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THE HUSIMI DISTRIBUTION FUNCTION AND SUPERPOSITION OF Q-HARMONIC OSCILLATOR STATIONARY STATES

Abstract

We study a superposition of four stationary states of the q-deformed quantum harmonic oscillator in phase space, through the Husimi quasi-distribution function. This function is analytically computed and it consists of 16 parts, which are explicitly given in terms of q-shifted factorials. Our results are consistent with the known observation that the smoothing of the Wigner quasi-probability distribution through a filter of a size larger than $\hbar$ restricts the appearance of any sub-Planck structures when $q \to 0$.

1. Introduction

Quasi-probability distribution functions are known to play an important role in the study of quantum-classical transition properties of quantum dynamical systems. Although it is possible to get sufficient information about the dynamics of the quantum system through examining its properties in the configuration representation, to understand its behaviour completely one however needs to know them in phase space too. To this end only the quasi-probability distribution functions are helpful, because phase space in the quantum approach is constructed through them. The construction of the correct phase space and the computation of an explicit form of a certain quasi-probability distribution function enables one to understand what is the correspondence between the quantum and classical dynamics of the problem under study [1].

The first definition of the quasi-probability distribution functions in the quantum approach goes back to the 1930s, when the famous Wigner function was proposed to study quantum corrections in classical statistical mechanics [2]. The Wigner function satisfies almost all properties of the probability distribution, but due to the uncertainty principle in quantum mechanics, distributions defined by this function are capable of both negative and positive values. Due to such a behaviour of the Wigner function, Gauss smoothing of this distribution has been proposed; this allowed researchers to consider various smoothed quasi-probability distribution functions, not coincident with the Wigner function [3]. Nowadays, there are many papers that study evolution of the quantum dynamical system comparing them in terms of the various quasi-probability distribution functions [4]-[9].

Recently, a q-deformed quantum harmonic oscillator model has been studied in phase space by employing the Wigner and Husimi quasi-probability distribution functions [10]. It was shown that, depending on the values of q-parameter within the condition $0 < q < 1$, this q-oscillator model behaves in phase space as a usual non-relativistic harmonic oscillator if $q \to 1$ and becomes a coherent-like dynamical system if $q \to 0$. Such behaviour was valid for both the Wigner and Husimi quasi-probability distribution functions. As a next step, the new kind of superposition based on q-oscillator stationary states was proposed and was studied in detail in phase space by employing Wigner quasi-probability distribution functions [11]. The main goal for such proposal for a new kind of superposition was to observe an appearance of sub-Planck structures during evolution of this dynamical system as functions of the q-parameter values. The reason for this expectation comes from Zurek's seminal paper [12], where he shows that phase space structures for a quantum system associated with sub-Planck structures exist and they can be observed through the superposition of four minimum-uncertainty Gaussians (so-called 'compass states') [13]-[16]. Then, it has been shown that the Wigner function of the proposed superposition based on q-oscillator stationary states, behaves as the standard non-relativistic quantum harmonic oscillator if $q \to 1$, whilst one observes an appearance of the compass states and sub-Planck structures similar to [12] in the limit as $q \to 0$.

Therefore an interesting question arises on what kind of picture one will have if a superposition of q-oscillator stationary states is constructed on phase space in terms of the Husimi function. As already mentioned in [10], the behaviour of the Husimi function is similar in general to that of the Wigner function for the q-oscillator, *i.e.*, it is the standard non-relativistic harmonic oscillator for $q \to 1$ and a Gaussian, similar to the quantum harmonic oscillator coherent states, for $q \to 0$. Then it seems that the evolution of the Husimi function for the proposed superposition, based on q-oscillator stationary states, gives the same pictures for sub-Planck structures as in the case of the Wigner function. However, the definition of the Husimi function is based on smoothing of the Wigner function to scales larger than $\hbar$, which allows one to obtain completely positive distribution function for the dynamical system under consideration. Also, it is necessary to mention that the Husimi function, even though positive, fails to enjoy the marginalitiy properties. Then, taking into account these controversies it becomes interesting to compute an explicit form of the Husimi function for a superposition of four stationary states of the q-oscillator, and then to analyze what kind of differences will such a superposition exhibit in the case of a completely positive distribution. In this paper we investigate this problem and compute an explicit expression of the Husimi function for a superposition of four stationary states of the q-oscillator.

