Microscopic and macroscopic properties of A-superstatistics

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Abstract

The microscopic and the macroscopic properties of A-superstatistics, related to the class $A(0, n-1) \equiv sl(1|n)$ of simple Lie superalgebras are investigated. The algebra $sl(1|n)$ is described in terms of generators $f_1^\pm, \ldots, f_n^\pm$, which satisfy certain triple relations and are called Jacobson generators. The Fock spaces of A-superstatistics are investigated and the Pauli principle of the corresponding statistics is formulated. Some thermal properties of A-superstatistics are constructed under the assumption that the particles interact only via statistical interaction imposed by the Pauli principle. The grand partition function and the average number of particles are written down explicitly in the general case and in two particular examples : 1) the particles have one and the same energy and chemical potential; 2) the energy spectrum of the orbitals is equidistant.

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1 Introduction

In this paper we consider the microscopic and the macroscopic properties of a class of generalized statistics, referred to as A-superstatistics. In our approach, the starting point is a certain symmetry principle, characterized by an algebra of creation and annihilation operators (CAO's) with Fock type representations.

The idea behind these investigations is based on a few observations. The first one is that any n pairs of Bose CAO's $B_1^{\pm}, B_2^{\pm}, \ldots, B_n^{\pm}$ generate a representation, the Bose representation ρ_B, of the orthosymplectic Lie superalgebra $osp(1|2n) = B(0,n)$. The representation independent generators $\hat{B}_1^{\pm}, \hat{B}_2^{\pm}, \ldots, \hat{B}_n^{\pm}$ of $osp(1|2n)$, which in the Bose representation coincide with the Bose operators $\rho_B(\hat{B}_i^{\pm}) = B_i^{\pm}$, are para-Bose CAO's [1] and satisfy the relations

$$[\{\hat{B}_i^{\xi}, \hat{B}_j^{\eta}\}, \hat{B}_k^{\epsilon}] = (\epsilon - \xi)\delta_{ik}\hat{B}_j^{\eta} + (\epsilon - \eta)\delta_{jk}\hat{B}_i^{\xi}, \quad \xi, \eta, \epsilon = \pm \text{ or } \pm 1. \tag{1.1}$$

These triple relations give one possible definition of the Lie superalgebra $osp(1|2n)$ [2], and are considered as the defining relations of para-Bose statistics. So ordinary Bose statistics corresponds to one particular realization of para-Bose statistics.

The situation with Fermi statistics and its generalization, para-Fermi statistics [1], is similar. So the second observation is that any n pairs of Fermi CAO's $F_1^{\pm}, F_2^{\pm}, \ldots, F_n^{\pm}$ give a representation, the Fermi representation ρ_F, of the orthogonal Lie algebra $so(2n+1) = B_n$ and any n pairs of para-Fermi CAOs $\hat{F}_i^{\pm}$, with

$$[[\hat{F}_i^{\xi}, \hat{F}_j^{\eta}], \hat{F}_k^{\epsilon}] = \frac{1}{2}(\eta - \epsilon)^2\delta_{jk}\hat{F}_i^{\xi} - \frac{1}{2}(\xi - \epsilon)^2\delta_{ik}\hat{F}_j^{\eta}, \quad \xi, \eta, \epsilon = \pm \text{ or } \pm 1, \tag{1.2}$$

generate the algebra B_n [3].

Both algebras B_n (every Lie algebra is also a Lie superalgebra) and $B(0|n)$ belong to the class B of basic classical Lie superalgebras in the classification of Kac [4]. Hence ordinary Bose and Fermi statistics and their generalizations para-Bose and para-Fermi statistics could be referred to as B-(super)statistics.

On the ground of these facts statistics related to the other classes of basic Lie superalgebras were introduced : A-, B-, C- and D-(super)statistics [5]. So far only A-statistics (corresponding to the Lie algebra $sl(n+1)$) was studied in detail from the microscopic [5], [6] and macroscopic point of view [7]. In the present paper we investigate the properties of A-superstatistics, introduced in Ref. [8], namely the statistics arising from the Lie superalgebra $sl(1|n) = A(0, n-1)$ or $gl(1|n)$.

Section 2 is devoted to the microscopic properties of this new class of generalized statistics. First we recall the definitions of the Lie superalgebras $sl(1|n)$ and $gl(1|n)$ and their Fock representations. Like for para-Bose and para-Fermi statistics, the generators $f_i^{\pm}$, $i = 1, \ldots, n$ of $sl(1|n)$ satisfy certain triple relations. These relations completely define the Lie superalgebra $sl(1|n)$, just as the triple relations for the generators $a_i^{\pm}$, $i = 1, \ldots, n$ of A-statistics do for the Lie algebra $sl(n+1)$ [6]. For $sl(n+1)$, this property of the operators $a_i^{\pm}$ was first observed by Jacobson [9] and therefore the generators $a_i^{\pm}$ are referred as Jacobson generators of $sl(n+1)$ [6]. By analogy we call the generators $f_i^{\pm}$ of $sl(1|n)$ also Jacobson generators of $sl(1|n)$.

The Fock representations of A-superstatistics are constructed in the same way as in parastatistics. They are generated by the operators f_i^+ and labeled by a positive integer $p = 1, 2, \ldots$. Within the corresponding module $W(p,n)$ the generator f_i^+ (respectively f_i^-) can be interpreted as a creation (respectively annihilation) operator of a "particle" on the ith orbital ($i = 1, 2, \ldots, n$). The Pauli principle of A-superstatistics is formulated and in Section 3 it is indicated that in the limit $p \to \infty$ the representation dependent operators $F(p)_i^\pm = \frac{f_i^\pm}{\sqrt{p}}$ coincide with ordinary Fermi CAO's. We complete this section by showing that the Jacobson generators of $sl(1|n)$ are implicitly present in certain physical models.

The following sections are devoted to the macroscopic properties of A-superstatistics. In Section 4 we construct explicitly the $sl(1|n)$ grand partition function $Z(p,n)$ (GPF), the average number of particles in the system $\bar{N}(p,n)$ and the average number of particles on each orbital $\bar{\theta}_i$ under the assumption that the energy of each particle on orbital i is ϵ_i. All these thermal properties of the system are described by means of the elementary symmetric functions. The fact that symmetric functions appear naturally in the description of (grand) partition functions in statistical mechanics got attention recently [10]. For quantum systems with Bose or Fermi statistics, see [10]. For quantum systems with A-statistics, the relevant symmetric functions are the so-called complete symmetric functions [7]. Here in the case of A-superstatistics, the relevant symmetric functions are the so-called elementary symmetric functions.

Then we consider two specializations of the general case of Section 4. In Section 5 the energy and the chemical potential of each orbital are assumed to be the same. This is the so-called degenerate case. The thermodynamical functions simplify, and many of these can be expressed as hypergeometric series. Furthermore, these functions can be seen as deformations of the corresponding ones in the case of Fermi statistics. In Section 6 we investigate a model with equidistant energy levels. Also under this specialization, the thermodynamical functions assume a simple form, usually in terms of q-generalized or basic hypergeometric functions.

2 Microscopic properties of A-superstatistics

A-superstatistics is defined in the context of the Lie superalgebra $sl(1|n)$ or $gl(1|n)$. A convenient basis of $gl(1|n)$ is given with the Weyl generators e_{ij}, with $i,j \in \{0, 1, \ldots, n\}$. The grading of $gl(1|n)$ is as follows : the *even* elements are given by e_{00} and e_{ij} with $i,j \in \{1, \ldots, n\}$; the *odd* elements are e_{0i} and e_{i0} ($i = 1, \ldots, n$). The Lie superalgebra bracket is determined by

$$[\![e_{ij}, e_{kl}]\!] \equiv e_{ij}e_{kl} - (-1)^{\deg(e_{ij})\deg(e_{kl})} e_{kl}e_{ij} = \delta_{jk}e_{il} - (-1)^{\deg(e_{ij})\deg(e_{kl})}\delta_{il}e_{kj}, \quad (2.1)$$

where $\deg(e_{ij})$ is 0 (resp. 1) if e_{ij} is even (resp. odd). One can define $sl(1|n)$ as the (super)commutator algebra of $gl(1|n)$; its basis consists of all elements e_{ij} ($i \neq j$) and the Cartan elements $e_{00} + e_{ii}$ ($i = 1, \ldots, n$).

