Calculating bounds on expected return and first passage times in finite-state imprecise birth-death chains

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Birth-death chains

Birth-death chain $\Rightarrow$ special type of Markov chain

Finite state space $\mathscr{X} := \{0, \ldots, L\}$, with $L \in \mathbb{N}$

Random variable X_n and a sequence of variables $X_{k:n}$, with $k, n \in \mathbb{N}$ and $k \leq n$

A sequence can be infinite as well $X_{k:\infty}$

Sequence of state values $x_{1:n} := x_1, \ldots, x_n$ in $\mathscr{X}^n$

Markov condition $\Rightarrow$ $E_{n+1}(\cdot|x_{1:n}) = E_{n+1}(\cdot|x_n), \forall x_{1:n} \in \mathscr{X}^n$

where $E_{n+1}(\cdot|x_n)$ is the expectation operator with p.m.f $p(X_{n+1}|x_n)$

for p time-homogeneous $\Rightarrow$ $P = \begin{pmatrix} r_0 & p_0 & 0 & \cdots & \cdots & 0 \\ q_1 & r_1 & p_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & q_{L-1} & r_{L-1} & p_{L-1} \\ 0 & \cdots & \cdots & 0 & q_L & r_L \end{pmatrix}$

Imprecise birth-death chains

Consider a matrix P with p.m.f. not precisely known

For every $i \in \mathscr{X}$, the p.m.f. of the i row belong to a credal set $\mathscr{M}_i$

and consists of elements ϕ_i of the form

$$\phi_i(j) = \begin{cases} q_i & \text{if } j = i-1 \\ r_i & \text{if } j = i \\ p_i & \text{if } j = i+1 \\ 0 & \text{otherwise} \end{cases} \quad i \in \mathscr{X} \setminus \{0, L\} \qquad \phi_0(j) = \begin{cases} r_0 & \text{if } j = 0 \\ p_0 & \text{if } j = 1 \\ 0 & \text{otherwise} \end{cases} \quad \phi_L(j) = \begin{cases} q_L & \text{if } j = L-1 \\ r_L & \text{if } j = L \\ 0 & \text{otherwise} \end{cases}$$

<u>Positivity assumption</u>: r_0, p_0, r_L, q_L and q_i, r_i, p_i for all $i \in \mathscr{X} \setminus \{0, L\}$ strictly positive

Imprecise Markov condition

Lower and upper expectations of real-valued function f on $\mathscr{X}$

$$\underline{E}(f|i) \coloneqq \min_{\phi_i \in \mathscr{M}_i} E_{\phi_i}(f) = \min_{\phi_i \in \mathscr{M}_i} \left\{ \sum_{j \in \mathscr{X}} \phi_i(j) f(j) \right\}$$

$$\overline{E}(f|i) \coloneqq \max_{\phi_i \in \mathscr{M}_i} E_{\phi_i}(f) = \max_{\phi_i \in \mathscr{M}_i} \left\{ \sum_{j \in \mathscr{X}} \phi_i(j) f(j) \right\}$$

and for all $x_{1:n} \in \mathscr{X}^n$, the imprecise Markov condition is

$$\underline{E}_{n+1}(\cdot|x_{1:n}) = \underline{E}_{n+1}(\cdot|x_n) \coloneqq \underline{E}(\cdot|x_n)$$

Global uncertainty models

Based on the notion of *submartingales,* we derive global uncertainty models

These models satisfy a version of the *Law of Iterated expectation*

For every $n \in \mathbb{N}$ and every real-valued function g on $\mathscr{X}^{\mathbb{N}}$

$\underline{E}_{n+1:\infty}(g(X_{n+1:\infty})|i) = \underline{E}_{n+2:\infty}(g(X_{n+2:\infty})|i)$ (time-homogeneity)

By defining f' on $\mathscr{X}$ by $f'(i') := \underline{E}_{n+2:\infty}(g(i', X_{n+2:\infty})|i')$ for all $i' \in \mathscr{X}$, then

$$\underline{E}_{n+1:\infty}(g(X_{n+1:\infty})|i) = \underline{E}_{n+1}(f'|i) = \underline{E}(f'|i)$$

