GENERALISED SYMMETRIES AND BASES FOR DUNKL MONOGENICS

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ABSTRACT. We introduce a family of commuting generalised symmetries of the Dunkl–Dirac operator inspired by the Maxwell construction in harmonic analysis. We use these generalised symmetries to construct bases of the polynomial null-solutions of the Dunkl–Dirac operator. These polynomial spaces form representation spaces of the Dunkl–Dirac symmetry algebra. For the $\mathbb{Z}_2^d$ case, the results are compared with previous investigations.

1. INTRODUCTION

Since their introduction by Dunkl in 1989 [8], the family of commutative differential-difference operators associated with a reflection group W, now known as Dunkl operators, have enjoyed a great deal of interest of mathematical nature and also for applications in physics. Due to their properties, it is possible to replace partial derivatives with Dunkl operators in classical differential equations and operators appearing in many physical systems. A great deal of work has been done in the study of the resulting differential operators, most notably on the Dunkl version of the Laplace operator and its harmonic functions.

This work focuses on the kernel of the Dunkl version of the Dirac operator, which, like its classical analogue, is a square root of the Dunkl Laplacian. Polynomials in the kernel of the Dunkl–Dirac operator are called Dunkl monogenics and they form solutions of the Dunkl version of the homogeneous Dirac equation.

The study of the Dunkl–Dirac operator $\underline{D}$ and its kernel can take many ways. A recent fruitful path to its understanding resides in the consideration of the symmetry algebra linked to the $\mathfrak{osp}(1|2)$ realisation generated by $\underline{D}$ and its dual symbol $\underline{x}$ [13]. This symmetry algebra consists of elements supercommuting with $\underline{D}$ and the Dunkl monogenics form natural representation spaces.

The representation theory of the symmetry algebra was studied for some specific reflection groups, namely $W = \mathbb{Z}_2^d$ [5] where the link was made with the (higher-rank) Bannai–Ito algebra and for any reducible rank 3 reflection group $W = D_{2m} \times \mathbb{Z}_2$ [6], where the finite-dimensional representations were constructed. It was also shown that the Dunkl–Dirac symmetry algebra can be considered as a specialisation of an abstract algebra [7].

The goal of this article is to introduce a class of generalised symmetries of the Dunkl–Dirac operator, which satisfy a more general commutation relation than commuting or supercommuting with $\underline{D}$. As a consequence, these generalised symmetries preserve the kernel of $\underline{D}$ and can thus be used to construct natural bases for the spaces of monogenic polynomials. They are inspired by the Maxwell representation in harmonic analysis [12, p.69].

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This construction was translated to Dunkl harmonic analysis by Xu [15] and to Dunkl–Clifford analysis in [10,16]. Similar operators were also considered in the study of the conformal symmetries of the super Dirac operator [3] and on the radially deformed Dirac operator [4]. The last two were presented via Kelvin inverses; the generalised symmetries defined here are valid in a the more general context of [7], but also admit a presentation using a Clifford–Kelvin type transform when specialised to the Dunkl setting.

As an application, we use these generalised symmetries to give a new interpretation of the basis previously obtained by means of a Dunkl version of the Cauchy–Kovalevskaya extension Theorem in [5].

We now go through the structure of the paper and highlight the main results. In Section 2, we introduce the preliminaries on Dunkl operators and rewrite results of Xu [15] in terms of certain generalised symmetries to motivate our next investigation. Section 3 goes from the Dunkl harmonics to the Dunkl monogenics. We introduce certain operators and prove their main properties, mirroring the harmonic case. They are generalised symmetry of the Dunkl–Dirac operator (Proposition 3.6), they commute with each others (Proposition 3.8), they can be written by the a Dunkl–Clifford–Kelvin transform on polynomials (Proposition 3.9) and they are linked to the projection operator (Propositions 3.11 and 3.12). A basis of the monogenic representation for any reflection group is then constructed in Section 4 (Theorem 4.2). Finally, we study in Section 5 the case of the group $W = \mathbb{Z}_2^d$ and retrieve a known basis (Proposition 5.11).

2. Dunkl operators

2.1. **Preliminaries.** Let W be a reflection group acting on $\mathbb{R}^d$ and $\langle -,- \rangle$ be the canonical bilinear form of $\mathbb{R}^d$. Let $R \subset \mathbb{R}^d$ denote the root system linked to W and R^+ is a fixed set of positive roots. The reflection $\sigma_\alpha : \mathbb{R}^d \to \mathbb{R}^d$ associated with a root $\alpha = (\alpha_1, \dots, \alpha_d) \in R^+$ is

$$\sigma_\alpha(y) := y - 2\frac{\langle y, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha. \tag{2.1}$$

The group W is generated as a Coxeter group by the reflections σ_α and its elements act on functions of $x \in \mathbb{R}^d$ by

$$\omega f(x) = f(\omega^{-1}x), \quad \omega \in W. \tag{2.2}$$

From now on we will assume that the roots of R are normalised. We consider a W-invariant function $\kappa : R \to \mathbb{C}$. We will usually assume κ to be a positive real function to avoid problems possibly resulting from specific negative values. Let $\xi_1, \dots, \xi_d$ denote the canonical basis of $\mathbb{R}^d$. The Dunkl operator associated with ξ_j is then defined by

$$\mathcal{D}_j f(x) = \partial_{x_j} f(x) + \sum_{\alpha \in R^+} \kappa(\alpha) \frac{f(x) - \sigma_\alpha f(x)}{\langle \alpha, x \rangle} \alpha_j. \tag{2.3}$$

A major, and non-trivial, property of these operators is that they commute [8]: $[\mathcal{D}_j, \mathcal{D}_k] = 0$.

With normalised roots, the commutation relations between the Dunkl operators and the variables are given by

$$[\mathcal{D}_i, x_j] = \delta_{ij} + 2\sum_{\alpha \in R^+} \kappa(\alpha)\alpha_i\alpha_j\sigma_\alpha, \tag{2.4}$$

and one readily sees that $[\mathcal{D}_i, x_j] = [\mathcal{D}_j, x_i]$.

The algebra generated by $x_1,\dots,x_d$, $\mathcal{D}_1,\dots,\mathcal{D}_d$ and the group algebra $\mathbb{C}W$ is a realisation of the faithful polynomial representation of a rational Cherednik algebra [14]. We denote it by $\mathcal{A}_\kappa$, with the index κ indicating the Dunkl realisation.

Remark 2.1. *This is an example of the algebra $\mathcal{A}$ considered in [7, Ex. 4.2].*

For a multi-index $\beta = (\beta_1,\dots,\beta_d) \in \mathbb{N}^d$, its 1-norm is $|\beta|_1 := \beta_1 + \dots + \beta_d$ and we denote the monomial $x^\beta := x_1^{\beta_1}\cdots x_d^{\beta_d}$ and also $\mathcal{D}^\beta := \mathcal{D}_1^{\beta_1}\cdots\mathcal{D}_d^{\beta_d}$.

The Dunkl–Laplace operator Δ_κ, the squared norm and the norm are respectively given by

$$\Delta_\kappa := \sum_{j=1}^d \mathcal{D}_j^2, \qquad |x|^2 := \sum_{j=1}^d x_j^2, \qquad |x| := \sqrt{\sum_{j=1}^d x_j^2}, \tag{2.5}$$

which are all invariant under the action of W. A consequence if this invariance and application of the Dunkl-Leibniz rule for radial functions is the property

$$[\mathcal{D}_j, |x|^a] = a|x|^{a-2}x_j, \quad \text{for } a \in \mathbb{R}. \tag{2.6}$$

The classical Euler operator $\mathbb{E}$, which measures the degree of a homogeneous polynomial, is also W-invariant. A direct computation shows that Δ_κ, $|x|^2$ and

$$H := \frac{1}{2}\sum_{j=1}^d \{\mathcal{D}_j, x_j\} = \mathbb{E} + d/2 + \gamma, \qquad \text{where} \qquad \mathbb{E} := \sum_{j=1}^d x_j\partial_{x_j}, \qquad \gamma := \sum_{\alpha\in R^+}\kappa(\alpha), \tag{2.7}$$

form a $\mathfrak{sl}_2$-triple in the algebra $\mathcal{A}_\kappa$ as the following relations hold [7]:

$$\left[H, |x|^2\right] = 2|x|^2, \qquad [H, \Delta_\kappa] = -2\Delta_\kappa, \qquad \left[\Delta_\kappa, |x|^2\right] = 4H. \tag{2.8}$$

Moreover, we also have the relations

$$[H, x_j] = x_j, \qquad [H, \mathcal{D}_j] = -\mathcal{D}_j, \qquad [\Delta_\kappa, x_j] = 2\mathcal{D}_j, \qquad \left[|x|^2, \mathcal{D}_j\right] = -2x_j. \tag{2.9}$$

2.2. **Dunkl Harmonics.** We will denote by $\mathcal{P} = \mathcal{P}(\mathbb{R}^d)$ the space of complex-valued polynomials on $\mathbb{R}^d$ and by $\mathcal{P}_n = \mathcal{P}_n(\mathbb{R}^d)$ the space of homogeneous polynomials of degree n. The space $\mathcal{H}$ of Dunkl harmonic polynomials consists of all polynomials in the kernel of the Dunkl–Laplace operator Δ_κ. We further denote $\mathcal{H}_n = \mathcal{H}\cap\mathcal{P}_n$.

In a classical construction of harmonic analysis, the Maxwell representation [12] allows one to construct bases of polynomial harmonics by means of the Kelvin transformation. This was extended by Xu to Dunkl harmonics [15]. For $\beta = (\beta_1,\dots,\beta_d) \in \mathbb{N}^d$, Xu considered the harmonic polynomials $H_\beta(x)$ given by (compare with [15, Def. 2.2])

$$H_\beta(x) := \mathcal{K}_\kappa\mathcal{D}_1^{\beta_1}\dots\mathcal{D}_d^{\beta_d}\mathcal{K}_\kappa(1), \qquad \Delta_\kappa H_\beta(x) = 0, \tag{2.10}$$

where a Dunkl version of the Kelvin transform is used

$$\mathcal{K}_\kappa f(x) := |x|^{-(2\gamma+d-2)} f\left(\frac{x}{|x|^2}\right), \quad \mathcal{K}_\kappa\mathcal{K}_\kappa f(x) = f(x). \tag{2.11}$$

2.3. **Generalised symmetries.** It is possible to express Xu's construction by means of a generalised symmetry of the Dunkl–Laplace operator.

Definition 2.2. *We define* $\mathfrak{m}_j \in \mathcal{A}_\kappa$ *to be*

$$\mathfrak{m}_j = 2x_j(H-1) - |x|^2\mathcal{D}_j. \tag{2.12}$$

For a multi-index $\beta = (\beta_1, \ldots, \beta_d) \in \mathbb{N}^d$, *we write* $\mathfrak{m}^\beta := \mathfrak{m}_1^{\beta_1} \ldots \mathfrak{m}_d^{\beta_d}$.

