Representations of the Lie Superalgebra $gl(1|n)$ and Wigner Quantum Oscillators

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Abstract. An explicit construction of all finite-dimensional irreducible representations of the Lie superalgebra $gl(1|n)$ in a Gel'fand-Zetlin basis is given. The notion of Wigner Quantum Oscillators (WQOs) is recalled. The star type I representations of $gl(1|n)$ are physical state spaces of the WQO. These solutions have remarkable properties following from the spectrum of the Hamiltonian and of the position and momentum operators.

1 Introduction

We investigate the properties of an N-particle D-dimensional Wigner Quantum Oscillator (WQO) whose mathematical background is the Lie superalgebra (LS) $gl(1|DN)$. After the construction of all finite dimensional irreducible representations of $gl(1|n)$ we consider the consequences for all representations of physical relevance – the star type I representations. For more details on the physical properties of the $gl(1|DN)$ WQOs see [1]

In Section 2 we construct all the finite-dimensional irreducible representations of the Lie superalgebra $gl(1|n)$ with a specification of the GZ basis vectors $|m)$ and the explicit action of a set of $gl(1|n)$ generators on these vectors. The WQO is introduced in Section 3. The WQO requirement that Hamilton's equations and the Heisenberg equations coincide as operator equations leads to compatibility conditions on the creation and ahhilation operators of the oscillator that have a non-canonical solution allowing them to be identified with the odd generators, e_{j0} and e_{0j}, of $gl(1|n)$. The latter is used to determine the physical properties of WQO models, including their energy spectrum and the eigenvalues of their spatial coordinate operators.

2 The $gl(1|n)$ representations

As a basis in $gl(1|n)$ we choose the Weyl matrices $e_{ij},\ i,j=0,1,\ldots,n$, where the odd elements are $\{e_{i0},e_{0i}|i=1,\ldots,n\}$, and the remaining elements are even. The Lie superalgebra bracket is determined by

$$[\![e_{ij},e_{kl}]\!]\equiv e_{ij}e_{kl}-(-1)^{\deg(e_{ij})\deg(e_{kl})}e_{kl}e_{ij}=\delta_{jk}e_{il}-(-1)^{\deg(e_{ij})\deg(e_{kl})}\delta_{il}e_{kj}.$$

The finite-dimensional simple modules of $gl(1|n)$ are characterized by their highest weight Λ [2] with coordinates $[m]_{n+1}=[m_{0,n+1},m_{1,n+1},\ldots,m_{n,n+1}]$, for which $m_{i,n+1}-m_{j,n+1}\in Z_+,\ \forall i\le j=1,\ldots,n$.

Proposition 1 *Consider the $gl(1|n)$ module $W([m]_{n+1})$ as a $gl(n)$ module. Then $W([m]_{n+1})$ can be represented as a direct sum of simple $gl(n)$ modules: $W([m]_{n+1})=\sum_i\oplus V_i([m]_n)$, where*

I. All $V_i([m]_n)$ carry inequivalent representations of $gl(n)$ $[m]_n=[m_{1n},m_{2n},\ldots,m_{nn}]$, $m_{in}-m_{i+1,n}\in Z_+$.

II. 1. $m_{in}-m_{i,n+1}=\theta_i\in\{0,1\},\quad 1\le i\le n$,

2. if for $k\in\{1,\ldots,n\}$ $m_{0,n+1}+m_{k,n+1}=k-1$, then $\theta_k=0$.