Our paper is organized as follows: in Section 2 we recall common information about the Wigner and Husimi quasi-probability distribution functions in the quantum approach. Section 3 is devoted to defining the q-deformed quantum harmonic oscillator and the superposition of its four stationary states. Finally, the computation of the Husimi function for such a superposition and a discussion of the derived results are given in Section 4.

Throughout this exposition we employ standard notations of the theory of special functions (see, for example, [17]).

2. The Wigner and Husimi distribution functions

The Wigner function is a quasi-probability distribution in terms of momentum p and position x, determined as follows:

$$F_W(p,x) = \frac{1}{2\pi\hbar}\int\limits_{-\infty}^{+\infty} \psi^*\left(x - \tfrac{1}{2}\,y\right)\,\psi\left(x + \tfrac{1}{2}\,y\right) e^{-ipy/\hbar} dy$$

$$= \frac{1}{2\pi\hbar} \int\limits_{-\infty}^{+\infty} \tilde{\psi}^* \left(p - \tfrac{1}{2}\,v\right)\, \tilde{\psi}\left(p + \tfrac{1}{2}\,v\right) e^{ixq/\hbar} dv\,. \tag{1}$$

It is bounded by the restriction $|F_W(p,x)| \leq (\pi\hbar)^{-1}$, which means that such a function generally takes both negative and positive values (but note that the Wigner function for the ground state of the linear harmonic oscillator takes only positive values, see (4) below). For this reason, the Gaussian smoothing approach was proposed in order to define some class of completely positive joint probability functions of the following form:

$$\bar{F}(p,x) = \frac{1}{\pi\hbar} \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} e^{-\frac{v^2}{2\Delta_p^2} - \frac{y^2}{2\Delta_x^2}} F_W(p+v, x+y)\, dv\, dy \geq 0. \tag{2}$$

Here, $\Delta_p\Delta_x$ defines a finite region in the phase plane. Choosing some value for this definition, one can obtain Gaussian smoothing of the Wigner function. The simplest way corresponds to the value $\Delta_p\Delta_x = \hbar/2$. Then (2) is well-known as the Husimi function of the form:

$$F_H(p,x) = \frac{1}{(2\pi)^{3/2}\,\hbar\,\Delta_x} \left| \int\limits_{-\infty}^{+\infty} \psi(y) \cdot e^{-\frac{ip\,y}{\hbar} - \frac{(x-y)^2}{4\Delta_x^2}} dy \right|^2 . \tag{3}$$

Unlike the Wigner quasi-distribution function, the Husimi quasi-distribution function, defined by (3), is bounded by the restriction $0 \leq F_H(p,x) \leq (\pi\hbar)^{-1}$. In other words, one observes from this restriction that it takes only positive values, therefore it represents a distribution function rather than quasi-distribution one. Here, we use the $1/\pi\hbar$ normalization of [18] for the Husimi function, where it is called as the smoothed Wigner function.

Both the Wigner and the Husimi functions are explicitly computed for the case of the non-relativistic quantum harmonic oscillator stationary states (for $n = 0, 1, 2, ...$) in canonical approach and they are given as [18]

$$F_W^{(HO)}(p,x) = \frac{(-1)^n}{\pi\hbar} L_n\left(\frac{4}{\hbar\omega}\left(\frac{p^2}{2m} + \frac{m\omega^2x^2}{2}\right)\right) e^{-\frac{2}{\hbar\omega}\left(\frac{p^2}{2m} + \frac{m\omega^2x^2}{2}\right)}\,, \tag{4}$$