Proposition 2.3. *The operator* $\mathfrak{m}$ *is a generalised symmetry of the Dunkl–Laplace operator*

$$[\Delta_\kappa, \mathfrak{m}_j] = 4x_j\Delta_\kappa \quad \text{or} \quad \Delta_\kappa\mathfrak{m}_j = (\mathfrak{m}_j + 4x_j)\Delta_\kappa. \tag{2.13}$$

Proof. It follows from the relations (2.8) and (2.9)

$$\begin{aligned}
\Delta_\kappa\mathfrak{m}_j &= \Delta_\kappa(2x_jH - 2x_j - |x|^2\mathcal{D}_j) \\
&= 2x_j\Delta_\kappa H + 4\mathcal{D}_jH - 2x_j\Delta_\kappa - 4\mathcal{D}_j - |x|^2\mathcal{D}_j\Delta_\kappa - 4H\mathcal{D}_j \\
&= (2x_jH - 2x_j - |x|^2\mathcal{D}_j)\Delta_\kappa + 4x_j\Delta_\kappa + 4[\mathcal{D}_j, H] - 4\mathcal{D}_j \\
&= \mathfrak{m}_j\Delta_\kappa + 4x_j\Delta_\kappa.
\end{aligned}$$

■

The next result also follows from [15, Thm 2.3].

Proposition 2.4. *When acting on* $\mathcal{P}$,

$$\mathfrak{m}_j = -\mathcal{K}_\kappa\mathcal{D}_j\mathcal{K}_\kappa = -H_{\xi_j}. \tag{2.14}$$

Proof. By linearity, it is sufficient to prove it for a homogeneous polynomial $p \in \mathcal{P}_n$. Apply the Dunkl–Leibniz rule (2.6) to get

$$\begin{aligned}
\mathcal{D}_j\mathcal{K}_\kappa p(x) &= \mathcal{D}_j|x|^{-(2\gamma+d-2-2n)}p(x) \\
&= |x|^{-(2\gamma+d-2-2n)}\mathcal{D}_jp(x) - (2\gamma+d-2-2n)|x|^{-(2\gamma+d-2n)}x_jp(x).
\end{aligned}$$

Both terms have degree of homogeneity $-2\gamma - d + 1 - n$; we can apply again the Kelvin transform $\mathcal{K}_\kappa$ on the two sides to obtain

$$\begin{aligned}
\mathcal{K}_\kappa\mathcal{D}_j\mathcal{K}_\kappa &= |x|^{-(2\gamma+d-2-2\gamma-2d+2-2n)}|x|^{-(2\gamma+d-2-2n)}\mathcal{D}_jp(x) \\
&\quad - (2\gamma+d+2n-2)|x|^{-(2\gamma+d-2-4\gamma-2d+2-2n)}|x|^{-(2\gamma+d+2n)}x_jp(x) \\
&= |x|^2\mathcal{D}_jp(x) + 2x_jp - (2\gamma+d+2n)x_jp(x),
\end{aligned}$$

and this is precisely $-\mathfrak{m}_jp(x) = -(2x_jH - x_j - |x|^2\mathcal{D}_j)p(x)$. ■

Proposition 2.5. *The generalised symmetries* $\mathfrak{m}_j$ *commute amongst themselves when acting on* $\mathcal{P}$

$$[\mathfrak{m}_j, \mathfrak{m}_k] = 0. \tag{2.15}$$

Proof. By Proposition 2.4, when acting on $\mathcal{P}$,

$$\mathfrak{m}_j\mathfrak{m}_k = \mathcal{K}_\kappa\mathcal{D}_j\mathcal{K}_\kappa\mathcal{K}_\kappa\mathcal{D}_k\mathcal{K}_\kappa = \mathcal{K}_\kappa\mathcal{D}_j\mathcal{D}_k\mathcal{K}_\kappa = \mathcal{K}_\kappa\mathcal{D}_k\mathcal{D}_j\mathcal{K}_\kappa = \mathcal{K}_\kappa\mathcal{D}_k\mathcal{K}_\kappa\mathcal{I}_\kappa\mathcal{D}_j\mathcal{K}_\kappa = \mathfrak{m}_k\mathfrak{m}_j. \tag{2.16}$$

■

Let $\operatorname{proj}_{\mathcal{H}}^{\mathcal{P}} \colon \mathcal{P} \to \mathcal{H}$ denote the projection operator that, when restricted to $\mathcal{P}_n$, reduces to $\operatorname{proj}_{\mathcal{H}_n}^{\mathcal{P}_n}$ given by [15, (2.5)]

$$\operatorname{proj}_{\mathcal{H}_n}^{\mathcal{P}_n} p(x) = \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{|x|^{2j} \Delta_\kappa^j p(x)}{2^{2j} j! (-n-d/2-\gamma+2)_j}, \tag{2.17}$$

where the notation for the Pochhammer symbol is used, which is defined as $(a)_0 = 1$, and $(a)_n = a(a+1)\cdots(a+n-1) = \Gamma(a+n)/\Gamma(a)$, with Γ the Gamma function.

Proposition 2.6. *With H given by (2.7) and x_j the operator that multiplies a polynomial by x_j, when acting on $\mathcal{H}$ we have*

$$\mathfrak{m}_j = 2(H-2) \circ \operatorname{proj}_{\mathcal{H}}^{\mathcal{P}} \circ x_j\,. \tag{2.18}$$

Proof. Let $h_{n-1} \in \mathcal{H}_{n-1}$, then $\Delta_\kappa^k x_j h_{n-1} = 0$ for $k \geq 2$, so using (2.17) we have

$$\begin{aligned} 2(h-2)\operatorname{proj}_{\mathcal{H}}^{\mathcal{P}}(x_j h_{n-1}) &= 2(H-2)\operatorname{proj}_{\mathcal{H}_n}^{\mathcal{P}_n}(x_j h_{n-1}) \\ &= 2(H-2)x_j h_{n-1} - 2(\mathbb{E}+d/2+\gamma-2)|x|^2 \Delta_\kappa(x_j h_{n-1})/(4(\gamma+n-2+d/2)) \\ &= 2x_j(H-1)h_{n-1} - |x|^2 \mathcal{D}_j h_{n-1}, \end{aligned}$$

where we used $[\Delta_\kappa, x_j] = 2\mathcal{D}_j$ and $\Delta_\kappa h_{n-1} = 0$. The last line is precisely (2.12). ∎

Theorem 2.7 ([15, Theorem 2.4])**.** *For $\beta \in \mathbb{N}^d$ with $|\beta|_1 = n$,*

$$H_\beta(x) = (-1)^n 2^n (\gamma - 1 + d/2)_n \operatorname{proj}_{\mathcal{H}_n}^{\mathcal{P}_n}(x^\beta). \tag{2.19}$$

Thus, from Proposition 2.4, the generalised symmetry is linked to the projection operator.

Corollary 2.8. *For $\beta \in \mathbb{N}^d$ with $|\beta|_1 = n$,*

$$\mathfrak{m}^\beta(1) = 2^n(\gamma-1+d/2)_n \operatorname{proj}_{\mathcal{H}_n}^{\mathcal{P}_n}(x^\beta)\,. \tag{2.20}$$

3. Dunkl Monogenics

3.1. **Clifford algebra.** Let $\varepsilon \in \{-1, +1\}$ be a sign and let $Cl(d)$ be the Clifford algebra associated with $\mathbb{R}^d$ and $\langle -, - \rangle$, which is generated by $e_1, \ldots, e_d$, the images of the canonical basis of $\mathbb{R}^d$: $\xi_j \mapsto e_j$, subject to the following anticommutation relations

$$\{e_i, e_j\} = e_i e_j + e_j e_i = 2\varepsilon \delta_{ij}. \tag{3.1}$$

Some specific elements in the tensor product $\mathcal{A}_\kappa \otimes Cl(d)$ are denoted as follows, with the tensor product omitted,

$$\underline{x} := \sum_{j=1}^d x_j e_j, \qquad \underline{D} := \sum_{j=1}^d \mathcal{D}_j e_j, \qquad \underline{\alpha} := \sum_{j=1}^d \alpha_j e_j, \tag{3.2}$$

where $\underline{x}$ is the *vector variable* and $\underline{D}$ the *Dunkl–Dirac operator*. Note that for $\alpha \in \mathbb{R}^d$, $\underline{\alpha} \in Cl(d)$ which we identify with $1 \otimes Cl(d) \subset \mathcal{A}_\kappa \otimes Cl(d)$.

Up to the sign ε, the square of the Dunkl–Dirac operator is the Dunkl–Laplace operator and its dual operator is the square of the vector variable:

$$\Delta_\kappa := \sum_{j=1}^d \mathcal{D}_j^2 = \varepsilon \underline{D}^2, \qquad |x|^2 := \sum_{j=1}^d x_j^2 = \varepsilon \underline{x}^2. \tag{3.3}$$

Moreover, we have

$$\{\underline{D}, \underline{x}\} = 2\varepsilon H = 2\varepsilon(\mathbb{E} + d/2 + \gamma), \tag{3.4}$$

and $\underline{D}$ and $\underline{x}$ are the odd generators of a realisation of the Lie superalgebra $\mathfrak{osp}(1|2)$ containing the Lie algebra $\mathfrak{sl}(2)$ as an even subalgebra realised by (2.8) [7, Theorem 3.4].

Unlike Δ_κ and $|x|^2$, the Dunkl–Dirac operator $\underline{D}$ and the vector variable $\underline{x}$ do not commute with the elements of W inside the $\mathcal{A}_\kappa$-part of $\mathcal{A}_\kappa \otimes Cl(d)$, as this W does not interact with $Cl(d)$. However, elements of the form

$$\widetilde{\sigma}_\alpha := \underline{\alpha}\sigma_\alpha, \quad \text{for } \alpha \in R, \tag{3.5}$$

anticommute (or supercommute as both have odd $\mathbb{Z}_2$-degree) with $\underline{D}$ and $\underline{x}$. These elements generate a group

$$\widetilde{W}^\varepsilon := \langle \widetilde{\sigma}_\alpha := \underline{\alpha}\sigma_\alpha \mid \alpha \in R^+ \rangle \subset \mathcal{A}_\kappa \otimes Cl(d). \tag{3.6}$$

It is a double covering of the reflection group W. In the realisation that we are considering, it can be seen as a subgroup of the group $\mathcal{O}(d)$ through the pullback of the projection of one of its double coverings $\mathrm{Pin}^\varepsilon(d)$ onto $\mathcal{O}(d)$. An abstract presentation by generators and relations from the ones of W can be also be given, see [11]. The dependency of $\widetilde{W}^\varepsilon$ on ε is apparent from the inclusion of the Clifford elements in $\underline{\alpha}$. The two different values of ε will yield in general the two non-isomorphic double coverings of W: $\widetilde{W}^+$ and $\widetilde{W}^-$. However depending on the group W, they may be trivial, as for example for $W = S_3$ [11, Thm 4.1].