Proposition 1 follows from the character formula for simple $gl(1|n)$ modules [3]. If for some $k\in\{1,\ldots,n\}$ the condition $m_{0,n+1}+m_{k,n+1}=k-1$ is satisfied, then the representation is *atypical of type k*. Otherwise, it is typical. A GZ-basis for the $gl(n)$ module is well known [4]. Using it and Proposition 1 we have

Proposition 2 *The set of vectors*

$$|m)=\left|\begin{array}{cccccc} m_{0,n+1} & m_{1,n+1} & \cdots & m_{n-2,n+1} & m_{n-1,n+1} & m_{n,n+1}\\ & m_{1n} & \cdots & \cdots & m_{n-1,n} & m_{nn}\\ & m_{1,n-1} & \cdots & \cdots & m_{n-1,n-1} & \\ & \vdots & \cdot^{\cdot^{\cdot}} & & & \\ & m_{11} & & & & \end{array}\right) \qquad (1)$$

satisfying the conditions

1. $m_{i,n+1}$ are fixed and $m_{i,n+1}-m_{j,n+1}\in Z_+\quad 1\le i\le j\le n$,

2. $m_{in}-m_{i,n+1}=\theta_i\in\{0,1\},\quad 1\le i\le n$,

3. if for $k\in\{1,\ldots,n\}$ $m_{0,n+1}+m_{k,n+1}=k-1$, then $\theta_k=0$,

4. $m_{i,j+1}-m_{ij}\in Z_+$ and $m_{i,j}-m_{i+1,j+1}\in Z_+,\quad 1\le i\le j\le n-1$

constitute a basis in $W([m]_{n+1})$.

The action of a set of $gl(1|n)$ generators is given by:

$$e_{00}|m) = \left(m_{0,n+1} - \sum_{j=1}^{n}\theta_j\right)|m);\ e_{kk}|m) = \left(\sum_{j=1}^{k} m_{jk} - \sum_{j=1}^{k-1} m_{j,k-1}\right)|m),\quad 1\le k\le n;$$

$$e_{k-1,k}|m) = \sum_{j=1}^{k-1}\left(-\frac{\prod_{i=1}^{k}(l_{ik}-l_{j,k-1})\prod_{i=1}^{k-2}(l_{i,k-2}-l_{j,k-1}-1)}{\prod_{i\neq j=1}^{k-1}(l_{i,k-1}-l_{j,k-1})(l_{i,k-1}-l_{j,k-1}-1)}\right)^{1/2}|m)_{+j,k-1},\quad 2\le k\le n;$$

$$e_{k,k-1}|m) = \sum_{j=1}^{k-1}\left(-\frac{\prod_{i=1}^{k}(l_{ik}-l_{j,k-1}+1)\prod_{i=1}^{k-2}(l_{i,k-2}-l_{j,k-1})}{\prod_{i\neq j=1}^{k-1}(l_{i,k-1}-l_{j,k-1})(l_{i,k-1}-l_{j,k-1}+1)}\right)^{1/2}|m)_{-j,k-1},\quad 2\le k\le n;$$

$$\begin{aligned}
e_{0j}|m) &= \sum_{i_n=1}^{n}\sum_{i_{n-1}=1}^{n-1}\cdots\sum_{i_j=1}^{j}\theta_{i_n}(-1)^{\theta_1+\ldots+\theta_{i_n-1}}(l_{i_n,n+1}+l_{0,n+1}+1)^{1/2}\\
&\times\prod_{r=j+1}^{n}S(i_r,i_{r-1})\left(\frac{\prod_{k\neq i_{r-1}=1}^{r-1}(l_{k,r-1}-l_{i_r,r})\prod_{k\neq i_r=1}^{r}(l_{kr}-l_{i_{r-1},r-1}+1)}{\prod_{k\neq i_r=1}^{r}(l_{kr}-l_{i_r,r})\prod_{k\neq i_{r-1}=1}^{r-1}(l_{k,r-1}-l_{i_{r-1},r-1}+1)}\right)^{1/2}\\
&\times\left(\prod_{k\neq i_n=1}^{n}\frac{(l_{kn}-l_{i_n,n})}{(l_{k,n+1}-l_{i_n,n+1})}\right)^{1/2}\left(\frac{\prod_{k=1}^{j-1}(l_{k,j-1}-l_{i_j,j})}{\prod_{k\neq i_j=1}^{j}(l_{kj}-l_{i_j,j})}\right)^{1/2}\\
&\times|m)_{-i_n,n;-i_{n-1},n-1;\ldots;-i_j,j},\qquad 1\le j\le n;
\end{aligned}$$