The Jacobson creation and annihilation operators $f_i^\pm$ of $sl(1|n)$ are given by

$$f_i^+ = e_{i0}, \qquad f_i^- = e_{0i}, \qquad (i = 1, \ldots, n). \tag{2.2}$$

It is known [8] that the linear envelope of

$$\left\{ f_i^\xi, \{f_j^\eta, f_k^\epsilon\} | i, j, k = 1, \ldots, n; \xi, \eta, \epsilon = \pm \right\} \tag{2.3}$$

is indeed the Lie superalgebra $sl(1|n)$.

A-superstatistics is determined by the relations that hold for the creation and annihilation operators. These are :

$$\{f_i^+, f_j^+\} = \{f_i^-, f_j^-\} = 0, \tag{2.4}$$
$$[\{f_i^+, f_j^-\}, f_k^+] = \delta_{jk} f_i^+ - \delta_{ij} f_k^+, \tag{2.5}$$
$$[\{f_i^+, f_j^-\}, f_k^-] = -\delta_{ik} f_j^- + \delta_{ij} f_k^-. \tag{2.6}$$

It is worth observing that these operators satisfy the compatibility conditions required in the context of Wigner quantum systems [11].

The Fock representations of A-superstatistics have been classified by Palev [8]. These representations are labeled by a positive integer p, the order of statistics. Let us denote the Fock representation of order p for $sl(1|n)$ by $W(p,n)$. The space $W(p,n)$ is characterized by a vacuum vector $|0\rangle$, such that

$$f_i^- |0\rangle = 0, \qquad (i = 1, \ldots, n), \tag{2.7}$$
$$f_i^- f_j^+ |0\rangle = p\, \delta_{ij} |0\rangle, \qquad (i, j = 1, \ldots, n). \tag{2.8}$$

These Fock spaces are finite-dimensional unitary irreducible $sl(1|n)$-modules. A set of basis vectors for the space $W(p,n)$ consists of all vectors

$$(f_1^+)^{\theta_1} (f_2^+)^{\theta_2} \ldots (f_n^+)^{\theta_n} |0\rangle, \qquad \theta_i \in \{0, 1\} \tag{2.9}$$

where

$$|\theta| \equiv \sum_{i=1}^n \theta_i \leq p. \tag{2.10}$$

The linear span of all vectors (2.9), without the restriction (2.10), also forms a $sl(1|n)$ module $\bar{W}(p,n)$. However, if $p < n$, $\bar{W}(p,n)$ is not irreducible : it contains a maximal invariant submodule, and $W(p,n)$ is the quotient module of $\bar{W}(p,n)$ with respect to this submodule. If $p \geq n$, we have that $\bar{W}(p,n) = W(p,n)$; in this case it is clear that the restriction (2.10) is superfluous.

One can define a Hermitian form $\langle\, , \rangle$ on $W(p,n)$ with the usual Fock space technique, by requiring

$$\langle 0|0 \rangle = 1, \tag{2.11}$$
$$\langle f_i^\pm v | w \rangle = \langle v | f_i^\mp w \rangle, \qquad \forall v, w \in W(p,n). \tag{2.12}$$

With respect to this form, the different vectors in (2.9) are orthogonal, and the following vectors form an orthonormal basis of $W(p,n)$:

$$|p;\theta\rangle \equiv |p;\theta_1,\ldots,\theta_n\rangle = \sqrt{\frac{(p-|\theta|)!}{p!}}(f_1^+)^{\theta_1}(f_2^+)^{\theta_2}\ldots(f_n^+)^{\theta_n}|0\rangle,$$
$$\theta_i\in\{0,1\}, \quad |\theta|\leq p. \tag{2.13}$$

Furthermore, the Hermitian conjugate of $f_i^\pm$ is $f_i^\mp$ in this module, which is an important physical requirement.

The transformation of the basis (2.13) under the action of the creation and annihilation operators reads :

$$f_i^-|p;\theta\rangle = \theta_i(-1)^{\theta_1+\cdots+\theta_{i-1}}\sqrt{p-|\theta|+1}\;|p;\theta_1,\ldots,\theta_i-1,\ldots,\theta_n\rangle, \tag{2.14}$$
$$f_i^+|p;\theta\rangle = (1-\theta_i)(-1)^{\theta_1+\cdots+\theta_{i-1}}\sqrt{p-|\theta|}\;|p;\theta_1,\ldots,\theta_i+1,\ldots,\theta_n\rangle. \tag{2.15}$$

Our Hamiltonian will be an element from (the Cartan subalgebra of) $gl(1|n)$. Therefore we first extend $W(p,n)$ to a $gl(1|n)$ module. For this purpose, we set $N_i=e_{ii}$ $(i=0,1,\ldots,n)$, and have :

$$N_0|p;\theta\rangle = (p-|\theta|)|p;\theta\rangle, \tag{2.16}$$
$$N_i|p;\theta\rangle = \theta_i|p;\theta\rangle. \tag{2.17}$$

We shall be studying macroscopic properties of A-superstatistics for a Hamiltonian of the following form :

$$H=\sum_{i=1}^n \epsilon_i N_i. \tag{2.18}$$

Via creation and annihilation operators, this can be rewritten as

$$H=\sum_{i=1}^n \epsilon_i\left(\{f_i^+,f_i^-\}+\frac{1}{n-1}\Big(p-\sum_{k=1}^n\{f_k^+,f_k^-\}\Big)\right). \tag{2.19}$$

Clearly, $H|0\rangle=0$ (so the vacuum has zero energy), and

$$[H,f_i^\pm]=\pm\epsilon_i f_i^\pm. \tag{2.20}$$

Therefore, each f_i^+ (resp. f_i^-) can be interpreted as an operator creating (resp. annihilating) a particle (or quasiparticle, or excitation) on orbital i (with energy ϵ_i). Since

$$H|p;\theta\rangle = \left(\sum_{i=1}^n \epsilon_i\theta_i\right)|p;\theta\rangle, \tag{2.21}$$

$|p;\theta\rangle$ is interpreted as a state with θ_i particles on orbital i.

The CAO's of $sl(1|n)$ together with the Hamiltonian (2.18) generate $gl(1|n)$. Then, in view of (2.19), $f_1^\pm,\ldots,f_n^\pm$, considered as operators in any $W(p,n)$, generate

$gl(1|n)$. For this reason we call the CAO's of $sl(1|n)$ also Jacobson generators [6] of $gl(1|n)$.

The *Pauli principle* for A-superstatistics follows basically from equation (2.10) and the form of the vectors (2.13). These relations imply that the system can accommodate up to $\min(p,n)$, but no more than $\min(p,n)$ particles, in such a way that every orbital contains at most one particle.

Note that in the atypical cases ($p < n$) the Pauli principle introduces an interaction, a statistical interaction, between the orbitals. For instance if the system is in the state with $\theta_1 = \ldots = \theta_p = 1$ and $\theta_{p+1} = \ldots = \theta_n = 0$, then it cannot accommodate more particles, not even on the empty orbitals with $i > p$. Therefore the orbitals are not filled independently. The filling of a given orbital depends on fillings of all other orbitals, which is the main feature of exclusion statistics [12]. Observe also that A-superstatistics is closely related to ordinary Fermi statistics : the only extra condition comes from the order of statistics p.

3 Quasi-Fermi creation and annihilation operators

In the present section we discuss some differences and similarities between A-superstatistics and ordinary Fermi statistics. First of all, observe that there is an important difference between the Fock spaces $W(p,n)$ with $p < n$ and those with $p \geq n$. For $p < n$, the irreducible $sl(1|n)$ representations are *atypical*, and their dimension is given by :

$$\dim W(p,n) = \sum_{k=0}^{p} \binom{n}{k}. \tag{3.1}$$

For $p \geq n$, the representations $W(p,n)$ are all typical, with

$$\dim W(p,n) = 2^n. \tag{3.2}$$

For $p_1 \neq p_2$ and $p_1, p_2 \geq n$, the representations $W(p_1,n)$ and $W(p_2,n)$ are certainly not isomorphic (even though they have the same dimension), since they have a different highest weight.