$$F_H^{(HO)}(p,x) = \frac{1}{2\pi\hbar\, n!}\left[\frac{1}{\hbar\omega}\left(\frac{p^2}{2m} + \frac{m\omega^2x^2}{2}\right)\right]^n e^{-\frac{1}{\hbar\omega}\left(\frac{p^2}{2m} + \frac{m\omega^2x^2}{2}\right)}\,, \tag{5}$$

where $L_n(z)$ are the Laguerre polynomials. Explicit expression of the Husimi function (5) is computed for value $\Delta_x^2 = \frac{\hbar}{2m\omega}$ and connected with explicit expression of the Wigner function (4) through (2).

3.A superposition of four stationary states of the q-oscillator

The q-deformed models of the non-relativistic harmonic oscillator that we will employ in order to construct a new type of the superposition, were introduced and studied in a number of papers [19]-[24]. The main idea of introducing the q-oscillator realization is based on the assumption that its annihilation and creation operators satisfy the q-deformed Heisenberg commutation relations $[b^-, b^+]_q = b^-b^+ - q\,b^+b^- = 1$, which generalize the standard Heisenberg commutation relations, and recover the latter in the limit $q \to 1$ [25]. Then, using this property, one can introduce four pairs (North, South, East and West) of the q-creation and annihilation operators, which satisfy the q-Heisenberg commutation relations [11]:

$$b_N^{\pm} = \pm \frac{i}{\sqrt{1-q}} e^{\mp\lambda x^2} \left(q^{\pm 1} e^{2\lambda\kappa x} e^{\mp\kappa\partial_x} - q^{\frac{1}{2}} e^{\mp\frac{\kappa}{2}\partial_x} \right) e^{\pm\lambda x^2}, \tag{6}$$

$$b_S^{\pm} = \mp \frac{i}{\sqrt{1-q}} e^{\mp\lambda x^2} \left(q^{\pm 1} e^{-2\lambda h x} e^{\pm\kappa\partial_x} - q^{\frac{1}{2}} e^{\pm\frac{\kappa}{2}\partial_x} \right) e^{\pm\lambda x^2}, \tag{7}$$

$$b_E^{\pm} = \mp \frac{i}{\sqrt{1-q}} e^{\mp\lambda x^2} \left(e^{\pm 2i\lambda\kappa x} - q^{\frac{1}{2}} e^{\frac{i\kappa}{2}\partial_x} \right) e^{\pm\lambda x^2}, \tag{8}$$

$$b_W^{\pm} = \pm \frac{i}{\sqrt{1-q}} e^{\mp\lambda x^2} \left(e^{\mp 2i\lambda\kappa x} - q^{\frac{1}{2}} e^{-\frac{i\kappa}{2}\partial_x} \right) e^{\pm\lambda x^2}. \tag{9}$$

Here, λ is a parameter given in terms of the mass m and the frequency ω as $\lambda = m\omega/2\hbar$. It is necessary to mention that q-deformation of the non-relativistic quantum harmonic oscillator changes it from being a dynamical system, described by a differential equation, to a system that is described by a finite-difference equation. Therefore, two additional interconnected parameters κ and q appear in (6)-(9), where κ is the mesh (or grid) parameter of the finite-difference method, $0 < \kappa < \infty$, and q is the deformation parameter related to κ as $q = \exp\left(-\lambda\kappa^2\right)$, and therefore $0 < q < 1$. Such a q-deformation of the quantum harmonic oscillator was further generalized into so-called f-oscillators, where it is shown that one can consider the deformation of them without using difference equations [26], [27].