$$\begin{aligned}
e_{j0}|m) &= \sum_{i_n=1}^{n}\sum_{i_{n-1}=1}^{n-1}\cdots\sum_{i_j=1}^{j}(1-\theta_{i_n})(-1)^{\theta_1+\ldots+\theta_{i_n-1}}(l_{i_n,n+1}+l_{0,n+1}+1)^{1/2}\\
&\times\prod_{r=j+1}^{n}S(i_r,i_{r-1})\left(\frac{\prod_{k\neq i_{r-1}=1}^{r-1}(l_{k,r-1}-l_{i_r,r}-1)\prod_{k\neq i_r=1}^{r}(l_{kr}-l_{i_{r-1},r-1})}{\prod_{k\neq i_r=1}^{r}(l_{kr}-l_{i_r,r})\prod_{k\neq i_{r-1}=1}^{r-1}(l_{k,r-1}-l_{i_{r-1},r-1}-1)}\right)^{1/2}\\
&\times\left(\prod_{k\neq i_n=1}^{n}\frac{(l_{kn}-l_{i_n,n})}{(l_{k,n+1}-l_{i_n,n+1})}\right)^{1/2}\left(\frac{\prod_{k=1}^{j-1}(l_{k,j-1}-l_{i_j,j}-1)}{\prod_{k\neq i_j=1}^{j}(l_{kj}-l_{i_j,j})}\right)^{1/2}\\
&\times|m)_{+i_n,n;+i_{n-1},n-1;\ldots;+i_j,j},\qquad 1\le j\le n;
\end{aligned}$$

where $l_{ij} = m_{ij} - i$; a symbol $\pm i_k, k$ attached as a subscript to $|m)$ indicates a replacement $m_{i_k,k} \to m_{i_k,k} \pm 1$, and $S(k,l) = \begin{cases} 1 & \text{for} \quad k\le l \\ -1 & \text{for} \quad k>l. \end{cases}$

In order to deduce the above formulas, we have used the paper of Palev [5] and the fact that $gl(n|1)$ and $gl(1|n)$ are isomorphic.

The representations of physical relevance are the *star type I* representations classified in [6].

Proposition 3 *The representation* $W([m]_{n+1})$ *is a star type I representation if and only if*

(a) The highest weight is real and $m_{0,n+1}+m_{n,n+1}-n+1>0$. *In this case, the representation is typical.*

(b) The highest weight is real and there exists a $k\in\{1,2,\ldots,n\}$ *such that* $m_{0,n+1}+m_{k,n+1}=k-1, \quad m_{k,n+1}=m_{k+1,n+1}=\cdots=m_{n,n+1}$. *In this case, the representation is atypical of type k.*

3 The N-particle D-dimensional WQO

Let $\hat{H}$ be the Hamiltonian of an N-particle D-dimensional oscillator:

$$\hat{H}=\sum_{\alpha=1}^{N}\Big(\frac{\hat{\mathbf{P}}_\alpha^2}{2m}+\frac{m\omega^2}{2}\hat{\mathbf{R}}_\alpha^2\Big). \tag{2}$$

We consider this oscillator as a Wigner quantum system [7]: this means that the canonical commutation relations are not required, but are replaced by compatibility conditions between Hamilton's equations and the Heisenberg equations. These compatibility conditions are such that

$$[\hat{H},\hat{\mathbf{P}}_\alpha]=i\hbar m\omega^2\hat{\mathbf{R}}_\alpha,\ \ [\hat{H},\hat{\mathbf{R}}_\alpha]=-\frac{i\hbar}{m}\hat{\mathbf{P}}_\alpha \ \text{ for }\ \alpha=1,2,\ldots,N. \tag{3}$$