Lemma 3.1. *The Dunkl–Dirac operator $\underline{D}$ and its dual symbol $\underline{x}$ respect the following commutation relations*

$$\{\widetilde{\sigma}_\alpha, \underline{x}\} = 0 = \{\widetilde{\sigma}_\alpha, \underline{D}\}; \tag{3.7}$$

$$[\mathcal{D}_j, \underline{x}] = [\underline{D}, x_j] = e_j + 2\sum_{\alpha \in R^+} \kappa(\alpha)\alpha_j \widetilde{\sigma}_\alpha; \tag{3.8}$$

$$[\underline{D}, \sigma_\alpha] = 2\langle \mathcal{D}, \alpha \rangle \widetilde{\sigma}_\alpha, \qquad [\underline{x}, \sigma_\alpha] = 2\langle x, \alpha \rangle \widetilde{\sigma}_\alpha; \tag{3.9}$$

$$\left[\underline{D}, |x|^2\right] = 2\underline{x}, \quad [\underline{x}, \Delta] = -2\underline{D}; \tag{3.10}$$

$$[\underline{D}, H] = \underline{D}, \quad [\underline{x}, H] = -\underline{x}. \tag{3.11}$$

Proof. These relations follow by direct computation using (2.1), (2.4), (2.9) and the Clifford algebra relations. ■

To denote more compactly specific linear combinations of elements in $\widetilde{W}^\varepsilon$ such as the one appearing in the right-hand side of (3.8), we write (see also [7, Ex. 4.2])

$$O_j := \frac{\varepsilon}{2}([\underline{D}, x_j] - e_j) = \varepsilon \sum_{\alpha \in R^+} \kappa(\alpha)\alpha_j \widetilde{\sigma}_\alpha. \tag{3.12}$$

Lemma 3.2. *The following holds*

$$\sum_{j=1}^{d} O_j e_j = \sum_{\alpha \in R^+} \kappa(\alpha)\sigma_\alpha. \tag{3.13}$$

Proof. Replacing O_j by its expression (3.12), and using the anticommutation of Clifford elements and the fact that the roots are normalised, in particular that $\underline{\alpha}^2 = \varepsilon$, yields

$$\sum_{j=1}^{d} O_j e_j = \varepsilon \sum_{j=1}^{d} \sum_{\alpha \in R^+} \kappa(\alpha)\alpha_j \underline{\alpha}\sigma_\alpha e_j = \varepsilon \sum_{\alpha \in R^+} \kappa(\alpha)\underline{\alpha}^2 \sigma_\alpha = \sum_{\alpha \in R^+} \kappa(\alpha)\sigma_\alpha. \tag{3.14}$$

■

3.2. **Dunkl Monogenics.** Let V be an irreducible representation of $Cl(d)$, also called a spinor representation. There is a natural action of $\mathcal{A}_\kappa \otimes Cl(d)$ on the space $\mathcal{P}\otimes V$. The space of Dunkl monogenic polynomials consists of the elements of $\mathcal{P}\otimes V$ that are in the kernel of the Dunkl–Dirac operator, and will be denoted by $\mathcal{M} := \mathcal{M}(\mathbb{R}^d;V)$. We denote $\mathcal{M}_n := \mathcal{M}_n(\mathbb{R}^d;V) = \mathcal{M}\cap(\mathcal{P}_n\otimes V)$ for the $\mathcal{M}$-subspace of (spinor valued) homogeneous polynomials of degree n, and we have $\mathcal{M}(\mathbb{R}^d;V) = \bigoplus_{n\geq 0}\mathcal{M}_n(\mathbb{R}^d;V)$.

There is a projection $\mathrm{proj}_{\mathcal{M}}^{\mathcal{P}\otimes V} : \mathcal{P}(\mathbb{R}^d)\otimes V \longrightarrow \mathcal{M}(\mathbb{R}^d;V)$ that, when restricted to $\mathcal{P}_n\otimes V$, is given by [13, Lem. 4.6]

$$\mathrm{proj}_{\mathcal{M}_n}^{\mathcal{P}_n\otimes V} : \mathcal{P}_n(\mathbb{R}^d)\otimes V \longrightarrow \mathcal{M}_n(\mathbb{R}^d;V)$$

$$p \longmapsto p - \varepsilon \sum_{j=0}^{\lfloor (n+1)/2\rfloor} \frac{(-1)^j \underline{x}^{2j+1}\underline{D}^{2j+1}p}{2^{2j+1}j!(n-j-1+d/2+\gamma)_{j+1}} + \sum_{j=1}^{\lfloor n/2\rfloor} \frac{(-1)^j|x|^{2j}\Delta_\kappa^j p}{2^{2j}j!(n-j+d/2+\gamma)_j}. \tag{3.15}$$

Remark 3.3. *Each Dunkl monogenic polynomial is a highest weight vector for the $\mathfrak{osp}(1|2)$ realisation containing the Dunkl–Dirac operator as positive root vector. For the $\mathfrak{osp}(1|2)$ extremal projector $\mathrm{proj}_{\mathcal{M}}^{\mathcal{P}\otimes V}$, we have [1, (3.8a)]*

$$\mathrm{proj}_{\mathcal{M}}^{\mathcal{P}\otimes V} = \mathrm{proj}_{\mathcal{M}}^{\mathcal{H}}\,\mathrm{proj}_{\mathcal{H}}^{\mathcal{P}\otimes V} = \mathrm{proj}_{\mathcal{H}}^{\mathcal{P}\otimes V}\,\mathrm{proj}_{\mathcal{M}}^{\mathcal{H}}, \tag{3.16}$$

where $\mathrm{proj}_{\mathcal{H}}^{\mathcal{P}\otimes V} = \mathrm{proj}_{\mathcal{H}}^{\mathcal{P}}$ as considered above for $\mathfrak{sl}(2)$, and $\mathrm{proj}_{\mathcal{M}}^{\mathcal{H}}$, when restricted to $\mathcal{P}_n\otimes V$, is given by

$$\mathrm{proj}_{\mathcal{M}_n}^{\mathcal{H}_n} = \left(1 - \frac{\varepsilon\underline{x}\underline{D}}{2(n-1+d/2+\gamma)}\right). \tag{3.17}$$

3.3. **Generalised symmetries.**

Definition 3.4. *We define $\mathfrak{z}_j \in \mathcal{A}\otimes Cl(d)$ for $1\leq j\leq d$ by*

$$\mathfrak{z}_j := 2\varepsilon x_j H - \underline{x}\mathcal{D}_j\underline{x}. \tag{3.18}$$

For a vector $\beta = (\beta_1,\ldots,\beta_d)\in\mathbb{N}^d$, we write $\mathfrak{z}^\beta := \mathfrak{z}_1^{\beta_1}\cdots\mathfrak{z}_d^{\beta_d}$.

We begin by giving alternative formulations of $\mathfrak{z}_j$ in terms of elements of $\mathcal{A}_\kappa\otimes Cl(d)$ that follow from Lemma 3.1 the expressions (3.12) of O_j.

Lemma 3.5. *The operator $\mathfrak{z}_j := 2\varepsilon x_j H - \underline{x}\mathcal{D}_j\underline{x}$ has the following expressions in the Dunkl case for a reflection group W with root system R*

$$\mathfrak{z}_j = x_j\{\underline{D},\underline{x}\} - \underline{x}[\underline{D},x_j] - \varepsilon|x|^2\mathcal{D}_j; \tag{3.19}$$

$$\mathfrak{z}_j = x_j\{\underline{D},\underline{x}\} - \underline{x}(e_j + 2\varepsilon O_j) - \varepsilon|x|^2\mathcal{D}_j; \tag{3.20}$$

$$\mathfrak{z}_j = 2\varepsilon x_j(\mathbb{E}+d/2+\gamma) - \underline{x}(e_j + 2\sum_{\alpha\in R^+}\kappa(\alpha)\alpha_j\widetilde{\sigma}_\alpha) - \varepsilon|x|^2\mathcal{D}_j; \tag{3.21}$$

$$\mathfrak{z}_j = 2\varepsilon x_j(\mathbb{E}+d/2+\gamma) - \underline{x}[\underline{D},x_j] - \varepsilon|x|^2\mathcal{D}_j. \tag{3.22}$$

We now consider some of the main properties of $\mathfrak{z}_j$ that are useful for our purposes.

Proposition 3.6. *The operator $\mathfrak{z}_j$ is a generalised symmetry of the Dunkl–Dirac operator*

$$[\underline{D},\mathfrak{z}_j] = 2\varepsilon x_j\underline{D} \quad \text{or} \quad \underline{D}\mathfrak{z}_j = (\mathfrak{z}_j + 2\varepsilon x_j)\underline{D}. \tag{3.23}$$

Proof. This follows from the relations of Proposition 3.1. First anticommute $\underline{D}$ and $\underline{x}$ and commute $\underline{D}$ and x_j

$$\underline{D}\mathfrak{z}_j = \underline{D}(x_j\{\underline{D}, \underline{x}\} - \underline{x}\mathcal{D}_j\underline{x}) = (x_j\underline{D} + [\underline{D}, x_j])\{\underline{D}, \underline{x}\} - (-\underline{x}\,\underline{D} + \{\underline{x}, \underline{D}\})\mathcal{D}_j\underline{x},$$

now employ (3.8) and the commutation of $\mathcal{D}_j$

$$= x_j\underline{D}\{\underline{D}, \underline{x}\} + [\mathcal{D}_j, \underline{x}]\{\underline{D}, \underline{x}\} + \underline{x}\mathcal{D}_j\underline{D}\,\underline{x} - \{\underline{D}, \underline{x}\}\mathcal{D}_j\underline{x},$$

then apply (2.9), and anticommute a second time $\underline{D}$ and $\underline{x}$

$$= x_j\{\underline{D}, \underline{x}\}\underline{D} + 2\varepsilon x_j\underline{D} + [\mathcal{D}_j, \underline{x}]\{\underline{D}, \underline{x}\} - \underline{x}\mathcal{D}_j\underline{x}\,\underline{D} + \underline{x}\mathcal{D}_j\{\underline{D}, \underline{x}\} - \{\underline{D}, \underline{x}\}\mathcal{D}_j\underline{x},$$

finally, $\{\underline{D}, \underline{x}\}$ commutes with $\mathcal{D}_j\underline{x}$ because of $[H, \mathcal{D}_j] = -\mathcal{D}_j$, $[H, \underline{x}] = \underline{x}$ and $\varepsilon^2 = 1$ so

$$\begin{aligned} &= (x_j\{\underline{D}, \underline{x}\} - \underline{x}\mathcal{D}_j\underline{x} + 2\varepsilon x_j)\underline{D} + [\mathcal{D}_j, \underline{x}]\{\underline{D}, \underline{x}\} - \mathcal{D}_j\underline{x}\{\underline{D}, \underline{x}\} + \underline{x}\mathcal{D}_j\{\underline{D}, \underline{x}\} \\ &= \mathfrak{z}_j\underline{D} + 2\varepsilon x_j\underline{D}. \end{aligned}$$