Let n be fixed, $p \geq n$, and let us define representation-dependent operators $F(p)_i^\pm$ in $W(p,n)$ by

$$F(p)_i^\pm = \frac{f_i^\pm}{\sqrt{p}}, \qquad i = 1, 2, \ldots, n. \tag{3.3}$$

The action of these new creation and annihilation operators on the vectors (2.13) reads :

$$F(p)_i^- |p;\theta\rangle = \theta_i (-1)^{\theta_1+\cdots+\theta_{i-1}} \sqrt{1 + \frac{1-|\theta|}{p}}\, |p;\theta_1,\ldots,\theta_i - 1,\ldots,\theta_n\rangle, \tag{3.4}$$

$$F(p)_i^+ |p;\theta\rangle = (1-\theta_i)(-1)^{\theta_1+\cdots+\theta_{i-1}} \sqrt{1 - \frac{|\theta|}{p}}\, |p;\theta_1,\ldots,\theta_i + 1,\ldots,\theta_n\rangle. \tag{3.5}$$

On the other hand, one can consider a set of n ordinary Fermi creation and annihilation operators $F_i^\pm$ $(i=1,\ldots,n)$, satisfying :

$$\{F_i^+,F_j^+\}=\{F_i^-,F_j^-\}=0, \qquad \{F_i^-,F_j^+\}=\delta_{ij}, \tag{3.6}$$

and its 2^n-dimensional Fock space $W(n)$ with orthonormal basis vectors

$$|\theta\rangle=|\theta_1,\ldots,\theta_n\rangle=(F_1^+)^{\theta_1}\cdots(F_n^+)^{\theta_n}|0\rangle, \qquad \theta_i\in\{0,1\}, \tag{3.7}$$

and action

$$F_i^-|\theta\rangle = \theta_i(-1)^{\theta_1+\cdots+\theta_{i-1}}|\theta_1,\ldots,\theta_i-1,\ldots,\theta_n\rangle, \tag{3.8}$$
$$F_i^+|\theta\rangle = (1-\theta_i)(-1)^{\theta_1+\cdots+\theta_{i-1}}|\theta_1,\ldots,\theta_i+1,\ldots,\theta_n\rangle. \tag{3.9}$$

We can now identify the basis vectors $|p;\theta\rangle$ of $W(p,n)$ with the vectors $|\theta\rangle$ of $W(n)$. Let us therefore define, for a given positive integer $p\geq n$, in $W(n)$ a realization of the operators $F(p)_i^\pm$, denoted by $\rho(F(p)_i^\pm)$, through the action :

$$\rho(F(p)_i^-)|\theta\rangle = \theta_i(-1)^{\theta_1+\cdots+\theta_{i-1}}\sqrt{1+\frac{1-|\theta|}{p}}\,|\theta_1,\ldots,\theta_i-1,\ldots,\theta_n\rangle, \tag{3.10}$$

$$\rho(F(p)_i^+)|\theta\rangle = (1-\theta_i)(-1)^{\theta_1+\cdots+\theta_{i-1}}\sqrt{1-\frac{|\theta|}{p}}\,|\theta_1,\ldots,\theta_i+1,\ldots,\theta_n\rangle. \tag{3.11}$$

Both $\rho(F(p)_i^\pm)$ and $F_i^\pm$ are now operators acting in the finite-dimensional Fock space $W(n)$, and from their action it follows immediately that

$$\lim_{p\to\infty}\rho(F(p)_i^\pm)=F_i^\pm. \tag{3.12}$$

For this reason, the operators $F(p)_i^\pm$ are said to be quasi-Fermi creation and annihilation operators. For large p-values, they tend to ordinary Fermi creation and annihilation operators.

It is interesting to consider the anti-commutators of the quasi-Fermi creation and annihilation operators, acting in the Fermi Fock space $W(n)$. From (3.10)-(3.11) one obtains

$$\{\rho(F(p)_i^+),\rho(F(p)_j^+)\}|\theta\rangle=0, \tag{3.13}$$
$$\{\rho(F(p)_i^-),\rho(F(p)_j^-)\}|\theta\rangle=0, \tag{3.14}$$
$$\{\rho(F(p)_i^-),\rho(F(p)_i^+)\}|\theta\rangle=\left(1+\frac{\theta_i-|\theta|}{p}\right)|\theta\rangle, \tag{3.15}$$
$$\{\rho(F(p)_i^-),\rho(F(p)_j^+)\}|\theta\rangle=-\frac{1}{p}(-1)^{\theta_i+\ldots+\theta_j}\theta_i(1-\theta_j)$$
$$\times|\ldots,\theta_i-1,\ldots,\theta_j+1,\ldots\rangle, \quad i<j, \tag{3.16}$$
$$\{\rho(F(p)_i^-),\rho(F(p)_j^+)\}|\theta\rangle=-\frac{1}{p}(-1)^{\theta_j+\ldots+\theta_i}\theta_i(1-\theta_j)$$
$$\times|\ldots,\theta_j+1,\ldots,\theta_i-1,\ldots\rangle, \quad i>j. \tag{3.17}$$

Compare again with (3.6) to see that the above anti-commutators tend to ordinary Fermi anticommutators when p tends to infinity. Also in this sense, the quasi-Fermi creation and annihilation operators $\rho(F(p)_i^{\pm})$ can be considered as "deformations" of ordinary Fermi creation and annihilation operators, with the integer p (the order of statistics) as a deformation parameter.

A comparison of (2.14)-(2.15) for $p \geq n$ with (3.8)-(3.9) shows that the Jacobson generators of $sl(1|n)$ can be expressed as functions of ordinary Fermi creation and annihilation operators $F_1^{\pm}, F_2^{\pm}, \ldots, F_n^{\pm}$. Let $W(p,n)$ be a typical representation (so with $p \geq n$), and identify again its basis vectors $|p;\theta_1,\ldots,\theta_n\rangle$ with the basis vectors $|\theta_1,\ldots,\theta_n\rangle \equiv |\theta\rangle$ of the Fermi Fock space $W(n)$. Since $F_i^+F_i^-$ is a number operator for fermions in a state i,

$$F_i^+F_i^-|\theta\rangle = \theta_i|\theta\rangle, \qquad i=1,\ldots,n, \tag{3.18}$$

we can write for (2.15)

$$f_i^+|\theta\rangle = e_{i0}|\theta\rangle = (1-\theta_i)(-1)^{\theta_1+\cdots+\theta_{i-1}}\sqrt{p+1-\sum_{k=1}^n F_k^+F_k^-}\;|\theta_1,\ldots,\theta_i+1,\ldots,\theta_n\rangle. \tag{3.19}$$

The latter can be represented as

$$f_i^+|\theta\rangle = \sqrt{p+1-\sum_{k=1}^n F_k^+F_k^-}\;F_i^+|\theta\rangle = F_i^+\sqrt{p-\sum_{k=1}^n F_k^+F_k^-}\;|\theta\rangle. \tag{3.20}$$

Equation (3.20) holds for any $|\theta\rangle$. Therefore

$$f_i^+ = e_{i0} = F_i^+\sqrt{p-\sum_{k=1}^n F_k^+F_k^-}, \qquad i=1,\ldots,n. \tag{3.21}$$

In a similar way one derives from (2.14)

$$f_i^- = e_{0i} = \sqrt{p-\sum_{k=1}^n F_k^+F_k^-}\;F_i^-, \qquad i=1,\ldots,n. \tag{3.22}$$

Evidently (see 2.16),

$$e_{00} = p-\sum_{k=1}^n F_k^+F_k^-, \tag{3.23}$$

and simple calculations lead to

$$e_{ij} = F_j^+F_i^-, \qquad i,j=1,\ldots,n. \tag{3.24}$$

In such a way we have expressed all Weyl generators $\{e_{ij}|i,j=0,1,\ldots,n\}$ of $gl(1|n)$ via n pairs of Fermi operators. Let us mention that the "ferminization" (3.21)-(3.22) of the Jacobson generators of $sl(1|n)$ is not new. It is known as the Holstein-Primakoff realization of $sl(1|n)$ [13].

Finally, we wish to mention that the Jacobson generators of $gl(1|n)$ and the considered Fock representations $W(p,n)$ (or equivalently, the quasi-Fermi operators) are implicitly present in certain physical models.

