Computational Methods for Imprecise Continuous-Time Birth-Death Processes: a Preliminary Study of Flipping Times

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We introduce the notion of flipping times for imprecise continuous-time birth-death processes, show how to obtain them, and explain how they lead to new computational methods.

The Precise Case Consider a continuous-time Markov processes where, at any time t, the stochastic matrix of the process P_t is derived from a transition rate matrix Q. When Q is bounded, P_t satisfies the Kolmogorov backward equation

$$\frac{d}{dt}P_t = QP_t. \tag{1}$$

If we let $f_t(x) := E_t(f|X_0 = x)$, with f a real-valued function on the finite state space $\mathcal{X}$ and $x \in \mathcal{X}$ an initial state, then we can rewrite Equation (1) as follows:

$$\frac{d}{dt}f_t = Qf_t. \tag{2}$$

Combined with the boundary condition $f_0 = f$, the unique solution of Equation (2) is $f_t = e^{Qt}f$.

Instead of considering a time-invariant Q, we can also let Q_t be a function of the time t. In that case, Equation (2) can be rewritten as

$$\frac{d}{dt}f_t = Q_t f_t. \tag{3}$$

In general, Equation (3) has no analytical solution.

The Imprecise Case We focus on the case where every state in $\mathcal{X} := \{0, \ldots, L\}$, has an interval-valued birth and/or death rate. The transition rate matrix is then a tridiagonal matrix of the form

$$\begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \mu_i & -(\mu_i + \lambda_i) & \lambda_i & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & \mu_L & -\mu_L \end{pmatrix}$$

where, for all $i \in \{0, \ldots, L-1\}$ and $j \in \{1, \ldots, L\}$, $\lambda_i \in [\underline{\lambda}, \overline{\lambda}]$ and $\mu_j \in [\underline{\mu}, \overline{\mu}]$. We use $\mathcal{Q}$ to denote the the set that consists of all these transition rate matrices.

At any time t, the only assumption we make about Q_t is that it is an element of $\mathcal{Q}$. Every such possible choice of non-stationary transition rate matrices will, by Equation (3), result in a—possibly different—solution f_t. Our goal is to calculate exact lower and upper bounds for the set of all these solutions f_t, as denoted by $\underline{f}_t$ and $\overline{f}_t$; we focus on the lower bound here. As proved by Škulj [1], $\underline{f}_t$ is the solution to

$$\frac{d}{dt}\underline{f}_t = \min_{Q \in \mathcal{Q}} Q\underline{f}_t, \tag{4}$$

with boundary condition $\underline{f}_0 = f$. If $\mathcal{Q}$ is the convex hull of a *finite* number of extreme transition rate matrices—as in our case—then since the solution to the above differential equation is continuous, we find that there must be time points $0 = t_0 < t_1 < \ldots < t_i < t_{i+1} < \ldots$ such that, for all $t \in [t_i, t_{i+1}]$, the minimum in Equation (4) is obtained by the same extreme transition rate matrix $Q_i \in \mathcal{Q}$. We call these time points t_i *flipping times*. The differential equation (4) is then piecewise linear, and the solution is therefore given by

$$\underline{f}_t = e^{Q_i(t-t_i)}e^{Q_{i-1}(t_i-t_{i-1})} \ldots e^{Q_1(t_2-t_1)}e^{Q_0(t_1)}f,$$

for $t \in [t_i, t_{i+1}]$. The difficult part is now to find the flipping times t_i and the corresponding extreme transition rate matrices Q_i. We provide computational methods that are able to do so.

Keywords. Imprecise continuous-time Markov process, birth-death process, flipping time. birth-death process, flipping time.

References

[1] Damjan Škulj. Efficient computation of the bounds of continuous time imprecise markov chains. *Applied Mathematics and Computation*, 250:165–180, 2015.