■

Proposition 3.7. *The operator $\mathfrak{z}_k$ respects the following commutation relations*

$$[\underline{x}, \mathfrak{z}_k] = -2\varepsilon x_k\underline{x} + \underline{x}(e_k + 2\varepsilon O_k)\underline{x}; \tag{3.24}$$

$$[x_j, \mathfrak{z}_k] = -2\varepsilon x_j x_k - \underline{x}[x_j, \mathcal{D}_k]\underline{x}; \tag{3.25}$$

$$[e_j, \mathfrak{z}_k] = 2\varepsilon(\underline{x}\mathcal{D}_k x_j - x_j\mathcal{D}_k\underline{x}); \tag{3.26}$$

$$[\mathcal{D}_j, \mathfrak{z}_k] = 2\varepsilon(x_k\mathcal{D}_j - x_j\mathcal{D}_k) + 2\varepsilon[\mathcal{D}_j, x_k]H + e_j[\underline{x}, \mathcal{D}_k] - 2\varepsilon(O_j\mathcal{D}_k\underline{x} + \underline{x}\mathcal{D}_k O_j); \tag{3.27}$$

$$\widetilde{\sigma}_\alpha\mathfrak{z}_k = \mathfrak{z}_{\sigma_\alpha(\xi_k)}\widetilde{\sigma}_\alpha := \sum_{j=1}^{d}\langle\sigma(\xi_k), \xi_j\rangle\mathfrak{z}_j. \tag{3.28}$$

Proof. Equation (3.24) follows from a small calculation using Proposition 3.1 and equation (2.9)

$$\begin{aligned} [\underline{x}, \mathfrak{z}_k] &= 2\varepsilon\underline{x}x_k H - \underline{x}\,\underline{x}\mathcal{D}_k\underline{x} = 2\varepsilon x_k\underline{x}H - \underline{x}\mathcal{D}_k\underline{x}\,\underline{x} - \underline{x}[\underline{x}, \mathcal{D}_k]\underline{x} - \mathfrak{z}_k\underline{x} \\ &= -2\varepsilon x_k\underline{x} + \underline{x}(e_k + 2\varepsilon O_k)\underline{x}. \end{aligned}$$

Equation (3.25) follows from the commutation relation (2.9) between x_j and H:

$$\begin{aligned} [x_j, \mathfrak{z}_k] &= 2\varepsilon x_j x_k H - x_j\underline{x}\mathcal{D}_k\underline{x} - 2\varepsilon\mathfrak{z}_k x_j = 2\varepsilon x_k H x_j - 2\varepsilon x_k x_j - (\underline{x}\mathcal{D}_k x_j\underline{x} + \underline{x}[x_j, \mathcal{D}_k]\underline{x}) - \mathfrak{z}_k x_j \\ &= -2\varepsilon x_j x_k - \underline{x}[x_j, \mathcal{D}_k]\underline{x}. \end{aligned}$$

Equation (3.26) comes from $\{e_j, \underline{x}\} = 2\varepsilon x_j$:

$$\begin{aligned} e_j\mathfrak{z}_k &= 2\varepsilon e_j x_k H - e_j\underline{x}\mathcal{D}_k\underline{x} = 2\varepsilon x_k H e_j + \underline{x}e_j\mathcal{D}_k\underline{x} - 2\varepsilon x_j\mathcal{D}_k\underline{x} \\ &= \mathfrak{z}_k e_j + 2\varepsilon\underline{x}\mathcal{D}_k x_j - 2\varepsilon x_j\mathcal{D}_k\underline{x}. \end{aligned}$$

Slightly more tedious computations yield equation (3.27). First develop

$$\begin{aligned} [\mathcal{D}_j, \mathfrak{z}_k] &= 2\varepsilon\mathcal{D}_j x_k H - \mathcal{D}_j\underline{x}\mathcal{D}_k\underline{x} - \mathfrak{z}_k\mathcal{D}_j \\ &= 2\varepsilon x_k\mathcal{D}_j H + 2\varepsilon[\mathcal{D}_j, x_k]H - (\underline{x}\mathcal{D}_j\mathcal{D}_k\underline{x} + [\mathcal{D}_j, \underline{x}]\mathcal{D}_k\underline{x}) - (2\varepsilon x_k H\mathcal{D}_j - \underline{x}\mathcal{D}_k\mathcal{D}_j\underline{x} - \underline{x}\mathcal{D}_k[\underline{x}, \mathcal{D}_j]), \end{aligned}$$

then employ $[H, \mathcal{D}_j] = -\mathcal{D}_j$ and cancel some terms

$$= 2\varepsilon x_k \mathcal{D}_j + 2\varepsilon[\mathcal{D}_j, x_k]H - [\mathcal{D}_j, \underline{x}]\mathcal{D}_k \underline{x} + \underline{x}\mathcal{D}_k[\underline{x}, \mathcal{D}_j]$$
$$= 2\varepsilon x_k \mathcal{D}_j + 2\varepsilon[\mathcal{D}_j, x_k]H - (e_j + 2\varepsilon O_j)\mathcal{D}_k \underline{x} - \underline{x}\mathcal{D}_k(e_j + 2\varepsilon O_j),$$

now use $\{\underline{x}, e_j\} = 2\varepsilon x_j$ to get

$$= 2\varepsilon x_k \mathcal{D}_j + 2\varepsilon[\mathcal{D}_j, x_k]H - e_j\mathcal{D}_k \underline{x} + e_j \underline{x}\mathcal{D}_k - 2\varepsilon x_j \mathcal{D}_k - 2\varepsilon(O_j\mathcal{D}_k \underline{x} + \underline{x}\mathcal{D}_k O_j)$$
$$= 2\varepsilon(x_k \mathcal{D}_j - x_j \mathcal{D}_k) + 2\varepsilon[\mathcal{D}_j, x_k]H + e_j[\underline{x}, \mathcal{D}_k] - 2\varepsilon(O_j\mathcal{D}_k \underline{x} + \underline{x}\mathcal{D}_k O_j).$$

The last relation follows from $\widetilde{W}^{\varepsilon}$ supercommuting with $\underline{x}$ and $\underline{D}$, and from the action of W on x_j and $\mathcal{D}_j$. ∎

The following important property can be proved by a direct computation using the results of Proposition 3.7. Furthermore, (3.29) holds even in the more general setting as considered in [7, Ex. 4.2]. The alternative representation of the operators $\mathfrak{z}_j$ in the next section allows for a very short proof of (3.29) when acting on (spinor-valued) polynomials, see (3.35).

Proposition 3.8. *The operators $\mathfrak{z}_j$ commute amongst themselves*

$$[\mathfrak{z}_j, \mathfrak{z}_\ell] = 0. \tag{3.29}$$

3.4. **Kelvin transformation.** Define the Dunkl–Clifford–Kelvin transform $\mathcal{I}_\kappa$ as

$$\mathcal{I}_\kappa f(x) = \underline{x}|x|^{-(2\gamma+d)} f\left(\frac{x}{|x|^2}\right). \tag{3.30}$$

Since $\kappa \geq 0$, the sum over positive roots γ is non-negative and $|x|^{-(2\gamma+d)}$ is thus well-defined. The operator $\mathcal{I}_\kappa$ is ε-idempotent, that is $\mathcal{I}_\kappa^2 = \varepsilon$. Indeed, using $\underline{x}\,\underline{x} = \varepsilon|x|^2$,

$$\mathcal{I}_\kappa \mathcal{I}_\kappa f(x) = \mathcal{I}_\kappa\left(\underline{x}|x|^{-(2\gamma+d)} f\left(\frac{x}{|x|^2}\right)\right) = \underline{x}|x|^{-(2\gamma+d)}\left(\frac{\underline{x}}{|x|^2}\frac{|x|^{2(2\gamma+d)}}{|x|^{(2\gamma+d)}} f\left(\frac{x}{|x|^2}\frac{|x|^4}{|x|^2}\right)\right) = \varepsilon f(x). \tag{3.31}$$

The relation between the two Kelvin-type transforms (2.11) and (3.30) is

$$\mathcal{I}_\kappa f = \varepsilon \underline{x}|x|^{-2}\mathcal{K}_\kappa f. \tag{3.32}$$

Remark that for $p(x) \in \mathcal{P}_n(\mathbb{R}^d)$ we have $p(x/|x|^2) = |x|^{-2n}p(x)$, and thus the action of the Dunkl–Clifford–Kelvin transform becomes

$$\mathcal{I}_\kappa p(x) = |x|^{-(2\gamma+d+2n)}\underline{x}p(x). \tag{3.33}$$

The transform (3.30) was considered before, for example see [16] and [10]. One of the main results of those two papers is to prove that, for any polynomial monogenic f, also $\mathcal{I}_\kappa \mathcal{D}_j \mathcal{I}_\kappa(f)$ is a polynomial monogenic. We give an interpretation in terms of generalised symmetries of the Dunkl–Dirac operator.

Proposition 3.9. *Acting on $\mathcal{P} \otimes V$,*

$$\mathfrak{z}_j = -\mathcal{I}_\kappa \mathcal{D}_j \mathcal{I}_\kappa. \tag{3.34}$$

Proof. Let $p \in \mathcal{P}_n(\mathbb{R}^d)$ be a homogeneous polynomial of degree n. Apply equation (2.6) to get

$$\begin{aligned}\mathcal{D}_j \mathcal{I}_\kappa p(x) &= \mathcal{D}_j |x|^{-(2\gamma+d+2n)} \underline{x} p(x) \\ &= -(2\gamma+d+2n)|x|^{-(2\gamma+d+2n+2)} x_j \underline{x} p(x) + |x|^{-(2\gamma+d+2n)} \mathcal{D}_j \underline{x} p(x).\end{aligned}$$

Remark now that the first and second terms have degree of homogeneity $-2\gamma-d-n$. Thus applying another time the Dunkl–Clifford–Kelvin transform yields

$$\begin{aligned}\mathcal{I}_\kappa \mathcal{D}_j \mathcal{I}_\kappa p(x) &= -(2\gamma+d+2n)\underline{x}|x|^{-(2\gamma+d-4\gamma-2d-2n)}|x|^{-(2\gamma+d+2n+2)} x_j \underline{x} p(x) \\ &\quad + \underline{x}|x|^{-(2\gamma+d-4\gamma-2d-2n)}|x|^{-(2\gamma+d+2n)} \mathcal{D}_j \underline{x} p(x) \\ &= -(2\gamma+d+2n)\underline{x}^2 |x|^{-2} x_j p(x) + \underline{x}\mathcal{D}_j \underline{x} p(x) \\ &= -2\varepsilon(n+d/2+\gamma) x_j p(x) + \underline{x}\mathcal{D}_j \underline{x} p(x),\end{aligned}$$

which equals $-\mathfrak{z}_j p(x) = -(2\varepsilon x_j(\mathbb{E}+d/2+\gamma) - \underline{x}\mathcal{D}_j \underline{x}) p(x)$. ∎

Using the commutativity of the Dunkl operators, Proposition 3.9 now gives an alternative easy proof of Proposition 3.8 when acting on polynomials:

$$\mathfrak{z}_j \mathfrak{z}_k = \mathcal{I}_\kappa \mathcal{D}_j \mathcal{I}_\kappa \mathcal{I}_\kappa \mathcal{D}_k \mathcal{I}_\kappa = \varepsilon \mathcal{I}_\kappa \mathcal{D}_j \mathcal{D}_k \mathcal{I}_\kappa = \varepsilon \mathcal{I}_\kappa \mathcal{D}_k \mathcal{D}_j \mathcal{I}_\kappa = \mathcal{I}_\kappa \mathcal{D}_k \mathcal{I}_\kappa \mathcal{I}_\kappa \mathcal{D}_j \mathcal{I}_\kappa = \mathfrak{z}_k \mathfrak{z}_j. \tag{3.35}$$

Monomials in $\mathfrak{z}_j$ also have a Dunkl–Clifford–Kelvin transform expression.