Examples from condensed matter physics include mainly models related in one or another way to high-temperature superconductivity. We have in mind those lattice models of strongly correlated electron systems where (the electronic part of) the Hamiltonian is expressed in terms of Hubbard operators (X-operators) [14] as for instance in [15] or in [16]. In such models, each Hubbard operator is labeled by three indices, X_A^{ij}, where A refers to the lattice site and $i,j=0,1,\ldots,N$, if the (combined spin-flavor) degrees of freedom (the number of the orbitals) of the electrons at each fixed site are N. For any site the operators X_A^{i0} and X_A^{0i}, $i=1,\ldots,N$, are said to be fermion-like generators (or of odd degree), whereas X_A^{00}, X_A^{ij}, $i,j=1,\ldots,N$ are boson-like generators (or of even degree). The X-operators obey the relations

$$[X_A^{ij}, X_B^{kl}]_\pm = \delta_{AB}(\delta_{jk}X_A^{il} \pm \delta_{il}X_A^{kj}), \quad i,j,k,l=0,\ldots,N, \tag{3.25}$$

with the upper signs if both X-operators in the left hand side are fermion-like and with lower signs in all other cases. Clearly (3.25) can also be written as

$$[\![X_A^{ij}, X_B^{kl}]\!] = \delta_{AB}(\delta_{jk}X_A^{il} - (-1)^{\deg(X_A^{ij})\deg(X_A^{kl})}\delta_{il}X_A^{kj}), \tag{3.26}$$

which indicates, that for a fixed value of the lattice site A the Hubbard operators are the Weyl generators of $gl(1|N) \equiv gl(1|N)_A$ [17], cfr. (2.1). Since moreover for $A \neq B$ the X-operators supercommute, $[\![X_A^{ij}, X_B^{kl}]\!] = 0$, the conclusion is that all Hubbard operators constitute a basis in the algebra $\mathcal{L}$ which is a direct sum of all $gl(1|N)_A$, i.e.,

$$\mathcal{L} = \bigoplus_A gl(1|N)_A. \tag{3.27}$$

Each local state space per site A (we suppress the site index A whenever possible) has a basis consisting of all vectors

$$|n_0; n_1, \ldots, n_N\rangle, \quad n_1,\ldots,n_N \in \{0,1\}, \tag{3.28}$$

subject to the additional constraint $n_0 = p - \sum_{k=1}^N n_k$, with p fixed (in [16], $p=N/2$; and in [15], $p=1$). Since n_0 is required to be a non-negative integer, only such sets of numbers $n_1,\ldots,n_N$, are admitted for which

$$n_1 + \ldots + n_N \leq p. \tag{3.29}$$

The physical interpretation of the state (3.28) is that it corresponds to a configuration with n_1 electrons on the first orbital, n_2 electrons on the second orbital, etc. The action of the X-operators on the states (3.28) reads (throughout below

$i,j,k=1,\ldots,N$) [16] :

$$X_A^{kk}|n_0;n_1,\ldots,n_k,\ldots,n_N\rangle = n_k|n_0;n_1,\ldots,n_k,\ldots,n_N\rangle, \tag{3.30}$$

$$X_A^{jk}|n_0;n_1,\ldots,n_j,\ldots,n_k,\ldots,n_N\rangle = (-1)^{n_j+\cdots+n_{k-1}}|n_0;n_1,\ldots,n_j+1,\ldots,n_k-1,\ldots,n_N\rangle, \quad j\neq k, \tag{3.31}$$

$$X_A^{0k}|n_0;n_1,\ldots,n_k,\ldots,n_N\rangle = \sqrt{n_0+1}(-1)^{n_1+\cdots+n_{k-1}}|n_0+1;n_1,\ldots,n_k-1,\ldots,n_N\rangle, \tag{3.32}$$

$$X_A^{k0}|n_0;n_1,\ldots,n_k,\ldots,n_N\rangle = \sqrt{n_0}(-1)^{n_1+\cdots+n_{k-1}}|n_0-1;n_1,\ldots,n_k+1,\ldots,n_N\rangle. \tag{3.33}$$

In eqs. (3.30)-(3.33), the convention is that the vectors on the right hand sides with unacceptable arguments should be identified with zero.

Equations (3.32) and (3.33) clearly indicate that the operator X_A^{k0} (resp. X_A^{0k}) creates (resp. annihilates) an electron on the k-th orbital of site A. However, these operators are not Fermi creation and annihilation operators in the strict sense because $\{X_A^{0i},X_A^{j0}\}\neq\delta_{ij}$.

Comparison with the formulas from Section 2 now leads to the identification of the Hubbard model Hilbert space with the representation $W(p,n)$, where

$$N=n, \qquad (n_1,\ldots,n_N)=(\theta_1,\ldots,\theta_n), \qquad n_0=p-|\theta|. \tag{3.34}$$

Furthermore, the Hubbard operators (on a fixed site A) are expressed in terms of the Jacobson creation and annihilation operators $f_A^{\pm}$ of $gl(1|n)$:

$$X_A^{0k}=f_k^-, \qquad X_A^{k0}=f_k^+, \qquad X_A^{jk}=\{f_j^+,f_k^-\},\ (j<k). \tag{3.35}$$

So the conclusion is that the operators X_A^{k0} and X_A^{0k}, creating and annihilating electrons at site A, are not Fermi operators. They are creation and annihilation operators of the Lie superalgebra $gl(1|n)$. The statistics of the electrons or of any other (quasi)particles described with these operators is not Fermi statistics, it is A-superstatistics of order p.

In order to quote an example from nuclear physics, note that the $p=1$ Jacobson generators $f_1^{\pm},\ldots,f_n^{\pm}$ of $sl(1|n)$ together with N_0 satisfy the relations :

$$f_j^-f_i^- = f_i^+f_j^+ = 0, \tag{3.36}$$

$$f_i^-f_j^+ = \delta_{ij}N_0, \tag{3.37}$$

$$N_0f_i^+ = f_i^-N_0 = 0, \tag{3.38}$$

$$N_0^2 = N_0. \tag{3.39}$$

In nuclear shell model theory the operators with the above properties are called *ideal odd-particle* (IOP) creation and annihilation operators [18]. Okubo [19] refers to the algebra of IOP operators as to the Marshalek algebra. The IOP operators play a relevant role for the description of properties of odd nuclei in the frame of the nuclear shell model (see the review article [20] and references therein).

4 Macroscopic properties of A-superstatistics

To describe the macroscopic properties of A-superstatistics for the Hamiltonian H with orbitals i ($i = 1, \ldots, n$), see (2.18), it is for us irrelevant whether the different orbitals correspond to different particles, to different energy levels of particles, or to different internal states of the particles. The only assumption is that they satisfy the Pauli principle for A-superstatistics, which follows from the Fock space construction.

As usually, we assume that the system is in a thermal and diffusive contact and in a thermal and diffusive equilibrium with a much bigger reservoir. We denote by τ its (fundamental) temperature, by μ_i the chemical potential and by ϵ_i the energy for the particles on orbital i.

The probability $\mathcal{P}(p, n; \theta)$ for the system to be in a (quantum) state $\theta = (\theta_1, \ldots, \theta_n)$ with $|\theta| = \theta_1 + \cdots + \theta_n$ particles and energy $E = \theta_1\epsilon_1 + \cdots + \theta_n\epsilon_n$ is given by the expression

$$\mathcal{P}(p, n; \theta) = \frac{1}{Z(p, n)} \exp\left(\sum_{i=1}^{n} \frac{\mu_i - \epsilon_i}{\tau}\theta_i\right). \tag{4.1}$$

The numerator in this expression is the Gibbs factor of the system in the state θ, and $Z(p, n)$ is the grand partition function (GPF) of the system. In the case of Bose or Fermi statistics or their generalizations (Green parastatistics [1]), the GPF is simply a product of the GPFs of all orbitals. This is due to the fact that for those statistics the different orbitals can be considered as independent subsystems : the filling of each orbital is completely independent of the number of particles that have already been accommodated on the other orbitals. Here due to the new Pauli principle this is no longer the case if $p < n$. Therefore we have to compute directly the GPF for the whole system. The latter as usual is the sum of the Gibbs factors over all possible states of the system. So we have :

$$Z(p, n) = \sum_{\substack{0 \leq \theta_1 + \cdots + \theta_n \leq p \\ \theta_i \in \{0,1\}}} \left(\exp(\frac{\mu_1 - \epsilon_1}{\tau})\right)^{\theta_1} \cdots \left(\exp(\frac{\mu_n - \epsilon_n}{\tau})\right)^{\theta_n}. \tag{4.2}$$

In terms of the notation

$$x_i = \exp(\frac{\mu_i - \epsilon_i}{\tau}), \qquad i = 1, \ldots, n, \tag{4.3}$$

we have

$$Z(p, n) = \sum_{\substack{0 \leq \theta_1 + \cdots + \theta_n \leq p \\ \theta_i \in \{0,1\}}} x_1^{\theta_1} x_2^{\theta_2} \cdots x_n^{\theta_n} = \sum_{k=0}^{\min(p,n)} \sum_{\substack{\theta_1 + \cdots + \theta_n = k \\ \theta_i \in \{0,1\}}} x_1^{\theta_1} x_2^{\theta_2} \cdots x_n^{\theta_n}. \tag{4.4}$$

It follows that $Z(p, n) = Z(n, n)$ if $p > n$; therefore, we shall from now on assume that $p \leq n$, thus covering all possible cases. Since all macroscopic properties are encapsulated in the grand partition function, this observation also implies that *A-superstatistics with $p \geq n$ has the same macroscopic properties as ordinary Fermi* *statistics*. So the case $p = n-1$ can be considered as the smallest deviation from Fermi statistics, as far as the macroscopic properties are concerned. In the following, we shall sometimes pay special attention to the $p = n-1$ case, and compare it with the properties of Fermi statistics.