First passage time

The first passage time from i to j with $i, j \in \mathscr{X}$ is

$$\tau_{i\to j}(i, X_{n+1:\infty}) := \begin{cases} 1 & X_{n+1} = j \\ 1 + \tau_{X_{n+1}\to j}(X_{n+1}, X_{n+2:\infty}) & X_{n+1} \neq j \end{cases}$$

$$= 1 + \mathbb{I}_{j^c}(X_{n+1})\tau_{X_{n+1}\to j}(X_{n+1}, X_{n+2:\infty})$$

where $\mathbb{I}_{j^c}$ is the indicator function of $j^c := \mathscr{X} \setminus \{j\}$

For $i = j$, we have the return time

Due to time-homogeneity $\underline{\tau}_{i\to j,n} := \underline{E}_{n+1:\infty}(\tau_{i\to j}(i, X_{n+1:\infty})|i)$ and

$\overline{\tau}_{i\to j,n} := \overline{E}_{n+1:\infty}(\tau_{i\to j}(i, X_{n+1:\infty})|i)$ will be denoted by $\underline{\tau}_{i\to j}$ and $\overline{\tau}_{i\to j}$

Due to positivity assumption $\underline{\tau}_{i\to j}$ and $\overline{\tau}_{i\to j}$ are *real-valued* and *strictly positive*

and have the form $\underline{\tau}_{i\to j} = 1 + \underline{E}(\mathbb{I}_{j^c}\underline{\tau}_{\bullet\to j}|i)$ and $\overline{\tau}_{i\to j} = 1 + \overline{E}(\mathbb{I}_{j^c}\overline{\tau}_{\bullet\to j}|i)$

Lower expected upward first passage time

The first passage time from i to j with $i, j \in \mathscr{X}$ and $i < j$

- $\underline{\tau}_{0\to 1} = \dfrac{1}{\overline{p}_0}$
- For all $i \in \mathscr{X} \setminus \{0, L\}$, we have that $\min_{\phi_i \in \mathscr{M}_i} \{q_i \underline{\tau}_{i-1\to i} - p_i \underline{\tau}_{i\to i+1}\} = -1$

For all $\mathscr{M}_i$ satisfying the positivity assumption, with $i \in \mathscr{X} \setminus \{0, L\}$,

and c a real constant, then $\min_{\phi_i \in \mathscr{M}_i} \{qc - p\mu\}$ is strictly decreasing in μ

Lower expected upward first passage time

$$\min_{\phi_i \in \mathscr{M}_i} \{q_i \underline{\tau}_{i-1\to i} - p_i \underline{\tau}_{i\to i+1}\} = -1$$

We can calculate $\underline{\tau}_{i\to i+1}$ recursively

Using a bisection method, as long as we have calculated $\underline{\tau}_{i-1\to i}$ …

Moreover,

- For all $i \in \mathscr{X} \setminus \{0, L\}$, s.t $i+1 < j$, we have that $\underline{\tau}_{i\to j} = \underline{\tau}_{i\to i+1} + \underline{\tau}_{i+1\to j}$
- For all $i \in \mathscr{X}$, such that $i < j$, we have that $\underline{\tau}_{i\to j} = \sum_{k=i}^{j-1} \underline{\tau}_{k\to k+1}$

Lower expected downward first passage time

The first passage time from i to j with $i, j \in \mathscr{X}$ and $i > j$

Similarly to the upward case…

- $\underline{\tau}_{L \to L-1} = \dfrac{1}{\overline{q}_L}$
- For all $i \in \mathscr{X} \setminus \{0, L\}$, we have that $\min_{\phi_i \in \mathscr{M}_i} \{-q_i \underline{\tau}_{i \to i-1} + p_i \underline{\tau}_{i+1 \to i}\} = -1$
- For all $i \in \mathscr{X}$, such that $i > j$, we have that $\underline{\tau}_{i \to j} = \sum_{k=j}^{i-1} \underline{\tau}_{k+1 \to k}$

Lower expected return time

The first passage time from i to j with $i, j \in \mathscr{X}$ and $i = j$

Combining the results from expected upward with these of downward first passage times

- $\underline{\tau}_{0\to 0} = 1 + \min_{\phi_0 \in \mathscr{M}_0} \{p_0 \underline{\tau}_{1\to 0}\} = 1 + \underline{p}_0 \underline{\tau}_{1\to 0}$
- $\underline{\tau}_{L\to L} = 1 + \min_{\phi_L \in \mathscr{M}_L} \{q_L \underline{\tau}_{L-1\to L}\} = 1 + \underline{q}_L \underline{\tau}_{L-1\to L}$

and for all $i \in \mathscr{X} \setminus \{0, L\}$

- $\underline{\tau}_{i\to i} = 1 + \min_{\phi_i \in \mathscr{M}_i} \{q_i \underline{\tau}_{i-1\to i} + p_i \underline{\tau}_{i+1\to i}\}$