Corollary 3.10. *For $\beta \in \mathbb{N}^d$ with $|\beta|_1 = n$, when acting on $\mathcal{P} \otimes V$*

$$\mathfrak{z}^\beta = (-1)^n \varepsilon^{n-1} \mathcal{I}_\kappa \mathcal{D}^\beta \mathcal{I}_\kappa. \tag{3.36}$$

3.5. **Projection operator relation.**

Proposition 3.11. *With H given by (2.7) and x_j the operator that multiplies a polynomial by x_j, when acting on $\mathcal{M}$ we have*

$$\mathfrak{z}_j = 2\varepsilon H \circ \operatorname{proj}_{\mathcal{M}}^{\mathcal{P}\otimes V} \circ x_j. \tag{3.37}$$

Proof. Let $M_{n-1} \in \mathcal{M}_{n-1}$, then $\underline{D}^k x_j M_{n-1} = 0$ for $k \geq 3$, so using (3.15) we have

$$\begin{aligned}2\varepsilon H \operatorname{proj}_{\mathcal{M}}^{\mathcal{P}\otimes V}(x_j M_{n-1}) &= 2\varepsilon(\mathbb{E}+d/2+\gamma)\operatorname{proj}_{\mathcal{M}_n}^{\mathcal{P}_n\otimes V}(x_j M_{n-1}) \\ &= 2\varepsilon(\mathbb{E}+d/2+\gamma)x_j M_{n-1} - \underline{x}\underline{D}(x_j M_{n-1}) - \varepsilon|x|^2 \Delta_\kappa(x_j M_{n-1})/2 \\ &= 2\varepsilon(\mathbb{E}+d/2+\gamma)x_j M_{n-1} - \underline{x}[\underline{D}, x_j] M_{n-1} - \varepsilon|x|^2 \mathcal{D}_j M_{n-1},\end{aligned}$$

where we used $[\Delta_\kappa, x_j] = 2\mathcal{D}_j$ and $\underline{D}M_{n-1} = 0 = \Delta_\kappa M_{n-1}$. The last line is precisely (3.22). ∎

The monogenics of degree 0 are given by the spinor space: $\mathcal{M}_0 = V$. We can construct monogenics of higher degree by acting with the generalised symmetries $\mathfrak{z}_j$.

Now, let $\beta \in \mathbb{N}^d$ with $|\beta|_1 = n$. For the next result (see also [16, Prop 4.2]), we will use (3.16) and Xu's work on the equivalence between the projection operator of harmonics and the H_β. Moreover, we also need Proposition 3.9 and the correspondence (3.32) between $\mathcal{I}_\kappa$ and $\mathcal{K}_\kappa$.

Proposition 3.12. *Let $\beta \in \mathbb{N}^d$ with $|\beta|_1 = n$ and $x^\beta \in \mathcal{P}_n$. Then, acting on $\mathcal{M}_0 = V$*

$$\mathfrak{z}^\beta = \varepsilon^n 2^n (\gamma+d/2)_n \operatorname{proj}_{\mathcal{M}_n}^{\mathcal{P}_n\otimes V} \circ x^\beta. \tag{3.38}$$

Proof. By (3.16), we write

$$\mathrm{proj}_{\mathcal{M}_n}^{\mathcal{P}_n\otimes V} = \mathrm{proj}_{\mathcal{M}_n}^{\mathcal{H}_n}\,\mathrm{proj}_{\mathcal{H}_n}^{\mathcal{P}_n\otimes V}. \tag{3.39}$$

Let $s\in V$, then by (2.19) ([15, Theorem 2.4])

$$\mathrm{proj}_{\mathcal{H}_n}^{\mathcal{P}_n\otimes V}(x^\beta s) = (-1)^n H_\beta s/(2^n(\gamma-1+d/2)_n).$$

Next, we apply (3.17) and make use of (2.6)

$$\begin{aligned}
\mathrm{proj}_{\mathcal{M}_n}^{\mathcal{H}_n} H_\beta s &= H_\beta s - \frac{\varepsilon}{2n+d+2\gamma-2}\underline{x}\underline{D}\mathcal{K}_\kappa\mathcal{D}^\beta\mathcal{K}_\kappa(s)\\
&= H_\beta s - \frac{\varepsilon}{2n+d+2\gamma-2}\underline{x}\underline{D}|x|^{2\gamma+d-2+2n}\mathcal{D}^\beta|x|^{-2\gamma-d+2}s\\
&= H_\beta s - \frac{\varepsilon}{2n+d+2\gamma-2}(2\gamma+d-2+2n)\underline{x}\underline{x}|x|^{-2}|x|^{2\gamma+d-2+2n}\mathcal{D}^\beta|x|^{-2\gamma-d+2}s\\
&\quad+\frac{\varepsilon}{2n+d+2\gamma-2}\underline{x}|x|^{2\gamma+d-2+2n}\mathcal{D}^\beta\underline{D}|x|^{-2\gamma-d+2}s\\
&= H_\beta s - |x|^{2\gamma+d-2+n}\mathcal{D}^\beta|x|^{-2\gamma-d+2}v + \frac{2\varepsilon(\gamma+d/2-1)}{2n+d+2\gamma-2}\underline{x}|x|^{2\gamma+d-2+2n}\mathcal{D}^\beta\underline{x}|x|^{-2}|x|^{-2\gamma-d+2}s\\
&= \frac{\varepsilon(\gamma+d/2-1)}{n+d/2+\gamma-1}\mathcal{I}_\kappa\mathcal{D}^\beta\mathcal{I}_\kappa(s),
\end{aligned}$$

where we used (3.32). The result now follows by Corollary 3.10. ∎

4. Monogenic Bases

The goal of this section is to give a basis for the polynomial monogenics using the generalised symmetries of the previous section. The strategy we employ is inspired by the one applied by Xu in the Dunkl harmonic case [15]. Note that we will here also assume κ to be a positive real function.

For each multi-index $\beta=(\beta_1,\dots,\beta_d)\in\mathbb{N}^d$ with $|\beta|_1=n$ and each spinor $s\in V=\mathcal{M}_0$, we define a polynomial monogenic of degree n by

$$\mathsf{Z}_s^\beta := \mathfrak{z}^\beta s = \mathfrak{z}_1^{\beta_1}\dots\mathfrak{z}_d^{\beta_d}s. \tag{4.1}$$

It is direct to see from Corollary 3.10 that

$$\mathsf{Z}_s^\beta = (-1)^n\varepsilon^{n-1}\mathcal{I}_\kappa\mathcal{D}^\beta\mathcal{I}_\kappa(s) = (-1)^n\varepsilon^{n-1}\mathcal{I}_\kappa\mathcal{D}_1^{\beta_1}\dots\mathcal{D}_d^{\beta_d}\mathcal{I}_\kappa(s). \tag{4.2}$$

A simple application of the commutativity of the $\mathfrak{z}_j$ using their expression (3.5) gives a recurrence relation for the Z_s^β's.

Lemma 4.1. *Let $\beta\in\mathbb{N}^d$ with $|\beta|_1=n$. The spinor-valued polynomial Z_s^β respects*

$$\mathsf{Z}_s^{\beta+\xi_j} = (2\varepsilon(n+d/2+\gamma)x_j - 2\varepsilon\underline{x}O_j - \underline{x}e_j - \varepsilon|x|^2\mathcal{D}_j)\mathsf{Z}_s^\beta. \tag{4.3}$$

We can now turn our attention to the basis construction. If we consider only multi-indices $\mathbf{j}\in\mathbb{N}^d$ with zero as last index, we get a basis of the polynomial monogenics.

Theorem 4.2. *Let ν be a basis of V, the spinor representation of $Cl(d)$. The set*

$$\mathcal{B}_n = \{\mathsf{Z}_s^{\mathbf{j}} \mid \mathbf{j}=(j_1,\dots,j_{d-1},0)\in\mathbb{N}^d, |\mathbf{j}|_1=n,\ s\in\nu\} \tag{4.4}$$

is a basis of $\mathcal{M}_n(\mathbb{R}^d;V)$.

To prove this last theorem, we study a similar set for multi-indices without condition on their last entry and exhibit the relations respected by the polynomials.

Proposition 4.3. *The set*

$$\mathcal{C}_n = \{ \mathsf{Z}_s^\beta \mid \beta \in \mathbb{N}^d, |\beta|_1 = n,\ s \in \mathbb{V} \} \tag{4.5}$$

is a generating set for $\mathcal{M}_n(\mathbb{R}^d;V)$*, and the relations respected by its members are of the following form*

$$\sum_{j=1}^{d} \mathsf{Z}_{e_j\cdot s}^{\eta+\xi_j} = 0, \quad \textit{for} \quad \eta \in \mathbb{N}^d, \textit{ with } |\eta|_1 = n-1. \tag{4.6}$$

Proof. Proposition 3.6 states that the $\mathfrak{z}_j$ are generalised symmetries and thus $\underline{D}\mathsf{Z}_s^\beta = 0$. Since κ is assumed to be positive, induction on the recurrence relation of Lemma 4.1 is sufficient to show that the polynomial Z_s^β is never zero.

Alternatively, from Corollary 3.10, the polynomials Z_s^β are obtained from the Dunkl–Clifford–Kelvin transform. When $\kappa \geq 0$, the sum γ is non-negative, and thus the Dunkl–Clifford–Kelvin transform is well-defined and invertible. As such, would $\mathsf{Z}_s^\beta = (-1)^n \varepsilon^{n-1} \mathcal{I}_\kappa \mathcal{D}^\beta \mathcal{I}_\kappa(s)$ be 0, it would mean that $\mathcal{D}^\beta$ acts as zero on homogeneous polynomials, a contradiction.