Taking into account that q-creation and annihilation operators satisfy the requirement $b^+ b^- \psi_n(x) = [n]_q \psi_n(x)$, where $[n]_q$ is the basic number, $[n]_q = \frac{1-q^n}{1-q}$, one finds the following four wave functions satisfying the above equation [11]:

$$\psi_n^N(x) = c_n H_n\left(-q^{n-1} e^{2\lambda\kappa x} \middle| q^{-1} \right) e^{-\lambda x^2}, \tag{10}$$

$$\psi_n^S(x) = c_n H_n\left(-q^{n-1} e^{-2\lambda\kappa x} \middle| q^{-1} \right) e^{-\lambda x^2}, \tag{11}$$

$$\psi_n^E(x) = c_n H_n\left(-e^{2i\lambda\kappa x} \middle| q \right) e^{-\lambda x^2}, \tag{12}$$

$$\psi_n^W(x) = c_n H_n\left(-e^{-2i\lambda\kappa x} \middle| q \right) e^{-\lambda x^2}. \tag{13}$$

The normalization constant c_n in (10)-(13) can be computed from their orthogonality relations and it is equal to

$$c_n = \left(\frac{2\lambda}{\pi} \right)^{1/4} q^{n/2} (q;q)_n^{-1/2}.$$

In (10)-(13), $H_n(-\tilde{x}|\, q)$ is the Rogers-Szegö polynomial of the following form:

$$H_n(z|\, q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \left(q^{1/2} z \right)^k,$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is the q-binomial coefficient,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}} = \begin{bmatrix} n \\ n-k \end{bmatrix}_q,$$

and $(q;q)_n$ is the q-shifted factorial, defined as

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1} \left(1 - aq^k\right), \quad n \geq 1.$$

Then, it is possible to construct the following superposition of four stationary states in terms of (10)-(13):

$$\Psi_{\times,n}(x) = \frac{N_q}{2} \left[(-1)^n \psi_n^N(x) + \psi_n^S(x) + i^n \psi_n^E(x) + i^{-n} \psi_n^W(x) \right], \tag{14}$$

where the normalization constant N_q is found from the following overlap of states (14),

$$\int_{-\infty}^{+\infty} \Psi^*_{\times,m}(x) \cdot \Psi_{\times,n}(x)\, dx = \delta_{m,n}, \tag{15}$$

and it has the following form:

$$N_q = \left\{ 1 + \frac{q^n}{(q;q)_n} \sum_{k=0}^{n} \frac{(q^{-n};q)_k}{(q;q)_k} \left[(-1)^n \left(q^k;q\right)_n + i^n \left(q^{ik};q\right)_n + i^{-n} \left(q^{-ik};q\right)_n \right] q^{nk} \right\}^{-\frac{1}{2}}.$$

Despite its rather complicated form, in the limit $q \to 1$ this relation simply reduces to 1/2. In order to prove this, the use of the following summations is sufficient:

$$\frac{(-1)^n}{n!} \sum_{k=0}^{n} \frac{(-n)_k (k)_n}{k!} = 1,$$

$$\frac{(\pm i)^n}{n!} \sum_{k=0}^{n} \frac{(-n)_k (\pm ik)_n}{k!} = 1,$$

where $(a)_n$ is the standard shifted factorial,

$$(a)_0 = 1, \quad (a)_n = \prod_{k=0}^{n-1} (a+k), \quad n \geq 1.$$

It is well known that the Rogers-Szegö polynomials satisfy the following orthogonality relation on the unit circle [28]. However, [29] discusses slightly different orthogonality relation for the Rogers-Szegö and Stieltjes-Wigert polynomials on the full real axis

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} H_n\left(-e^{-2i\alpha y}|\tilde{q}\right) H_m\left(-e^{2i\alpha y}|\tilde{q}\right) e^{-y^2} dy = \frac{(\tilde{q};\tilde{q})_n}{\tilde{q}^n} \delta_{nm}, \tag{16}$$

