The exceptional Lie algebra $\mathfrak{g}_2$ is generated by three generators subject to quadruple relations

N.I. Stoilova[1]

Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences,
Boul. Tsarigradsko Chaussee 72, 1784 Sofia, Bulgaria

J. Van der Jeugt[2]

Department of Applied Mathematics, Computer Science and Statistics, Ghent University,
Krijgslaan 281-S9, B-9000 Gent, Belgium.

Abstract

In this short communication we show how the Lie algebra $\mathfrak{g}_2$ can easily be described as a free Lie algebra on 3 generators, subject to some simple quadruple relations for these generators.

The exceptional Lie groups and algebras and their descriptions have always attracted much attention. Many presentations and realizations can be found in the literature and in books [6–9], especially for the Lie algebra $\mathfrak{g}_2$ (often also denoted by G_2). The Lie group G_2 is commonly defined as the group of automorphisms of the octonions [2], and thus the Lie algebra $\mathfrak{g}_2$ is identified as the derivation algebra of octonions. There are also realizations of $\mathfrak{g}_2$ as a derivation algebra of other non-associative algebras [3,5], combinatorial constructions [10] or constructions starting from different automorphism groups [11]. For a survey, see [4] or [1].

The common description of $\mathfrak{g}_2$ as a Lie algebra with generators and relations is in terms of the Chevalley generators and the Serre relations [7]. As far as we know a presentation of $\mathfrak{g}_2$ as a Lie algebra with 3 generators subject to some quadruple relations has not been given; at least, we were not able to find it in the literature. Therefore we think it is worthwhile to correspond our simple result to the mathematics community.

Note that our result holds for any ground field K of characteristic different from 2 and 3.

Theorem 1. *The Lie algebra $\mathfrak{g}$ on three generators x_1, x_2, x_3 subject to the following quadruple relations is equal to $\mathfrak{g}_2$:*

$$[x_i, [x_j, [x_i, x_k]]] = 2\epsilon_{ijk} x_i, \tag{1}$$

$$[x_i, [x_j, [x_j, x_k]]] = 6\epsilon_{ijk} x_j. \tag{2}$$

Herein, ϵ_{ijk} is the common Levi-Civita symbol in three dimensions: ϵ_{ijk} is 1 if (i,j,k) is an even permutation of $(1,2,3)$, -1 if it is an odd permutation of $(1,2,3)$, and 0 otherwise.

Observe that another quadruple relation follows using the Jacobi identity on the triple (x_i, x_j, x_k):

$$[x_i, [x_i, [x_j, x_k]]] + [x_i, [x_j, [x_k, x_i]]] + [x_i, [x_k, [x_i, x_j]]] = 0;$$

[1]E-mail: stoilova@inrne.bas.bg

[2]E-mail: Joris.VanderJeugt@UGent.be

using (1) this leads to

$$\begin{aligned}[x_i,[x_i,[x_j,x_k]]] &= [x_i,[x_j,[x_i,x_k]]] - [x_i,[x_k,[x_i,x_j]]] \\ &= 2\epsilon_{ijk}x_i - 2\epsilon_{ikj}x_i,\end{aligned}$$

and hence

$$[x_i,[x_i,[x_j,x_k]]] = 4\epsilon_{ijk}x_i. \tag{3}$$

Note that the list (1), (2) and (3) describes all possible quadruple relations for four elements with three indices from $\{1,2,3\}$, since among 4 indices at least two should be equal.

The most straightforward proof of Theorem 1 is by identifying the independent elements of $\mathfrak{g}$, and computing their commutation relations.

Let us first count the number of independent elements of $\mathfrak{g}$. There are three elements of degree 1 in the generators: x_1, x_2 and x_3. Using the anti-symmetry of the Lie bracket, there are also three elements of degree 2 in the generators: $[x_2,x_3]$, $[x_1,x_3]$ and $[x_1,x_2]$. Next, the elements of degree 3 are of the form $[x_i,[x_j,x_k]]$, where $i \in \{1,2,3\}$ and $(j,k) \in \{(2,3),(1,3),(1,2)\}$. However, among these 9 elements there is one linear relation following from the Jacobi identity, namely

$$[x_1,[x_2,x_3]] + [x_2,[x_3,x_1]] + [x_3,[x_1,x_2]] = 0.$$

This implies that there are 8 linearly independent elements of degree 3 in the generators. The quadruple relations imply that there are no independent elements of degree 4 or higher. So all together, $\mathfrak{g}$ has dimension 14.

In order to identify $\mathfrak{g}$ with $\mathfrak{g}_2$ it is sufficient to show that $\mathfrak{g}$ is a simple Lie algebra. We shall do more, and give the complete table of brackets among the 14 basis elements (from which it also follows that $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$). For this purpose, let us introduce some further notation. The elements of degree 2 are denoted by:

$$y_1 = \frac{1}{2}[x_2,x_3], \qquad y_2 = \frac{1}{2}[x_3,x_1], \qquad y_3 = \frac{1}{2}[x_1,x_2]. \tag{4}$$

The elements of degree 3 in the generators are denoted by:

$$\begin{aligned} a_{12} &= \frac{1}{3}[x_2,y_1], \qquad a_{23} = \frac{1}{3}[x_3,y_2], \qquad a_{13} = \frac{1}{3}[x_3,y_1], \\ a_{21} &= \frac{1}{3}[x_1,y_2], \qquad a_{32} = \frac{1}{3}[x_2,y_3], \qquad a_{31} = \frac{1}{3}[x_1,y_3], \\ h_1 &= \frac{1}{3}([x_1,y_1]-[x_2,y_2]), \qquad h_2 = \frac{1}{3}([x_2,y_2]-[x_3,y_3]). \end{aligned} \tag{5}$$

It is now a simple matter to determine the table of brackets between these elements, using the definitions (4) and (5), the quadruple relations (1)-(3), and the Jacobi identity to rewrite elements of degree 4 in the form of relations (1)-(3).