Thus, the set $\mathcal{C}_n$ contains polynomial monogenics of degree n. To see that it contains them all, take a polynomial monogenic M. It is expressible as a sum of monomials multiplied with spinors. Each of those monomials is labelled by its spinors and a multi-index. These are the constituents of Z_s^β and we can thus define an endomorphism of $\mathcal{M}_n(\mathbb{R}^d;V)$ by factoring through $\mathcal{P}_n(\mathbb{R}^d)\otimes V$ as follows:

$$\begin{array}{ccccc} \mathcal{M}_n(\mathbb{R}^d;V) & \hookrightarrow & \mathcal{P}_n(\mathbb{R}^d)\otimes V & \longrightarrow & \mathcal{M}_n(\mathbb{R}^d;V) \\ M & \longmapsto & \sum_{\eta,s} a_\eta^s x^\eta \otimes s & \longmapsto & \sum_{\eta,s} a_\eta^s \mathsf{Z}_s^\eta . \end{array} \tag{4.7}$$

This map is invertible with the inverse given by the projection (3.15)

$$\begin{array}{ccccc} \mathcal{M}_n(\mathbb{R}^d;V) & \longrightarrow & \mathcal{P}_n(\mathbb{R}^d)\otimes V & \longrightarrow & \mathcal{M}_n(\mathbb{R}^d;V) \\ \mathsf{Z}_s^\eta & \longmapsto & x^\eta \otimes s & \longmapsto & \operatorname{proj}_{\mathcal{M}_n}^{\mathcal{P}_n\otimes V}(x^\eta\otimes s). \end{array} \tag{4.8}$$

We now show that the $\dim \mathcal{P}_{n-1}(\mathbb{R}^d) \times \dim V$ relations of the form (4.6) hold. Let $\eta \in \mathbb{N}^d$ with $|\eta|_1 = n-1$. Expand the added $\mathfrak{z}_j$ to the right using Lemma 3.5

$$\begin{aligned} \sum_{j=1}^{d} \mathsf{Z}_{e_j\cdot s}^{\eta+\xi_j} = \sum_{j=1}^{d} \mathfrak{z}^\eta \mathfrak{z}_j e_j s = \mathfrak{z}^\eta \sum_{j=1}^{d} (2\varepsilon(d/2+\gamma)x_j - 2\varepsilon \underline{x} O_j - \underline{x} e_j - \varepsilon|x|^2 \mathcal{D}_j) e_j s \\ = \mathfrak{z}^\eta (2\varepsilon(d/2+\gamma)\underline{x} - 2\varepsilon \underline{x} \sum_{j=1}^{d} O_j e_j - d\varepsilon \underline{x} - \varepsilon |x|^2 \underline{D}) s \\ = \mathfrak{z}^\eta (2\varepsilon(\gamma)\underline{x} - 2\varepsilon \underline{x} \sum_{j=1}^{d} O_j e_j) s = 0, \end{aligned}$$

where Lemma 3.2 was used in the last equality.

Each relation is unique. Indeed, each relation corresponds to the following expression in $\mathcal{A}_\kappa \otimes Cl(d)$ by Proposition 3.9

$$\sum_{j=1}^{d} Z_{e_j \cdot s}^{\eta+\xi_j} = (-1)^n \varepsilon^{n-1} \mathcal{I}_\kappa \mathcal{D}^\eta \underline{D} \mathcal{I}_\kappa(s) = 0. \tag{4.9}$$

The set $\{\mathcal{D}^\eta \underline{D} \mid \eta \in \mathbb{N}^d,\ |\eta|_1 = n-1\}$ contains only linearly independent elements and the Dunkl–Clifford–Kelvin transform is invertible, hence all relations are linearly independent. These are all the relations, as can be seen from the dimensions of the spaces involved. ■

Theorem 4.2 follows by expressing all instances of $\mathfrak{z}_d$ by using equations (4.6). Indeed, $|\mathcal{C}_n| = \dim \mathcal{P}_n(\mathbb{R}^d) \times \dim V$ and there are $\dim \mathcal{P}_{n-1}(\mathbb{R}^d) \times \dim V$ different linear relations between its members; the dimension of $\mathcal{M}_n(\mathbb{R}^d;V)$ is precisely

$$\dim \mathcal{M}_n(\mathbb{R}^d;V) = (\dim \mathcal{P}_n(\mathbb{R}^d) - \dim \mathcal{P}_{n-1}(\mathbb{R}^d)) \times \dim V = \binom{d-1+n-1}{d-1} \times \dim V. \tag{4.10}$$

As the last equation shows, the dimension of $\mathcal{M}_n(\mathbb{R}^d;V)$ is equal to that of the tensor product $\mathcal{P}_n(\mathbb{R}^{d-1}) \otimes V$.

Remark 4.4. *Let $\eta \in \mathbb{N}^d$ with $|\eta|_1 = n-2$. Xu exhibited the relations satisfied by the harmonics of equation* (2.10) *[15, p. 500] (see also [9, pp. 212-213]):*

$$\sum_{j=1}^{d} H_{\eta+2\xi_j} = \mathcal{K}_\kappa \mathcal{D}^\eta \Delta_\kappa \mathcal{K}_\kappa(1) = 0. \tag{4.11}$$

The monogenics satisfy the same relation, as can be seen by applying twice relation (4.6)*, or by viewing it in the Dunkl–Clifford–Kelvin transform*

$$\sum_{j=1}^{d} Z_s^{\eta+2\xi_j} = (-1)^n \varepsilon^{n-1} \mathcal{I}_\kappa \mathcal{D}^\eta \Delta_\kappa \mathcal{I}_\kappa(s) = 0. \tag{4.12}$$

5. Examples: the abelian cases

5.1. **Reducible reflection groups.** In this section, we consider the cases when the reflection group is reducible: $W = W_S \oplus W_T$, for S,T the two root subsystems with $R = S \oplus T$.

Let M be the rank of S. For simplicity, assume that S is restricted to the first M coordinates. We have an $\mathfrak{osp}(1|2)$ realisation given by the following operators

$$\underline{D}_{[M]} := \sum_{a=1}^{M} e_a \mathcal{D}_a, \qquad \underline{x}_{[M]} := \sum_{a=1}^{M} e_a x_a, \qquad \mathbb{E}_{[M]} := \sum_{a=1}^{M} x_a \partial_{x_a}, \quad \gamma_{[M]} := \sum_{\alpha \in S^+} \kappa(\alpha).$$

The odd elements $\underline{D}_{[M]}$ and $\underline{x}_{[M]}$ generate a realisation of the superalgebra $\mathfrak{osp}(1|2)$ with the following commutation relation

$$\left\{\underline{D}_{[M]}, \underline{x}_{[M]}\right\} = 2\varepsilon H_{[M]} = 2\varepsilon(\mathbb{E}_{[M]} + M/2 + \gamma_{[M]}). \tag{5.1}$$

Thus one can also define generalised symmetries in this "sub" $\mathfrak{osp}(1|2)$-realisation.

Definition 5.1. *Let $1 \leq j \leq M$. The partial generalised symmetry linked to $\underline{D}_{[M]}$ is given by*

$$\mathfrak{z}_{j,[M]} := 2\varepsilon x_j H_{[M]} - \underline{x}_{[M]} \mathcal{D}_j \underline{x}_{[M]}. \tag{5.2}$$

Naturally, $\mathfrak{z}_{j,[M]}$ satisfies the equivalent relations of Proposition 3.8, Lemma 3.5, Proposition 3.7 and Proposition 3.9 since it is also in an $\mathfrak{osp}(1|2)$ realisation.

5.2. **The abelian cases.** We turn to the study of the abelian case, so the Dunkl–Dirac symmetry algebra for the group $W = \mathbb{Z}_2^d$ acting on $\mathbb{R}^d$ with a W-invariant function given by the d-tuple of non-negative constants $(\kappa_1, \dots, \kappa_d)$. In the abelian case, the reflection σ_j sends x_j to $-x_j$ and leaves the other variables invariant. The Dunkl operators are given by

$$\mathcal{D}_i = \partial_{x_i} + \kappa_i \frac{1-\sigma_i}{x_i}, \tag{5.3}$$

and the commutation relation (2.4) becomes

$$[\mathcal{D}_i, x_j] = \delta_{ij}(1+2\kappa_i). \tag{5.4}$$

Albeit Theorem 4.2 gives a basis of $\mathcal{M}_n$, the specificity's of the group studied call for a slightly different approach. The complete reducibility of $W = \mathbb{Z}_2^d$ was used to construct the Cauchy–Kovalevskaya basis of [5]. We will retrieve this construction from the generalised symmetries (5.2).

5.3. **The Cauchy–Kovalevskaya basis.** In the abelian case, there exists a generalisation of the Cauchy-Kovalevskaya map. It can be used to construct a basis of the polynomial monogenics.

Proposition 5.2 ([5, Eq. (31)])**.** *Let V be an irreducible representation of the Clifford algebra $Cl(d)$. There is an isomorphism between the space of spinor-valued polynomials of degree n over $k-1$ variables and the monogenics of degree n over k variables given by*

$$\begin{gathered} \mathbf{CK}_{x_k}^{\kappa_k} : \mathcal{P}_n(\mathbb{R}^{k-1}) \otimes V \longrightarrow \mathcal{M}_n(\mathbb{R}^k; V) \\ p \mapsto \mathbf{CK}_{x_k}^{\kappa_k}(p) = \sum_{a=0}^{\lfloor n/2 \rfloor} \frac{x_k^{2a} \underline{D}_{[k-1]}^{2a}}{2^{2a} a! (\kappa_k + 1/2)_a} p - \varepsilon \frac{e_k x_k \underline{D}_{[k-1]}}{2} \sum_{a=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{x_k^{2a} \underline{D}_{[k-1]}^{2a}}{2^{2a} a! (\kappa_k + 1/2)_{a+1}} p. \end{gathered} \tag{5.5}$$

Note that in [5], the proposition is given for $\varepsilon = -1$. The proof for the two signs is the same up to minor modifications.

The map $\mathbf{CK}_{x_k}^{\kappa_k}$ is an isomorphism and has an inverse given by the map evaluating the last variable to 0:

$$\begin{gathered} R_k : \mathcal{M}_n(\mathbb{R}^k; V) \longrightarrow \mathcal{P}_n(\mathbb{R}^{k-1}) \otimes V \\ f(x_1, \dots, x_k) \longmapsto R_k(f)(x_1, \dots, x_{k-1}) = f(x_1, \dots, x_{k-1}, 0). \end{gathered} \tag{5.6}$$

Consider now the Fischer decomposition of the polynomial space.

Proposition 5.3 (Fischer decomposition [2])**.** *When κ is positive, the space of spinor-valued polynomials decomposes in monogenic spaces as*

$$\mathcal{P}_n(\mathbb{R}^d) \otimes V = \bigoplus_{k=0}^{n} \underline{x}_{[d]}^{n-k} \mathcal{M}_k(\mathbb{R}^d; V). \tag{5.7}$$

From this proposition and the tower of CK extensions and Fischer decompositions (see for example [5, Eq. (32)]), we get a basis of the space of monogenics.