Write the operators $\hat{\mathbf{P}}_\alpha$ and $\hat{\mathbf{R}}_\alpha$ for $\alpha=1,2,\ldots,N$ in terms of new operators (creation and annihilation operators):

$$A^{\pm}_{D(\alpha-1)+k}=\sqrt{\frac{(DN-1)m\omega}{4\hbar}}\hat{R}_{\alpha k}\pm i\sqrt{\frac{(DN-1)}{4m\omega\hbar}}\hat{P}_{\alpha k}, \quad k=1,\ldots,D. \tag{4}$$

The Hamiltonian $\hat{H}$ and the compatibility conditions take the form:

$$\hat{H}=\frac{\omega\hbar}{DN-1}\sum_{j=1}^{DN}\{A_j^+,A_j^-\},\ \sum_{j=1}^{DN}[\{A_j^+,A_j^-\},A_i^{\pm}]=\mp(DN-1)A_i^{\pm},\ i=1,\ldots,DN.$$

As a solution of the compatibility conditions one can choose:
$[\{A_i^+,A_j^-\},A_k^+]=\delta_{jk}A_i^+-\delta_{ij}A_k^+,\ [\{A_i^+,A_j^-\},A_k^-]=-\delta_{ik}A_j^-+\delta_{ij}A_k^-,$
$\{A_i^+,A_j^+\}=\{A_i^-,A_j^-\}=0.$

Proposition 4 *The operators* $A_j^{\pm}$, *for* $j=1,2,\ldots,DN$, *are the odd elements of* $gl(1|DN)$: $A_j^+=e_{j0}$, $A_j^-=e_{0j}$.

The Hilbert space (state space) of the WQO is a star type I representation space W of the Lie superalgebra $gl(1|DN)$. The Hamiltonian $\hat{H}$ is diagonal in the GZ-basis, i.e. $|m)$ are stationary states of the system.

$$\hat{H}|m) = E_q|m)\; E_q = \hbar\omega\left(\frac{nm_{0,n+1}+m_{1,n+1}+\ldots+m_{n,n+1}}{n-1}-q\right),\; q=\sum_{j=1}^{n}\theta_j.$$

The position operators $\hat{R}_{\alpha k}$ ($\alpha = 1,\ldots,N$, $k = 1,\ldots,D$) do not commute with each other $[\hat{R}_{\alpha i}, \hat{R}_{\beta j}] \neq 0$ for $\alpha i \neq \beta j$. Similarly $[\hat{P}_{\alpha i}, \hat{P}_{\beta j}] \neq 0$ for $\alpha i \neq \beta j$. These imply that the WQO belongs to the class of models of non-commutative quantum oscillators. However, the squares of the components of position and momentum operators commute $[\hat{R}^2_{\alpha i}, \hat{R}^2_{\beta j}] = [\hat{P}^2_{\alpha i}, \hat{P}^2_{\beta j}] = 0$ for $\alpha i \neq \beta j$. Furthermore, the GZ basis states $|m)$ are eigenstates of these operators,

$$\hat{R}^2_{\alpha i}|m) = \frac{\hbar}{(n-1)m\omega}(m_{0,n+1}+\ldots+m_{n,n+1}-m_{1,n}-\cdots-m_{n,n}$$
$$+m_{1,k}+\cdots+m_{k,k}-m_{1,k-1}-\cdots-m_{k-1,k-1})|m),\; k=D(\alpha-1)+i.$$

Thus the spectrum of the position operator component $\hat{R}_{\alpha i}$ is discrete

$$\pm\sqrt{\frac{\hbar}{(n-1)m\omega}(\sum_{j=0}^{n} m_{j,n+1}-\sum_{j=1}^{n} m_{j,n}+\sum_{j=1}^{k} m_{j,k}-\sum_{j=1}^{k-1} m_{j,k-1})},\; k=D(\alpha-1)+i.$$

Acknowledgments

NIS was supported by a project from the Fund for Scientific Research – Flanders (Belgium).

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