For example, let us compute a bracket corresponding to an element of degree 5 in the generators:

$$\begin{aligned}[y_1,a_{23}] &= \frac{1}{2}[[x_2,x_3],a_{23}] = \frac{1}{2}[x_2,[x_3,a_{23}]] - \frac{1}{2}[x_3,[x_2,a_{23}]] \\ &= \frac{1}{12}[x_2,[x_3,[x_3,[x_3,x_1]]]] - \frac{1}{12}[x_3,[x_2,[x_3,[x_3,x_1]]]] \\ &= \frac{1}{12}[x_2,0] - \frac{1}{12}[x_3,6x_3] = 0. \end{aligned} \tag{6}$$

And as a second example, we compute an element of degree 6:

$$
\begin{aligned}
[a_{12}, a_{23}] &= \frac{1}{3}[[x_2, y_1], a_{23}] = \frac{1}{3}[x_2, [y_1, a_{23}]] - \frac{1}{3}[y_1, [x_2, a_{23}]] \\
&= 0 - \frac{1}{3}[y_1, \frac{1}{6}[x_2, [x_3, [x_3, x_1]]]] = -\frac{1}{3}[y_1, x_3] = a_{13},
\end{aligned}
\tag{7}
$$

where the first term vanishes using the previous computation (6). Using such manipulations, the complete commutator table (Table 1) is computed.

$[\cdot,\cdot]$	h_1	h_2	a_{12}	a_{13}	a_{23}	a_{21}	a_{31}	a_{32}	x_1	x_2	x_3	y_1	y_2	y_3
h_1	0	0	$2a_{12}$	a_{13}	$-a_{23}$	$-2a_{21}$	$-a_{31}$	a_{32}	$-x_1$	x_2	0	y_1	$-y_2$	0
h_2		0	$-a_{12}$	a_{13}	$2a_{23}$	a_{21}	$-a_{31}$	$-2a_{32}$	0	$-x_2$	x_3	0	y_2	$-y_3$
a_{12}			0	0	a_{13}	h_1	$-a_{32}$	0	$-x_2$	0	0	0	y_1	0
a_{13}				0	0	$-a_{23}$	h_1+h_2	a_{12}	$-x_3$	0	0	0	0	y_1
a_{23}					0	0	a_{21}	h_2	0	$-x_3$	0	0	0	y_2
a_{21}						0	0	$-a_{31}$	0	$-x_1$	0	y_2	0	0
a_{31}							0	0	0	0	$-x_1$	y_3	0	0
a_{32}								0	0	0	$-x_2$	0	y_3	0
x_1									0	$2y_3$	$-2y_2$	$2h_1+h_2$	$3a_{21}$	$3a_{31}$
x_2										0	$2y_1$	$3a_{12}$	$-h_1+h_2$	$3a_{32}$
x_3											0	$3a_{13}$	$3a_{23}$	$-h_1-2h_2$
y_1												0	$2x_3$	$-2x_2$
y_2													0	$2x_1$
y_3														0

Table 1: Table of brackets among the 14 basis elements

From the commutator table, the subalgebra structure of the basis is obvious. Clearly, the elements h_i and a_{ij} satisfy the standard commutation relations of $\mathfrak{sl}(3)$: in the defining representation of $\mathfrak{sl}(3)$ in terms of 3×3-matrices, the elements a_{ij} can be realized as e_{ij} (a matrix with 1 on position (i, j) and zeros elsewhere) and h_i as $e_{ii} - e_{i+1,i+1}$. The elements x_1, x_2, x_3 are an $\mathfrak{sl}(3)$ triple, and y_1, y_2, y_3 a dual $\mathfrak{sl}(3)$ triple.

Acknowledgments

Both authors were supported by the Bulgarian National Science Fund, grant KP-06-N28/6.

References

[1] I. Agricola: *Old and New on the Exceptional Group G_2*, Notices of the AMS **55** (2008) 922–929.

[2] J.C. Baez: *The octonions*, Bull. Amer. Math. Soc. (N.S.) **39** (2002) 145–205.

[3] T. Basak: *Root space decomposition of $\mathfrak{g}_2$ from octonions*, J. Pure Appl. Algebra **222** (2018) 3036–3042.

[4] C. Draper Fontanals: *Notes on G_2: the Lie algebra and the Lie group*, Differential Geom. Appl. **57** (2018) 23–74.

[5] A. Elduque: *Order 3 elements in G_2 and idempotents in symmetric composition algebras*, Canad. J. Math. **70** (2018) 1038–1075.

[6] W. Fulton and J. Harris: *Representation theory*, Springer-Verlag, New York (1991).

[7] J.E. Humphreys: *Introduction to Lie algebras and representation theory*, Graduate Texts in Mathematics, Vol. 9, Springer-Verlag: New York-Berlin (1972).

[8] N. Jacobson: *Lie algebras*, Dover Publications: New York (1962).

[9] N. Jacobson: *Exceptional Lie algebras*, Lecture Notes in Pure and Applied Mathematics, Vol. 1, Marcel Dekker, Inc.: New York (1971).

[10] N.J. Wildberger: *A Combinatorial Construction of* G_2, J. Lie Theory **13** (2003) 155–165.

[11] R.A. Wilson: *On the compact real form of the Lie algebra* $\mathfrak{g}_2$, Math. Proc. Cambridge Philos. Soc. **148** (2010) 87–91.