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} S_n\left(\tilde{q}^{-1/2} e^{-2\alpha y};\tilde{q}\right) S_m\left(\tilde{q}^{-1/2} e^{-2\alpha y};\tilde{q}\right) e^{-y^2} dy = \frac{1}{(\tilde{q};\tilde{q})_n \tilde{q}^n} \delta_{nm}, \tag{17}$$

and provides a detailed proof of their connection with the mentioned above the orthogonality relation on the unit circle. Here,

$$S_n\left(\tilde{q}^{-1/2}e^{-2\alpha y};\tilde{q}\right)=\frac{1}{(\tilde{q};\tilde{q})_n}\sum_{k=0}^{n}\frac{(\tilde{q}^{-n};\tilde{q})_k}{(\tilde{q};\tilde{q})_k}\tilde{q}^{k(k/2+n)}e^{-2k\alpha y},\quad \tilde{q}=e^{-2\alpha^2},\ \alpha>0$$

is the Stieltjes-Wigert polynomial [17]. In order to compute an integral (15) of the overlap of states (14), we used orthogonality relations on the full real axis (16) and (17) instead the well-known one on the unit circle.

It should be noted that all four wave functions (10)-(13) together with their phase factors defined in (14), reduce to non-relativistic quantum harmonic oscillator wave function in conformity with [10]. Therefore, the superposition (14) also easily recovers the non-relativistic quantum harmonic oscillator stationary states in the limit as $q\to 1$.

4. Computation of the Husimi function and further discussions

To calculate the Husimi function for the superposition (14), we employ the same method as in [10]. The only difference is that the superposition (14) leads to the calculation of 16 integrals, and the result is:

$$\begin{aligned}F^{H}_{\times,n}(p,x)&=F^{NN}(p,x)+(-1)^nF^{NS}(p,x)+i^{-n}F^{NE}(p,x)+i^nF^{NW}(p,x)\\&+(-1)^nF^{SN}(p,x)+F^{SS}(p,x)+i^nF^{SE}(p,x)+i^{-n}F^{SW}(p,x)\\&+i^nF^{EN}(p,x)+i^{-n}F^{ES}(p,x)+F^{EE}(p,x)+(-1)^nF^{EW}(p,x)\\&+i^{-n}F^{WN}(p,x)+i^nF^{WS}(p,x)+(-1)^nF^{WE}(p,x)+F^{WW}(p,x)\,,\end{aligned}\qquad(18)$$

where the above components are explicitly given as

$$\begin{aligned}F^{NN}(p,x)=F^{SS}(p,-x)&=\gamma_n(p,x)\left(e^{-ia/2};q\right)_n\left(e^{ia^*/2};q\right)_n\,,\\F^{SN}(p,x)=F^{NS}(-p,-x)&=\gamma_n(p,x)\left(e^{ia/2};q\right)_n\left(e^{ia^*/2};q\right)_n\,,\\F^{ES}(p,x)=F^{NE}(p,-x)&=\gamma_n(p,x)\left(e^{a/2};q\right)_n\left(e^{-ia^*/2};q\right)_n\,,\\F^{SE}(p,x)=F^{EN}(p,-x)&=\gamma_n(p,x)\left(e^{ia/2};q\right)_n\left(e^{a^*/2};q\right)_n\,,\\F^{EE}(p,x)=F^{WW}(-p,-x)&=\gamma_n(p,x)\left(e^{a/2};q\right)_n\left(e^{a^*/2};q\right)_n\,,\\F^{EW}(p,x)=F^{WE}(p,-x)&=\gamma_n(p,x)\left(e^{a/2};q\right)_n\left(e^{-a^*/2};q\right)_n\,,\\F^{WS}(p,x)=F^{NW}(p,-x)&=\gamma_n(p,x)\left(e^{-a/2};q\right)_n\left(e^{-ia^*/2};q\right)_n\,,\\F^{SW}(p,x)=F^{WN}(p,-x)&=\gamma_n(p,x)\left(e^{ia/2};q\right)_n\left(e^{-a^*/2};q\right)_n\,,\end{aligned}$$

with $a=\frac{\kappa}{\hbar}p+2i\lambda\kappa x$ and the common factor $\gamma_n(p,x)$ for all 16 components, which is equal to

$$\gamma_n(p,x)=\frac{N_q^2}{8\pi\hbar}\frac{q^n}{(q;q)_n}e^{-\frac{1}{\hbar\omega}\left(\frac{m\omega^2x^2}{2}+\frac{p^2}{2m}\right)}\,.$$

In order to arrive at (18) explicitly, one needs just to substitute (14) into the general definition of the Husimi function (3) and to employ the well-known Gaussian integral and q-binomial theorem for each of the 16 integrals. This results in the analytical expression for the Husimi function of the proposed superposition, which is presented above.