In the present context, it is appropriate to introduce the elementary symmetric functions $e_k(x_1, \ldots, x_n)$, $k = 0, 1, \ldots$. The k-th elementary symmetric function [21] is the sum over all products of k distinct variables x_i, so that $e_0(x_1, \ldots, x_n) = 1$ and

$$e_k(x_1, \ldots, x_n) = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k} \tag{4.5}$$

$$= \sum_{\substack{\theta_1 + \cdots + \theta_n = k \\ \theta_i \in \{0,1\}}} x_1^{\theta_1} x_2^{\theta_2} \cdots x_n^{\theta_n}. \tag{4.6}$$

For instance, $e_1(x_1, x_2, x_3) = x_1 + x_2 + x_3$, $e_2(x_1, x_2, x_3) = x_1x_2 + x_1x_3 + x_2x_3$, $e_3(x_1, x_2, x_3) = x_1x_2x_3$, and $e_k(x_1, x_2, x_3) = 0$ for $k > 3$. The generating function for the e_k is given by [21]

$$\sum_{k=0}^{n} e_k(x_1, \ldots, x_n) t^k = (1 + x_1 t) \cdots (1 + x_n t). \tag{4.7}$$

In terms of the elementary symmetric functions, one finds

$$Z(p, n) = \sum_{k=0}^{p} e_k(x_1, \ldots, x_n). \tag{4.8}$$

This sum does not simplify if $p < n$; for $p = n$, (4.7) yields

$$Z(n, n) = (1 + x_1)(1 + x_2) \cdots (1 + x_n). \tag{4.9}$$

Also from (4.7), it follows that one can give the following description : for $p < n$, $Z(p, n)$ consists of those terms of $Z(n, n)$ that have total degree less than or equal to p.

Let us now consider some other thermodynamic quantities. The probability $\mathcal{P}(p, n; \theta)$ for the system to be in the state $\theta = (\theta_1, \ldots, \theta_n)$ with $|\theta|$ particles reads

$$\mathcal{P}(p, n; \theta) = \frac{x_1^{\theta_1} x_2^{\theta_2} \cdots x_n^{\theta_n}}{Z(p, n)}. \tag{4.10}$$

Therefore, the average number of particles in the system is

$$\bar{N}(p, n) = \sum_{\substack{0 \le \theta_1 + \cdots + \theta_n \le p \\ \theta_i \in \{0,1\}}} |\theta| \mathcal{P}(p, n; \theta) = \sum_{\substack{0 \le \theta_1 + \cdots + \theta_n \le p \\ \theta_i \in \{0,1\}}} |\theta| \frac{x_1^{\theta_1} x_2^{\theta_2} \cdots x_n^{\theta_n}}{Z(p, n)}. \tag{4.11}$$

This can be rewritten as

$$\bar{N}(p, n) = \sum_{i=1}^{n} x_i \partial_{x_i} \ln(Z(p, n)). \tag{4.12}$$

In terms of symmetric functions, the numerator of (4.11) reads

$$\sum_{\substack{0\le\theta_1+\cdots+\theta_n\le p\\ \theta_i\in\{0,1\}}} |\theta| x_1^{\theta_1}x_2^{\theta_2}\cdots x_n^{\theta_n} = \sum_{k=0}^{p} k \sum_{\substack{\theta_1+\cdots+\theta_n=k\\ \theta_i\in\{0,1\}}} x_1^{\theta_1}x_2^{\theta_2}\cdots x_n^{\theta_n} = \sum_{k=0}^{p} k e_k(x_1,\ldots,x_n), \tag{4.13}$$

so we can write

$$\bar{N}(p,n) = \frac{\sum_{k=0}^{p} k e_k(x_1,\ldots,x_n)}{\sum_{k=0}^{p} e_k(x_1,\ldots,x_n)}, \tag{4.14}$$

or equivalently,

$$\bar{N}(p,n) = p - \frac{\sum_{k=0}^{p-1}(p-k) e_k(x_1,\ldots,x_n)}{\sum_{k=0}^{p} e_k(x_1,\ldots,x_n)}. \tag{4.15}$$

From this last equation it is clear that the average number of particles in the system is indeed less than p. Formula (4.14) simplifies when $p=n$. Indeed, from (4.7) we have

$$\begin{aligned}\sum_{k=0}^{n} k e_k(x_1,\ldots,x_n)t^k &= t\frac{\partial}{\partial t}\left(\sum_{k=0}^{n} e_k(x_1,\ldots,x_n)t^k\right)\\ &= t\frac{\partial}{\partial t}\left(\prod_{i=1}^{n}(1+x_it)\right) = t\sum_{r=1}^{n}\frac{x_r}{1+x_rt}\prod_{i=1}^{n}(1+x_it),\end{aligned} \tag{4.16}$$

and so

$$\bar{N}(n,n) = \sum_{r=1}^{n}\frac{x_r}{1+x_r}. \tag{4.17}$$

Next, we shall determine the equilibrium distribution of the particles on the orbitals. First of all, consider the orbital n. Either there are no particles on this orbital, or else there is just one particle. Denote by $\mathcal{P}(p,n;\theta_n=0)$, resp. $\mathcal{P}(p,n;\theta_n=1)$, the probability that there are no particles present (resp. that there is one particle present) on orbital n. From the sum of the corresponding Gibbs factors, one finds :

$$\mathcal{P}(p,n;\theta_n=0) = \frac{Z(p,n)|_{x_n=0}}{Z(p,n)} = \frac{\sum_{k=0}^{p} e_k(x_1,\ldots,x_{n-1})}{\sum_{k=0}^{p} e_k(x_1,\ldots,x_{n-1},x_n)}, \tag{4.18}$$

$$\mathcal{P}(p,n;\theta_n=1) = 1-\mathcal{P}(p,n;\theta_n=0) = x_n\frac{\sum_{k=0}^{p-1} e_k(x_1,\ldots,x_{n-1})}{\sum_{k=0}^{p} e_k(x_1,\ldots,x_{n-1},x_n)}. \tag{4.19}$$

The last relation follows from the trivial observation

$$e_k(x_1,\ldots,x_{n-1},x_n) = e_k(x_1,\ldots,x_{n-1}) + x_n e_{k-1}(x_1,\ldots,x_{n-1}). \tag{4.20}$$

By the symmetry (of the symmetric functions), these probabilities extend to any orbital i :

$$\mathcal{P}(p,n;\theta_i=0) = \frac{Z(p,n)|_{x_i=0}}{Z(p,n)} = \frac{\sum_{k=0}^{p} e_k(x_1,\ldots,\widehat{x}_i,\ldots,x_n)}{\sum_{k=0}^{p} e_k(x_1,\ldots,x_n)}, \tag{4.21}$$

$$\mathcal{P}(p,n;\theta_i=1) = 1-\mathcal{P}(p,n;\theta_i=0) = x_i\frac{\sum_{k=0}^{p-1} e_k(x_1 x_i,.,\hat{.},x_n)}{\sum_{k=0}^{p} e_k(x_1,\ldots,x_n)}. \tag{4.22}$$

Herein, $(x_1 x_i, ., \widehat{.}, x_n)$ stands for the $(n-1)$-tuple obtained by removing x_i from the n-tuple $(x_1, \ldots, x_n)$. It is now clear that the average number of particles on the i-th orbital, denoted by $\bar{\theta}_i$, is just $\mathcal{P}(p, n; \theta_i = 1)$. In other words :

$$\bar{\theta}_i = x_i \frac{\sum_{k=0}^{p-1} e_k(x_1, \ldots, \widehat{x}_i, \ldots, x_n)}{\sum_{k=0}^{p} e_k(x_1, \ldots, x_n)}. \tag{4.23}$$