Proposition 5.4 ([5, Prop. 6])**.** *Let $\{s\}_{s \in V}$ be a basis of the spinor representation V. The set of functions defined for all $\mathbf{j} = (j_1, \dots, j_{d-1}, 0) \in \mathbb{N}^d$ with $|\mathbf{j}|_1 = n$,*

$$\Psi_s^{\mathbf{j}}(x_1, \dots, x_d) = \mathbf{CK}_{x_d}^{\kappa_d} \left(\underline{x}_{d-1}^{j_{d-1}} \mathbf{CK}_{x_{d-1}}^{\kappa_{d-1}} \left(\dots \mathbf{CK}_{x_3}^{\kappa_3} \left(\underline{x}_2^{j_2} \mathbf{CK}_{x_2}^{\kappa_2} (x_1^{j_1}) \right) \dots \right) \right) s, \tag{5.8}$$

is a basis of $\mathcal{M}_n(\mathbb{R}^d;V)$.

5.4. **A new basis.** We will use the partial generalised symmetries of Subsection 5.1 to make full use of the completely reducible nature of $\mathbb{Z}_2^d$. The crucial point of the abelian case $W=\mathbb{Z}_2^d$ is a chain of inclusions

$$\mathbb{Z}_2 \subset \mathbb{Z}_2^2 \subset \cdots \subset \mathbb{Z}_2^{d-1} \subset \mathbb{Z}_2^d. \tag{5.9}$$

This gives in turn a tower of $\mathfrak{osp}(1|2)$ algebra realisations given by the pairs $(\underline{D}_{[k]}, \underline{x}_{[k]})$ for each $1 \le k \le d$. This feature of the group $\mathbb{Z}_2^d$ was used in Proposition 5.4 to give a basis. We give a basis proportional to the CK basis by replacing the operators $\mathfrak{z}_j$ in Theorem 4.2 by the partial ones $\mathfrak{z}_{j,[j]}$. This is done by linking the Cauchy-Kovalevskaya extension of each level to one partial generalised symmetries.

An important note, the commutation of the $\mathfrak{z}_{j,[j]}$ requires that they stay on the same level. Indeed, two partial generalised symmetries at a different level in the tower do not commute in general. So in the basis of the following proposition, the order of application matters.

Proposition 5.5. *The set of polynomials of the form*

$$\Phi_s^{\mathbf{j}} := \mathfrak{z}_{d,[d]}^{j_{d-1}} \mathfrak{z}_{d-1,[d-1]}^{j_{d-2}} \cdots \mathfrak{z}_{2,[2]}^{j_1} s, \quad \text{for } \mathbf{j}=(j_1,\dots j_{d-1},0)\in\mathbb{N}^d,\ |\mathbf{j}|_1=n, \text{ and } s\in v \tag{5.10}$$

constitutes a basis of $\mathcal{M}_n(\mathbb{R}^d;V)$.

We will give in Proposition 5.11 the change of basis from the $\Psi_s^{\mathbf{j}}$ to the $\Phi_s^{\mathbf{j}}$. To do so we will prove that, as operators, $\mathfrak{z}_{k,[k]}^{j}$ and $\mathbf{CK}_{x_k}^{\kappa_k}\underline{x}_{[k-1]}^{j}$ are Clifford proportional, meaning that they differ only by a Clifford number. There is a small difference between $k=2$ and $k>2$ and thus we separate the proof in two steps.

We begin by showing what will constitute the hard part of the induction proof of Proposition 5.7.

Lemma 5.6. *The* $\mathfrak{z}_{2,[2]}$ *operator and the* $\mathbf{CK}_{x_2}^{\kappa_2}$ *extension are linked by*

$$\mathfrak{z}_{2,[2]}\mathbf{CK}_{x_2}^{\kappa_2} x_1^m s = A_m \mathbf{CK}_{x_2}^{\kappa_2} x_1^{m+1} e_2 e_1 s, \tag{5.11}$$

with

$$A_m = 1+m+(1-(-1)^m)\kappa_1 + 2\kappa_2. \tag{5.12}$$

Proof. Recall that R_2 is the inverse of $\mathbf{CK}_{x_2}^{\kappa_2}$. Acting with R_2 on (5.11) thus yields

$$R_2\mathfrak{z}_{2,[2]}\mathbf{CK}_{x_2}^{\kappa_2} x_1^m s = A_m x_1^{m+1} e_2 e_1 s, \tag{5.13}$$

so it suffices to compute the left-hand side of (5.13). Begin by using the expression (3.21) of $\mathfrak{z}_{2,[2]}$

$$R_2(\mathfrak{z}_{2,[2]}\mathbf{CK}_{x_2}^{\kappa_2}(x_1^m s)) = R_2((2\varepsilon x_2(\mathbb{E}_2+1+\gamma_2) - \underline{x}_2(1+2\kappa_2\sigma_2)e_2 - \varepsilon|x|_2^2\mathcal{D}_2)\mathbf{CK}_{x_2}^{\kappa_2}(x_1^m s)).$$

Since R_2 sends x_2 to 0, this reduces to

$$\begin{aligned} &= (-x_1 e_1(1+2\kappa_2\sigma_2)e_2)x_1^m s + \varepsilon^2 x_1^2 \mathcal{D}_2 \frac{x_2 e_2 \mathcal{D}_1 e_1}{2}\frac{x_1^m s}{(1/2+\kappa_2)} \\ &= ((1+2\kappa_2\sigma_2)e_2 e_1)x_1^{m+1}s + x_1^2(1+2\kappa_2\sigma_2)\frac{\mathcal{D}_1 x_1^m e_2 e_1 s}{(1+2\kappa_2)}, \end{aligned}$$

and now we apply $\mathcal{D}_1 x_1^m s = (m+\kappa_1(1-(-1)^m))x_1^{m-1}s$, since $\mathcal{D}_1 s = 0$, to obtain

$$= (1+m+2\kappa_2+(1-(-1)^m)\kappa_1)x_1^{m+1}e_2e_1 s.$$

∎

Using this lemma, we can prove the general proposition.

Proposition 5.7. *Acting on a spinor s, we have*

$$\mathfrak{z}_{2,[2]}^j s = a_2^j \mathbf{CK}_{x_2}^{\kappa_2} x_1^j (e_2e_1)^j s, \tag{5.14}$$

with

$$a_2^j := 2^j(\kappa_2+1/2)_{\lfloor (j+1)/2 \rfloor}(\gamma_2+1)_{\lfloor j/2 \rfloor}. \tag{5.15}$$

Proof. We proceed by induction on j, the case $j=1$ being covered by Lemma 5.6 with $m=0$. Assume the induction hypothesis holds up to $j=m$. Now we consider the $(m+1)$th step and apply the induction hypothesis

$$\mathfrak{z}_{2,[2]}^{m+1} s = \mathfrak{z}_{2,[2]}\mathfrak{z}_{2,[2]}^{m} s = \mathfrak{z}_{2,[2]} a_2^m \mathbf{CK}_{x_2}^{\kappa_2} x_1^m (e_2e_1)^m s), \tag{5.16}$$

then we apply Lemma 5.6 to get $\mathfrak{z}_{2,[2]}^{m+1} s = a_2^{m+1}\mathbf{CK}_{x_2}^{\kappa_2} x_1^{m+1}(e_2e_1)^{m+1}s$ since $A_m a_2^m = a_2^{m+1}$. ∎

In general, for $k>2$, there is one additional difficulty: the CK map includes not only Dunkl derivatives, but also partial Dunkl–Dirac operators. We will thus need a small computation for the future.

Lemma 5.8 ([5, Lem. 13])**.** *Let $f \in \mathcal{M}_n(\mathbb{R}^k;V)$. The action of $\underline{D}_{[k]}$ on $\underline{x}_{[k]}^m f$ is given by*

$$\underline{D}_{[k]}\underline{x}_{[k]}^m f = \varepsilon(m+\frac{(1-(-1)^m)}{2}(2n+k-1+2\gamma_k))\underline{x}_{[k]}^{m-1})f. \tag{5.17}$$

Proof. Proceed by induction on m, first for even m. The base case $m=2$ comes from the commutation relation $\left[\underline{D}_{[k]}, |x|_k^2\right] = 2|x|_k^2$, equation (3.10). The induction step follows then from

$$\underline{D}_{[k]}\underline{x}_{[k]}^m f = \underline{D}_{[k]}\underline{x}_{[k]}^2\underline{x}_{[k]}^{m-2} f = \underline{x}_{[k]}^2\underline{D}_{[k]}\underline{x}_{[k]}^{m-2} f + 2\varepsilon\underline{x}_{[k]}^{m-1} f = \varepsilon m\underline{x}_{[k]}^{m-1} f. \tag{5.18}$$

The base case for odd m follows from the anticommutation relation $\left\{\underline{D}_{[k]}, \underline{x}_{[k]}\right\} = 2\varepsilon(\mathbb{E}_k + k/2+\gamma_k)$, $\underline{D}_{[k]} f = 0$ and $\mathbb{E}_k f = nf$. The induction step is then achieved by one application of equation (3.10)

$$\underline{D}_{[k]}\underline{x}_{[k]}^m f = \underline{D}_{[k]}\underline{x}_{[k]}^2\underline{x}_{[k]}^{m-2} f = \underline{x}_{[k]}^2\underline{D}_{[k]}\underline{x}_{[k]}^{m-2} f + 2\varepsilon\underline{x}_{[k]}^{m-1} f = \varepsilon(2n+m+k-1+2\gamma_k)\underline{x}_{[k]}^{m-1} f. \tag{5.19}$$

∎

Now to prove the relation between the partial generalised symmetry and the CK map for the other levels of the tower, we introduce a lemma that takes care of the difficult induction step.

Lemma 5.9. *Let $f \in \mathcal{M}_n(\mathbb{R}^{k-1};V)$ be a monogenic of degree n in the first $k-1$ variables. Then*

$$\mathfrak{z}_{k,[k]}\mathbf{CK}_{x_k}^{\kappa_k}\underline{x}_{[k-1]}^m f = B_{k,n}^m \mathbf{CK}_{x_k}^{\kappa_k}\underline{x}_{[k-1]}^{m+1} e_k f, \tag{5.20}$$

with

$$B_{k,n}^m = (-1)^{m+1}(m+1+\frac{(1-(-1)^m)}{2}(2n+k-2+2\gamma_{k-1})+2\kappa_k). \tag{5.21}$$

Proof. The proof proceeds in the same fashion as the one of Lemma 5.6, using $\mathbb{E}_{k-1}f = nf$, $\underline{D}_{k-1}f = 0$ and $\mathcal{D}_k f = 0$ instead of $\mathbb{E}s = 0$, $\mathcal{D}_1 s = 0$ and $\mathcal{D}_2 s = 0$ in the corresponding steps.