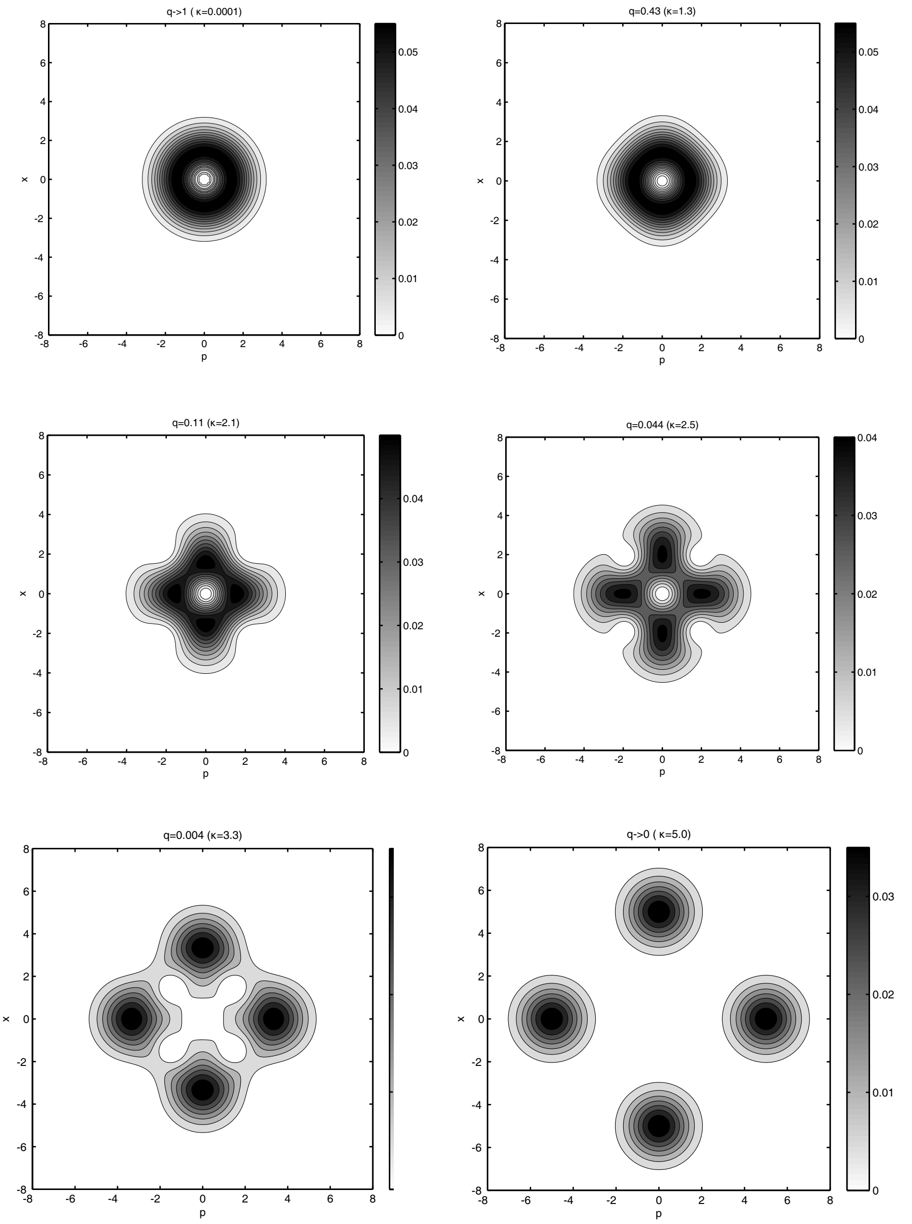

Fig. 1. Density plot of the Husimi function of the single photon state ($n = 1$) of (18), for values of $\kappa = 0.0001$, 1.3, 2.1,2.5, 3.3, 5.0 and $m = \omega = \hbar = 1$. The associated values of q are found through the relation $q = \exp(-\kappa^2/2)$.

