \mu_{15}-3 & \mu_{25}-1 & & & \\ \mu_{11} & & & & \end{array} ; \begin{array}{l} 10000 \\ 1000 \\ 000 \\ 00 \\ 0 \end{array} \right| \left. \begin{array}{lllll} \mu_{15} & \mu_{25}+1 & \mu_{35} & \mu_{45} & \mu_{55} \\ \mu_{15}-1 & \mu_{25} & \mu_{34} & \mu_{44} & \\ \mu_{15}-2 & \mu_{25}-1 & \mu_{33} & & \\ \mu_{15}-3 & \mu_{25}-1 & & & \\ \mu_{11} & & & & \end{array} \right) \\
&= \xi \left(\begin{array}{lllll} \mu_{15} & \mu_{25} & \mu_{35} & \mu_{45} & \mu_{55} \\ \mu_{15}-1 & \mu_{25}-1 & \mu_{34} & \mu_{44} & \end{array} \middle| \begin{array}{l} 1\dot{0} \\ 1\dot{0} \end{array} \middle| \begin{array}{lllll} \mu_{15} & \mu_{25}+1 & \mu_{35} & \mu_{45} & \mu_{55} \\ \mu_{15}-1 & \mu_{25} & \mu_{34} & \mu_{44} & \end{array} \right) \\
&\times (-1)^{\theta_{14}+\theta_{24}} \left(\begin{array}{llll} \mu_{15}-1 & \mu_{25}-1 & \mu_{34} & \mu_{44} \\ \mu_{15}-2 & \mu_{25}-1 & \mu_{33} & \end{array} \middle| \begin{array}{l} 1\dot{0} \\ \dot{0} \end{array} \middle| \begin{array}{llll} \mu_{15}-1 & \mu_{25} & \mu_{34} & \mu_{44} \\ \mu_{15}-2 & \mu_{25}-1 & \mu_{33} & \end{array} \right) \times 1 \\
&= -\xi \sqrt{\frac{(\mu_{25}+\mu_{35})(\mu_{25}+\mu_{45}-1)(\mu_{25}+\mu_{55}-2)(\mu_{25}+\mu_{33}-1)}{(\mu_{25}+\mu_{35}+1)(\mu_{25}+\mu_{45})(\mu_{25}+\mu_{55}-1)(\mu_{25}+\mu_{34}-1)(\mu_{25}+\mu_{44}-2)}}.
\end{aligned} \tag{4.18}
$$

$$
\begin{aligned}
&\left(\begin{array}{lllll} \mu_{15} & \mu_{25} & \mu_{35} & \mu_{45} & \mu_{55} \\ \mu_{15}-1 & \mu_{25}-1 & \mu_{34} & \mu_{44} & \\ \mu_{15}-2 & \mu_{25}-1 & \mu_{33} & & \\ \mu_{15}-3 & \mu_{25}-1 & & & \\ \mu_{11} & & & & \end{array} ; \begin{array}{l} 10000 \\ 1000 \\ 100 \\ 10 \\ 0 \end{array} \right| \left. \begin{array}{lllll} \mu_{15} & \mu_{25}+1 & \mu_{35} & \mu_{45} & \mu_{55} \\ \mu_{15}-1 & \mu_{25} & \mu_{34} & \mu_{44} & \\ \mu_{15}-2 & \mu_{25} & \mu_{33} & & \\ \mu_{15}-3 & \mu_{25} & & & \\ \mu_{11} & & & & \end{array} \right) \\
&= \left(\begin{array}{lllll} \mu_{15} & \mu_{25} & \mu_{35} & \mu_{45} & \mu_{55} \\ \mu_{15}-1 & \mu_{25}-1 & \mu_{34} & \mu_{44} & \end{array} \middle| \begin{array}{l} 1\dot{0} \\ 1\dot{0} \end{array} \middle| \begin{array}{lllll} \mu_{15} & \mu_{25}+1 & \mu_{35} & \mu_{45} & \mu_{55} \\ \mu_{15}-1 & \mu_{25} & \mu_{34} & \mu_{44} & \end{array} \right) \\
&\times \left(\begin{array}{llll} \mu_{15}-1 & \mu_{25}-1 & \mu_{34} & \mu_{44} \\ \mu_{15}-2 & \mu_{25}-1 & \mu_{33} & \end{array} \middle| \begin{array}{l} 1\dot{0} \\ 1\dot{0} \end{array} \middle| \begin{array}{llll} \mu_{15}-1 & \mu_{25} & \mu_{34} & \mu_{44} \\ \mu_{15}-2 & \mu_{25} & \mu_{33} & \end{array} \right) \\
&\times \left(\begin{array}{lll} \mu_{15}-2 & \mu_{25}-1 & \mu_{33} \\ \mu_{15}-3 & \mu_{25}-1 & \end{array} \middle| \begin{array}{l} 1\dot{0} \\ 1\dot{0} \end{array} \middle| \begin{array}{lll} \mu_{15}-2 & \mu_{25} & \mu_{33} \\ \mu_{15}-3 & \mu_{25} & \end{array} \right) \\
&\times \left(\begin{array}{ll} \mu_{15}-3 & \mu_{25}-1 \\ \mu_{11} & \end{array} ; \begin{array}{l} 10 \\ 0 \end{array} \middle| \begin{array}{ll} \mu_{15}-3 & \mu_{25} \\ \mu_{11} & \end{array} \right) \\
&= \sqrt{\frac{(\mu_{25}+\mu_{35})(\mu_{25}+\mu_{45}-1)(\mu_{25}+\mu_{55}-2)(\mu_{25}+\mu_{34})(\mu_{25}+\mu_{44}-1)(\mu_{11}-\mu_{25}+1)}{(\mu_{25}+\mu_{35}+1)(\mu_{25}+\mu_{45})(\mu_{25}+\mu_{55}-1)(\mu_{25}+\mu_{34}-1)(\mu_{25}+\mu_{44}-2)(\mu_{15}-\mu_{25}-1)}}.
\end{aligned} \tag{4.19}
$$

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