From (4.8) and (4.20) this is also :

$$\bar{\theta}_i = x_i \partial_{x_i}(\ln Z(p, n)). \tag{4.24}$$

For $p \geq n$ (4.23) gives the Fermi case

$$\bar{\theta}_i^f = \frac{x_i}{1 + x_i}, \tag{4.25}$$

which is consistent with (4.17). So $\bar{\theta}_i^f$ denotes the average number of particles on orbital i in the case of Fermi statistics. It is interesting to consider the deviation when $p = n - 1$. We can express the average number of particles on orbital i in the case of A-superstatistics of order $p = n - 1$ by means of the Fermi averages $\bar{\theta}_i^f$:

$$\bar{\theta}_i^{p=n-1} = \frac{\bar{\theta}_i^f - \prod_{j=1}^n \bar{\theta}_j^f}{1 - \prod_{j=1}^n \bar{\theta}_j^f}. \tag{4.26}$$

Clearly, these new averages are small deviations from the averages in the case of Fermi statistics. Also note that the deviation of the average on orbital i depends on the Fermi averages on all other orbitals. Similarly the average number of particles in the system for $p = n - 1$ is

$$\bar{N}(p = n - 1, n) = \frac{\bar{N}(n, n) - n \prod_{j=1}^n \bar{\theta}_j^f}{1 - \prod_{j=1}^n \bar{\theta}_j^f}, \tag{4.27}$$

where $\bar{N}(n, n)$ is the average particle number in the case of Fermi statistics.

The average energy of the particles on the i-th orbital is given by

$$\bar{E}_i = \epsilon_i \bar{\theta}_i = \epsilon_i x_i \partial_{x_i}(\ln Z(p, n)), \tag{4.28}$$

and the average energy of the total system is

$$\begin{aligned} \bar{E}(p, n) &= \sum_{i=1}^n \epsilon_i \bar{\theta}_i = \sum_{i=1}^n \epsilon_i x_i \partial_{x_i}(\ln Z(p, n)) \\ &= \frac{1}{Z(p, n)} \sum_{i=1}^n \epsilon_i x_i \sum_{k=0}^{p-1} e_k(x_1, \ldots, \widehat{x}_i, \ldots, x_n). \end{aligned} \tag{4.29}$$

We can again express the average energy in the case $p = n-1$ via the average energy in the Fermi case $\bar{E}(n, n)$, and the Fermi averages $\bar{\theta}_i^f$:

$$\bar{E}(p = n - 1, n) = \frac{\bar{E}(n, n) - \sum_{k=1}^n \epsilon_k \prod_{j=1}^n \bar{\theta}_j^f}{1 - \prod_{j=1}^n \bar{\theta}_j^f}. \tag{4.30}$$

Also other thermodynamical functions, such as the entropy $S(p,n)$ and the heat capacity $C_V(p,n)$, defined in terms of the grand partition function and the average energy, can be computed. The formulation of these expressions in terms of symmetric functions does not lead to further simplifications or insights, so we shall not deal with these.

5 A-superstatistics in the degenerate case

Let us consider as a particular example a Hamiltonian of the form

$$H = \epsilon \sum_{i=1}^{n} N_i. \tag{5.1}$$

Thus we assume that all orbitals have the same energy, and let us furthermore assume that they also have the same chemical potential, i.e. $\mu_1 = \mu_2 = \cdots = \mu_n = \mu$. Therefore $x_1 = x_2 = \cdots = x_n = x$, with

$$x = \exp(\frac{\mu - \epsilon}{\tau}). \tag{5.2}$$

In this case the orbitals label internal degrees of freedom of the particles such as spin, color, flavor, etc. The thermodynamical functions for this example follow from the formulas of the previous section, under the specialization $x_i = x$ $(i = 1, \ldots, n)$.

Since the number of terms in $e_k(x_1, \ldots, x_n)$ is given by $\binom{n}{k}$, we have

$$e_k(\underbrace{x, \ldots, x}_{n \text{ times}}) = \binom{n}{k} x^k. \tag{5.3}$$

Thus (4.8) yields

$$Z(p,n) = \sum_{k=0}^{p} \binom{n}{k} x^k. \tag{5.4}$$

For $p < n$ this cannot be rewritten in a closed form; for $p \geq n$, this is simply $(1+x)^n$, i.e. the GPF for a Fermi system with n distinct orbitals having the same energy.

For $p < n$ the sum in (5.4) can be rewritten as follows :

$$\begin{aligned} Z(p,n) &= \sum_{k=0}^{n} \binom{n}{k} x^k - \sum_{k=p+1}^{n} \binom{n}{k} x^k \\ &= (1+x)^n - \binom{n}{p+1} x^{p+1} \, {}_2F_1\left({1, p-n+1 \atop p+2} ; -x \right), \end{aligned} \tag{5.5}$$

where ${}_2F_1$ is the classical hypergeometric series

$${}_2F_1\left({a, b \atop c} ; x \right) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!}, \quad (d)_k = d(d+1)\ldots(d+k-1). \tag{5.6}$$

The first term in the right hand side of (5.5) is the Fermi GPF and therefore the second term gives the difference between Fermi statistics and A-superstatistics. In a sense, it describes the statistical interaction between the particles.

Using Euler's transformation formula for hypergeometric functions, i.e.

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; x\right) = (1-x)^{c-a-b}{}_2F_1\left(\begin{matrix} c-a, c-b \\ c \end{matrix}; x\right), \tag{5.7} $$

(5.5) can also be rewritten as

$$ Z(p,n) = (1+x)^n \left(1 - \binom{n}{p+1} x^{p+1}\, {}_2F_1\left(\begin{matrix} p+1, n+1 \\ p+2 \end{matrix}; -x\right)\right). \tag{5.8} $$

So (5.5) gives the deviation from the Fermi GPF in additive form, and (5.8) gives the deviation in multiplicative form.

The average number of particles follows from (4.14) :

$$ \bar{N}(p,n) = \frac{\sum_{k=0}^{p} k\binom{n}{k} x^k}{\sum_{k=0}^{p} \binom{n}{k} x^k}. \tag{5.9} $$

Using the definition of hypergeometric functions, equation (5.9) can be rewritten as

$$ \bar{N}(p,n) = \frac{nx(1+x)^{n-1} - (p+1)\binom{n}{p+1} x^{p+1}\, {}_2F_1\left(\begin{smallmatrix} 1, p-n+1 \\ p+1 \end{smallmatrix}; -x\right)}{(1+x)^n - \binom{n}{p+1} x^{p+1}\, {}_2F_1\left(\begin{smallmatrix} 1, p-n+1 \\ p+2 \end{smallmatrix}; -x\right)}. \tag{5.10} $$

For $p \geq n$, this becomes

$$ \bar{N}(p \geq n, n) = \frac{nx}{1+x}. \tag{5.11} $$

The average number of particles on the i-th orbital follows from (4.23) :

$$ \bar{\theta}_i = x \frac{\sum_{k=0}^{p-1} \binom{n-1}{k} x^k}{\sum_{k=0}^{p} \binom{n}{k} x^k}. \tag{5.12} $$

For $p \geq n$, this sum simplifies to $\frac{x}{1+x}$. In the general case, simple properties of binomial coefficients lead to :

$$ \bar{\theta}_i = \frac{x}{1+x} - \binom{n-1}{p} \frac{x^{p+1}}{(1+x)Z(p,n)}. \tag{5.13} $$

Obviously, this expression is the same for every orbital i. As a consequence, the average number of particles of the total system can be rewritten as

$$ \bar{N}(p,n) = \frac{nx}{1+x} - n\binom{n-1}{p} \frac{x^{p+1}}{(1+x)Z(p,n)}. \tag{5.14} $$

It is interesting to consider an example. Let $n = 5$; we shall examine the dependence of the average number of particles on the i-th orbital $\bar{\theta}_i$ upon the variable

$$ y = \frac{\epsilon - \mu}{\tau}, \text{ where } x = e^{-y}. \tag{5.15} $$

In Figure 1, we plot $\bar{\theta}_i$ for $p = 1, 2, \ldots, 5$. The case $p = n = 5$ yields the known Fermi-Dirac distribution function. For $p > n$, the distribution function is the same. For $p < n = 5$, the distribution function is different. The difference is most noticable for $\epsilon < \mu$ (or $y < 0$), as the average number of particles cannot exceed p/n.