In Fig.1 we show a density plot of the Husimi function of the single photon state ($n = 1$) of (18). We depict the behaviour of the Husimi function for the values $\kappa = 0.0001$, 1.3, 2.1, 2.5, 3.3 and 5.0, which correspond to the values $q = 1$, 0.43, 0.11, 0.044, 0.004 and 0. All plots are given in terms of the units $m = \omega = \hbar = 1$. Chosen values of depicted plots are completely same with plots of the Wigner function of the single photon state ($n = 1$) for such a q-oscillator superposition [11], which allows to make necessary comparisons between two quasi-probability functions. As one observes from the first plot, for values of q, close to 1 ($\kappa = 0.0001$), the superposition under discussion completely reduces to the non-relativistic quantum harmonic oscillator. This result is obvious from the following point of view. First, computation of the $q \to 1$ limit of the expressions for $F^{NN}(p,x)$, $F^{SS}(p,x)$, $F^{EE}(p,x)$ and $F^{EE}(p,x)$ components are similar to the $q \to 1$ limit of q-oscillator Husimi function, which is presented in [10]. This means that for all 4 above-mentioned components one can recover the non-relativistic quantum harmonic oscillator Husimi function (5) easily. The $q \to 1$ limit relations for the other 12 expressions, $F^{NS}(p,x)$, $F^{NE}(p,x)$, $F^{NW}(p,x)$, $F^{SN}(p,x)$, $F^{SE}(p,x)$, $F^{SW}(p,x)$, $F^{EN}(p,x)$, $F^{ES}(p,x)$, $F^{EW}(p,x)$, $F^{WN}(p,x)$, $F^{WS}(p,x)$ and $F^{WE}(p,x)$, can be readily computed as well. However, here it is necessary to apply different techniques of limit computation, namely, one needs to expand all q-depending factors in powers of κ and then to evaluate a limit for each term separately. Thus, all of them recover (5) in this limit too. Of course, here it is also necessary to take into account that the $q \to 1$ limit of N_q is $1/2$. While the value of q varies from 1 to 0, one observes that the transition from non-relativistic quantum harmonic oscillator stationary states to the so-called compass states but without any sign of sub-Planck structures. This behaviour proves that contrary to the cases of Wigner quasi-distribution functions and Kirkwood quasi-distribution functions, the Husimi function does not lead to the appearance of sub-Planck structures from the superposition of q-oscillator stationary states. In other words, its Gaussian smoothing is based on the suppression of all sub-Planck structures and therefore at the end one observes the pure classical case of the four shifted Gaussians, which are like the well-known quantum harmonic oscillator coherent states.

Quasi-distribution functions are powerful tools for possible description of quantum dynamical systems in the language of classical physics. Wigner function is most important both from mathematical point of view and a lot of experimental applications (we just refer to recent one about experimental observation of quantum chaos in a beam of light [30]). However, there are number of others, which are related with the Wigner function through the certain method of its Gaussian smoothing [31], [32]. In this paper, we applied one of them - namely the Husimi quasi-distribution function to superposition of the q-deformed oscillator stationary states in order to see what will be behaviour of the q-deformed superposed system in terms of the smoothed quasi-distribution function. Our results show that unlike the case of the Wigner function, where sub-Planck structures can be easily observed under the limit $q \to 0$, there is no any evidence of such a structures for the Husimi function, which is in complete agreement with statement that the Husimi function is not ideally adapted to the study any quantum coherence effects and computed interference terms are so small than in case of Wigner function [33].

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