The map $\mathbf{CK}^{\kappa_k}_{x_k}$ has an inverse R_k defined as evaluating x_k to 0. We compute, using (5.5),

$$
\begin{aligned}
R_k(\mathfrak{z}_{k,[k]}\mathbf{CK}^{\kappa_k}_{x_k}\underline{x}^m_{[k-1]}f) &= R_k(2\varepsilon x_k(\mathbb{E}_k + k/2 + \gamma_k) - \underline{x}_{[k]}(1+2\kappa_k\sigma_k)e_k - \varepsilon|x|^2_k\mathcal{D}_k)\mathbf{CK}^{\kappa_k}_{x_k}\underline{x}^m_{[k-1]}f \\
&= -\underline{x}_{[k-1]}(1+2\kappa_k\sigma_k)e_k\underline{x}^m_{[k-1]}f + \varepsilon^2|x|^2_{k-1}\mathcal{D}_k\frac{e_kx_k\underline{D}_{[k-1]}}{2(\kappa_k+1/2)}\underline{x}^n_{[k-1]}f \\
&= (-1)^{m+1}(1+2\kappa_k)\underline{x}^{m+1}_{[k-1]}e_kf + |x|^2_{k-1}e_k\frac{[\mathcal{D}_k, x_k]}{2(\kappa_k+1/2)}\underline{D}_{[k-1]}\underline{x}^n_{[k-1]}f
\end{aligned}
$$

and we use Lemma 5.8 for $f \in \mathcal{M}_n(\mathbb{R}^{k-1}; V)$ on the rightmost term to get

$$
\begin{aligned}
&= (-1)^{m+1}(1+2\kappa_k)\underline{x}^{m+1}_{[k-1]}e_kf \\
&\quad + \varepsilon(m + \frac{(1-(-1)^m)}{2}(2n+k-2+2\gamma_{k-1})|x|^2_{k-1}e_k\underline{x}^{m-1}_{[k-1]}f \\
&= (-1)^{m+1}(1+2\kappa_k + \varepsilon^2(m+\frac{(1-(-1)^m)}{2}(2n+k-2+2\gamma_{k-1})))\underline{x}^{m+1}_{[k-1]}e_kf.
\end{aligned}
$$

This allows to read the constant $B^m_{k,n}$ by comparing with (5.20). ∎

Proposition 5.10. *Let $f \in \mathcal{M}_n(\mathbb{R}^{k-1}; V)$ be a monogenic in $k-1$ variables of degree n. For $k > 2$,*

$$
\mathfrak{z}^j_{k,[k]}f = b^j_{k,n}\mathbf{CK}^{\kappa_k}_{x_k}\underline{x}^j_{[k-1]}e^j_kf, \tag{5.22}
$$

with

$$
b^j_{k,n} = (-1)^{\lfloor (j+1)/2\rfloor}2^j(\kappa_k+1/2)_{\lfloor (j+1)/2\rfloor}(\gamma_k+n+k/2)_{\lfloor j/2\rfloor}. \tag{5.23}
$$

Proof. We proceed by induction on j. The base case follows from Lemma 5.9 with $m = 0$. Assume the induction hypothesis holds up to j. The induction step follows from the induction hypothesis and Lemma 5.9

$$
\mathfrak{z}^{j+1}_{k,[k]}f = \mathfrak{z}_{k,[k]}\mathfrak{z}^j_{k,[k]}f = \mathfrak{z}_{k,[k]}b^j_{k,n}\mathbf{CK}^{\kappa_k}_{x_k}\underline{x}^j_{k-1}e^j_kf = B^j_{k,n}b^j_{k,n}\mathbf{CK}^{\kappa_k}_{x_k}\underline{x}^{j+1}_{k-1}e^{j+1}_kf. \tag{5.24}
$$

This shows the result since $B^j_{k,n}b^j_{k,n} = b^{j+1}_{k,n}$. ∎

Connecting this to the CK basis, we get the following correspondence.

Proposition 5.11. *Let $\mathbf{j} = (j_1, \ldots, j_{d-1}, 0) \in \mathbb{N}^d$ with $|\mathbf{j}|_1 = n$ and s be a spinor. The partial generalised symmetry basis is linked to the CK basis by*

$$
\Phi^{\mathbf{j}}_s = c_{\mathbf{j}}\Psi^{\mathbf{j}}_{\mathbf{j}\cdot s}, \tag{5.25}
$$

where the action $\mathbf{j}\cdot s$ in $\Psi^{\mathbf{j}}_{\mathbf{j}\cdot s}$ denotes the action $e_d^{j_{d-1}}\ldots e_3^{j_2}(e_2e_1)^{j_1}s$ on the spinor space in the expression of the polynomial Ψ and where the proportionality constant is given by

$$
\begin{aligned}
c_{\mathbf{j}} = &\left(\prod_{k=3}^{d-1}\prod_{l=2}^{k-1}(-1)^{j_kj_l}\right)2^n(1/2+\kappa_2)_{\lfloor (j_1+1)/2\rfloor}(1+\gamma_2)_{\lfloor j_1/2\rfloor}\times \\
&\prod_{i=2}^{d-1}((-1)^{\lfloor (j_i+1)/2\rfloor}(1/2+\kappa_{i+1})_{\lfloor (j_i+1)/2\rfloor}((i+1)/2+\gamma_{i+1}+\sum_{k=1}^{i-1}j_k)_{\lfloor j_i/2\rfloor}).
\end{aligned} \tag{5.26}
$$

Proof. The first steps are to apply one time Proposition 5.7 for the $\mathfrak{z}_{2,[2]}$ contribution and multiple times Proposition 5.10 for the remaining contributions of the $\mathfrak{z}_{k,[k]}$. This will give $c_{\mathbf{j}}$ up to the first sign. This sign is obtained when Clifford elements go to the right from their interaction with the vector variables. Note that e_k commutes with $\mathbf{CK}_{x_l}^{\kappa_l}$ when $k > l$, as can be clearly seen from the expression (5.5), so the only sign to consider is from the vector variable crossing. Step by step, this gives

$$\begin{aligned}
\Phi_{\mathbf{j}}^{s} &= \mathfrak{z}_{d,[d]}^{j_{d-1}}\mathfrak{z}_{d-1,[d-1]}^{j_{d-2}}\cdots\mathfrak{z}_{2,[2]}^{j_1}s\\
\text{(Prop. 5.7)}\quad &= a_2^{j_1}\mathfrak{z}_{d,[d]}^{j_{d-1}}\mathfrak{z}_{d-1,[d-1]}^{j_{d-2}}\cdots\mathfrak{z}_{3,[3]}^{j_2}\mathbf{CK}_{x_2}^{\kappa_2}(x_1^{j_1}(e_2e_1)^{j_1}s\\
\text{(Prop. 5.10)}\quad &= a_2^{j_1}\prod_{k=2}^{d-1}(b_{k+1,\sum_{j=1}^{k-1}j_k}^{j_k})\mathbf{CK}_{x_d}^{\kappa_d}(\underline{x}_{[d-1]}^{j_d}e_d^{j_{d-1}}\mathbf{CK}_{x_{d-1}}^{\kappa_{d-1}}\cdots e_3^{j_2}\mathbf{CK}_{x_2}^{\kappa_2}(x_1^{j_1}(e_2e_1)^{j_1}s))\\
&= \left(\prod_{k=3}^{d-1}\prod_{l=2}^{k-1}(-1)^{j_kj_l}\right)a_2^{j_1}\prod_{k=2}^{d-1}(b_{k+1,\sum_{j=1}^{k-1}j_k}^{j_k})\mathbf{CK}_{x_d}^{\kappa_d}(\underline{x}_{[d-1]}^{j_d}\cdots\mathbf{CK}_{x_2}^{\kappa_2}(x_1^{j_1}e_d^{j_{d-1}}\cdots(e_2e_1)^{j_1}s))\\
&= c_{\mathbf{j}}\Psi_{\mathbf{j}}^{\mathbf{j}\cdot s}.
\end{aligned}$$

■

With this last proposition, we have proved Proposition 5.5.

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REFERENCES

[1] F. A. Berezin and V. N. Tolstoy. The group with Grassmann structure UOSP(1.2). *Communications in Mathematical Physics*, 78(3):409–428, 1981.

[2] F. Brackx, R. Delanghe, and F. Sommen. *Clifford analysis*, volume 76 of *Research Notes in Mathematics*. Pitman Books Limited, 1982.

[3] K. Coulembier and H. De Bie. Conformal symmetries of the super Dirac operator. *Rev. Mat. Iberoam.*, 31(2):373–410, 2015. Number: 2.

[4] H. De Bie, N. De Schepper, and D. Eelbode. New Results on the Radially Deformed Dirac Operator. *Complex Analysis and Operator Theory*, 11(6):1283–1307, 2017.

[5] H. De Bie, V. X. Genest, and L. Vinet. The $\mathbb{Z}_2^n$ Dirac–Dunkl operator and a higher rank Bannai–Ito algebra. *Advances in Mathematics*, 303:390–414, Nov. 2016.

[6] H. De Bie, A. Langlois-Rémillard, R. Oste, and J. Van der Jeugt. Finite-dimensional representations of the symmetry algebra of the dihedral Dunkl–Dirac operator. *Journal of Algebra*, 591:170–216, 2022. arXiv: 2010.03381.

[7] H. De Bie, R. Oste, and J. Van der Jeugt. On the algebra of symmetries of Laplace and Dirac operators. *Lett. Math. Phys.*, 108(8):1905–1953, 2018.

[8] C. F. Dunkl. Differential-Difference Operators Associated to Reflection Groups. *Trans. Amer. Math. Soc.*, 311(1):167–183, 1989.

[9] C. F. Dunkl and Y. Xu. *Orthogonal polynomials of several variables*. Number 155. Cambridge University Press, 2014.

[10] M. Fei, P. Cerejeiras, and U. Kähler. Fueter's theorem and its generalizations in Dunkl–Clifford analysis. *Journal of Physics A: Mathematical and Theoretical*, 42(39):395209, 2009.

[11] A. O. Morris. Projective Representations of Reflection Groups. *Proc. Lond. Math. Soc.*, s3-32(3):403–420, 1976.

[12] C. Müller. *Analysis of Spherical Symmetries in Euclidean Spaces*. Applied Mathematical Sciences. Springer-Verlag, New York, 1998.

[13] B. Ørsted, P. Somberg, and V. Souček. The Howe Duality for the Dunkl Version of the Dirac Operator. *Adv. Appl. Clifford Algebr.*, 19(2):403–415, July 2009.

[14] R. Rouquier. Representations of rational Cherednik algebras. *Contemp. Math.*, 392:103, 2005.

[15] Y. Xu. Harmonic Polynomials Associated With Reflection Groups. *Canadian Mathematical Bulletin*, 43(4):496–507, 2000.

[16] C. Yacoub. On the Dunkl Version of Monogenic Polynomials. *Advances in Applied Clifford Algebras*, 21(4):839–847, 2011.

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