The case $p = 1$ and any n is also of interest :

$$\bar{N}(1, n) = \frac{n}{e^{(\epsilon-\mu)/\tau} + n}. \tag{5.16}$$

$\bar{N}(1, n)$ is always smaller than 1, i.e. the system can accommodate at most one particle. When $n = 1$ this corresponds to the Fermi-Dirac distribution function. When n increases the average number of particles of the system increases also for fixed $y = \frac{\epsilon-\mu}{\tau}$. This description ($p = 1$, $n > 1$) corresponds to a system consisting of "hard-core bosons". Such particles appear (as mentioned in Sect. 3) in some models of condensed matter physics [15] and nuclear physics [20]. This case is illustrated in Figure 2, where we take $p = 1$ fixed and let n vary.

6 Equidistant energy levels

Now we consider the Hamiltonian (2.18) with equidistant energies ϵ_i. Let the gap between the different energy levels be $\Delta > 0$. Then

$$\epsilon_i = \epsilon_1 + (i-1)\Delta, \qquad (i = 1, 2, \ldots, n). \tag{6.1}$$

We assume also that $\mu_1 = \mu_2 = \cdots = \mu_n = \mu$. Under these conditions the different orbitals correspond to different energy levels. According to notation of (4.3), we have

$$x_i = \exp\left(\frac{\mu - \epsilon_i}{\tau}\right) = \exp\left(\frac{\mu - \epsilon_1}{\tau}\right) \exp\left(-\frac{\Delta}{\tau}\right)^{i-1} = xq^{i-1}, \tag{6.2}$$

where we have used the notation

$$x = x_1 = \exp\left(\frac{\mu - \epsilon_1}{\tau}\right) \qquad \text{and} \qquad q = \exp\left(-\frac{\Delta}{\tau}\right). \tag{6.3}$$

Under this specialization the elementary symmetric functions simplify. For this purpose, consider their generating function (4.7). Using [21, p. 26] one finds,

$$\begin{aligned}&(1+xt)(1+qxt)\cdots(1+q^{n-1}xt) = \\ &\sum_{k=0}^{n} q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} x^k t^k = \sum_{k=0}^{n} q^{k(k-1)/2} \frac{(q^{n-k+1}; q)_k}{(q; q)_k} x^k t^k,\end{aligned} \tag{6.4}$$

where $\begin{bmatrix} n \\ k \end{bmatrix}$ denotes the Gaussian polynomial [21, p. 26] and $(a; q)_k$ the q-raising factorials [22]

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(1-q)(1-q^2)\cdots(1-q^k)}, \tag{6.5}$$

$$(a;q)_k = (1-a)(1-qa)\cdots(1-q^{k-1}a). \tag{6.6}$$

It follows from (4.7) and (6.2) that

$$e_k(x, qx, q^2x, \cdots, q^{n-1}x) = q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} x^k = q^{k(k-1)/2} \frac{(q^{n-k+1};q)_k}{(q;q)_k} x^k. \tag{6.7}$$

To write down the GPF, we use (4.8), the specialization (6.2) and (6.7)

$$Z(p,n) = \sum_{k=0}^{p} q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} x^k = \sum_{k=0}^{p} q^{k(k-1)/2} \frac{(q^{n-k+1};q)_k}{(q;q)_k} x^k. \tag{6.8}$$

Using the formula

$$\begin{bmatrix} n \\ k \end{bmatrix} = q^{-k(k-1)/2}(-q^n)^k \frac{(q^{-n};q)_k}{(q;q)_k}, \tag{6.9}$$

one obtains

$$Z(p,n) = \sum_{k=0}^{p} \frac{(q^{-n};q)_k}{(q;q)_k} (-q^n)^k x^k = {}_2\phi_1 \left(\begin{matrix} q^{-n}, q^{-p} \\ q^{-p} \end{matrix} ; -q^n x \right). \tag{6.10}$$

The last function is a terminating basic hypergeometric series [22],

$${}_2\phi_1 \left(\begin{matrix} a, b \\ c \end{matrix} ; x \right) = \sum_{k=0}^{\infty} \frac{(a;q)_k (b;q)_k}{(c;q)_k} \frac{x^k}{(q;q)_k}. \tag{6.11}$$

For $p = n$ (6.8) yields

$$Z(n,n) = (-x;q)_n \tag{6.12}$$

The average number of particles in the system follows from (4.14) :

$$\bar{N}(p,n) = \frac{\sum_{k=0}^{p} k q^{k(k-1)/2} \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right] x^k}{\sum_{k=0}^{p} q^{k(k-1)/2} \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right] x^k} = x \frac{\partial}{\partial x} (\ln Z(p,n)) \quad \left(= \tau \frac{\partial}{\partial \mu} (\ln Z(p,n)) \right). \tag{6.13}$$

For $p = n$ (6.13) becomes

$$\bar{N}(n,n) = x \sum_{i=0}^{n-1} \frac{q^i}{1 + q^i x}. \tag{6.14}$$

The average number of particles on the ith orbital can be written in the form :

$$\bar{\theta}_i = \frac{x q^{i-1}}{Z(p,n)} \sum_{k=0}^{p-1} \sum_{l=0}^{k} (-1)^l q^{l(i-1)+(k-l)(k-l-1)/2} \begin{bmatrix} n \\ k-l \end{bmatrix} x^k. \tag{6.15}$$

We used that :

$$e_k(x, qx, q^2x, \ldots, q^{i-2}x, q^i x, \ldots, q^{n-1}x) = \sum_{l=0}^{k} (-1)^l q^{l(i-1)+(k-l)(k-l-1)/2} \begin{bmatrix} n \\ k-l \end{bmatrix} x^k. \tag{6.16}$$

The conclusion from (6.15) is that the "population" of the orbitals depends mainly on their level i via q^{i-1} with $q = \exp(-\Delta/\tau) < 1$: as i increases, $\bar{\theta}_i$ decreases.

Consider the case $p = 1$ and any n. Then

$$Z(1,n) = 1 + (1 + q + q^2 + \ldots + q^{n-1})e^{(\mu-\epsilon_1)/\tau}, \qquad (6.17)$$

$$\bar{N}(1,n) = \frac{(1 + q + \ldots + q^{n-1})}{e^{(\epsilon_1-\mu)/\tau} + (1 + q + \ldots + q^{n-1})}. \qquad (6.18)$$

If $q = \exp(-\Delta/\tau) \ll 1$, i.e., for large gaps between the energy levels or very low temperature, one can neglect all positive powers of q in (6.18). What remains is the Fermi-Dirac distribution

$$\bar{N}(1,n) \approx \frac{1}{e^{(\epsilon_1-\mu)/\tau} + 1}. \qquad (6.19)$$

The expression for the average number of particles on orbital i reads

$$\bar{\theta}_i = \frac{q^{i-1}}{e^{(\epsilon_1-\mu)/\tau} + (1 + q + \ldots + q^{n-1})}, \quad i = 1, \ldots, n. \qquad (6.20)$$

For very low temperatures, or a large energy gap Δ, (6.20) reduces to

$$\bar{\theta}_1 \approx \frac{1}{e^{(\epsilon_1-\mu)/\tau} + 1} \quad \text{and} \qquad \bar{\theta}_i \approx 0 \ \text{ if } \ i > 1. \qquad (6.21)$$

Therefore if the system contains a particle, it is "sitting" permanently on the first (i.e. the lowest) energy orbital. This also explains why $\bar{N}(1,n) \approx \bar{\theta}_1$. To illustrate these ideas, we plot the values of the average number of particles $\bar{\theta}_i$ on orbital i, in the case $p = 1$ and $n = 5$ (five equidistant energy levels), as a function of q, see Figure 3. For any value of q $(0 < q < 1)$, $\bar{\theta}_1 > \bar{\theta}_2 > \bar{\theta}_3 > \bar{\theta}_4 > \bar{\theta}_5$. For small values of q, $\bar{\theta}_1$ is the large and the other averages close to zero. For increasing values of q, the averages on the other orbitals become larger.

7 Concluding remarks

In this paper we have studied the microscopic and thermal (macroscopic) properties of "free" particles, interacting only via statistical interaction. This interaction follows from the Pauli principle of A-superstatistics : the system cannot accommodate more than p particles if the order of statistics is p, irrespective of the number of available orbitals (which may even be infinite).

A-superstatistics is defined by the triple relations (2.4)-(2.6) which should hold for the creation and annihilation operators. The Fock spaces for A-superstatistics are naturally related to certain representations of the Lie superalgebra $sl(1|n)$, and are labeled by a positive integer p referred to as the order of statistics. It is the mathematical structure of the Fock spaces that gives rise to the Pauli principle of A-superstatistics.

It is shown that the creation and annihilation operators of A-superstatistics are fermion-like. Just like ordinary Fermi operators, the creation (resp. annihilation) operators anti-commute. However, they do not satisfy all traditional Fermi relations : only in the limit $p \to \infty$ do the remaining relations tend to the ordinary Fermi relations. It may be unusual and unconventional to consider such alternative creation and annihilation operators. However, we show that (representations of) such operators have already appeared in physical models, proving their applicability.

In the second part of the paper we focuss on the macroscopic properties of A-superstatistics, for a free Hamiltonian. The usual thermal functions (grand partition function, average number of particles, orbital distribution, average energy, ...) are expressed in terms of elementary symmetric functions, and can be considered as deviations of the usual Fermi case when $p < n$.

In addition to the general case, we have considered two specific examples. The case with identical energy levels per orbital (degenerate case) leads to further simplifications of the thermal functions, and the deviation from Fermi statistics (for $p < n$) becomes more apparent. The corresponding distribution functions are reminiscent of the Fermi-Dirac case, but there are also some striking differences (see Figures 1 and 2).

The case with equidistant energy levels is interesting from the mathematical point of view, since the thermal functions have simple expressions in terms of q-series (or basic hypergeometric series). This can be considered as a q-deformation of the degenerate case, and in fact the degenerate case can be deduced from the current one under the limit $q \to 1$. Also the situation with equidistant energy levels yields some interesting physical interpretations, e.g. concerning the orbital distribution (see Figure 3).

Both from the microscopic and macroscopic properties, it can be concluded that A-superstatistics is an interesting deformation of ordinary Fermi statistics.

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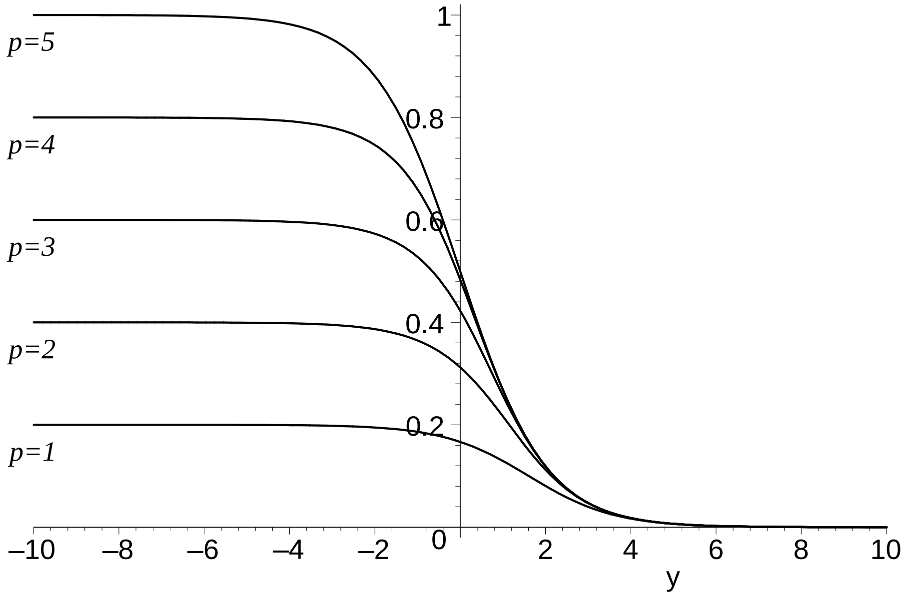

Figure 1: Dependence of the average number of particles on the i-th orbital $\bar{\theta}_i$ upon the variable $y = (\epsilon - \mu)/\tau$, for fixed $n = 5$, and $p = 1, 2, 3, 4, 5$. The distribution $\bar{\theta}_i$ is independent of i since we consider the degenerate case here. For $p = n = 5$, the distribution function coincides with the Fermi-Dirac distribution; for $p < n$ it is different and $\bar{\theta}_i$ cannot exceed p/n.

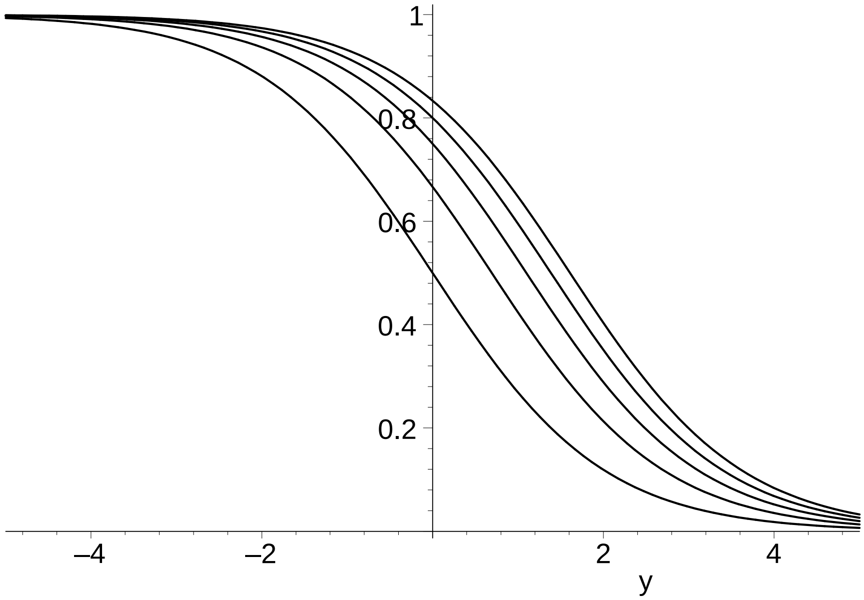

Figure 2: Dependence of the average number of particles $\bar{N}(p,n)$ upon the variable $y=(\epsilon-\mu)/\tau$, for fixed $p=1$, and $n=1,2,3,4,5$, in the degenerate case. The graph of $\bar{N}(1,1)$ is the closest to the y-axis, then $\bar{N}(1,2)$, etc. Observe that the graph of $\bar{N}(1,1)$ coincides with the Fermi-Dirac distribution.

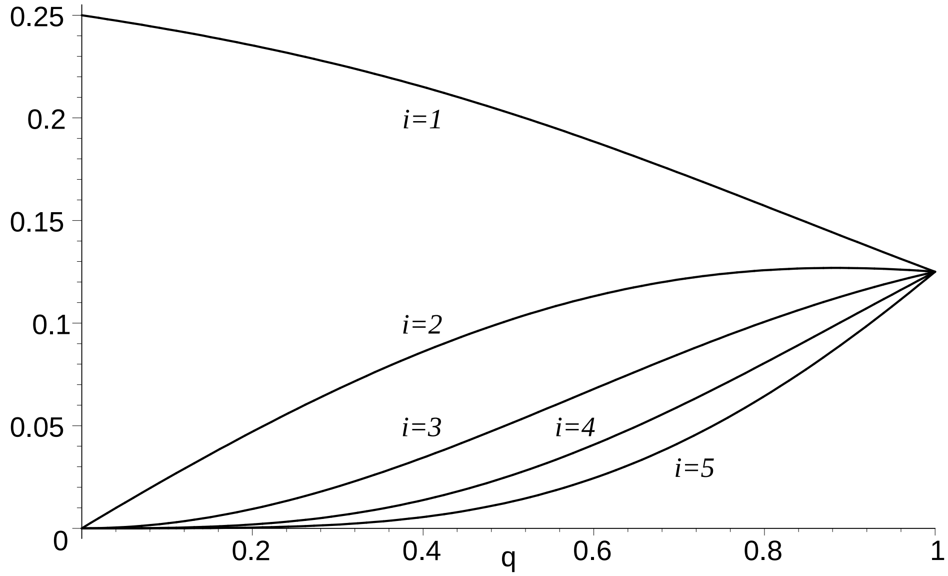

Figure 3: Graphs of the average number of particles $\bar{\theta}_i$ on the i-th orbital, for $p = 1$, $n = 5$, and $i = 1, 2, 3, 4, 5$. The distribution $\bar{\theta}_i$ is plotted as a function of $q = \exp(-\Delta/\tau)$, where Δ is the gap between the equidistant energy levels.