Linear vacuous mixtures

The set $\mathscr{M}_i$ is a subset of the simplex $\Sigma_{\mathscr{X}}$

For any $i \in \mathscr{X}$, $\Sigma_{\mathscr{X}_i}$ is the subset of $\Sigma_{\mathscr{X}}$ containing p.m.f. ϕ_i

Given precise $\phi_0^*, \phi_L^*, \phi_i^*$ and $\varepsilon_i \in [0,1)$ for any $i \in \mathscr{X}$

- $\mathscr{M}_0 = \left\{(1-\varepsilon_0)\phi_0^* + \varepsilon_0\phi_0' : \phi_0' \in \Sigma_{\mathscr{X}_0}\right\}$
- $\mathscr{M}_L = \left\{(1-\varepsilon_L)\phi_L^* + \varepsilon_L\phi_L' : \phi_L' \in \Sigma_{\mathscr{X}_L}\right\}$

  and for all $i \in \mathscr{X} \setminus \{0, L\}$
- $\mathscr{M}_i = \left\{(1-\varepsilon_i)\phi_i^* + \varepsilon_i\phi_i' : \phi_i' \in \Sigma_{\mathscr{X}_i}\right\}$

Linear vacuous mixtures

We can also define

$$\underline{q}_i \coloneqq (1-\varepsilon_i)q_i^* \text{ and } \overline{q}_i \coloneqq (1-\varepsilon_i)q_i^* + \varepsilon_i \text{ for all } i \in \mathscr{X} \setminus \{0\}$$

$$\underline{p}_i \coloneqq (1-\varepsilon_i)p_i^* \text{ and } \overline{p}_i \coloneqq (1-\varepsilon_i)p_i^* + \varepsilon_i \text{ for all } i \in \mathscr{X} \setminus \{L\}$$

Expected lower upward, downward first passage and return times

$$\underline{\tau}_{i\to i+1} = \textstyle\sum_{k=0}^{i} \frac{\prod_{\ell=k+1}^{i} \underline{q}_\ell}{\prod_{m=k}^{i} \overline{p}_m} \qquad \underline{\tau}_{i\to i-1} = \textstyle\sum_{k=i}^{L} \frac{\prod_{\ell=i}^{k-1} \underline{p}_\ell}{\prod_{m=i}^{k} \overline{q}_m}$$

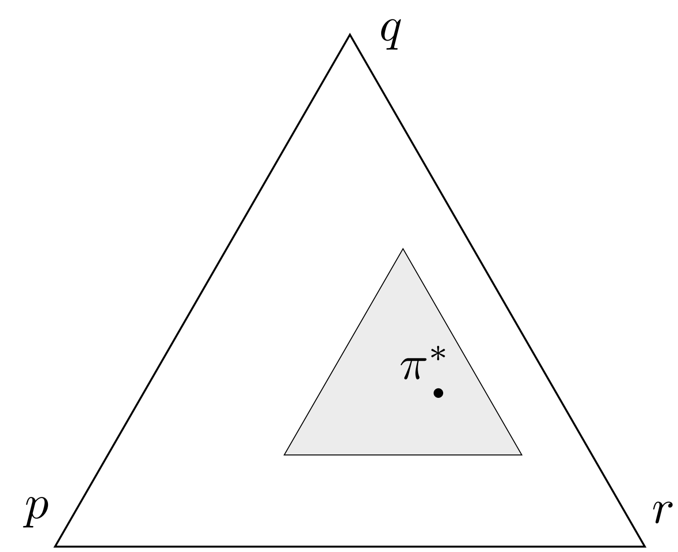
$$\underline{\tau}_{i\to i} = 1 + \underline{q}_i \underline{\tau}_{i-1\to i} + \underline{p}_i \underline{\tau}_{i+1\to i}$$

Linear vacuous mixtures

Consider state space $\mathscr{X} := \{0,\ldots,4\}$, $\varepsilon_i = \varepsilon = 0.4$ and

$$P^* = \begin{pmatrix} 0.55 & 0.45 & 0 & 0 & 0 \\ 0.3 & 0.5 & 0.2 & 0 & 0 \\ 0 & 0.3 & 0.5 & 0.2 & 0 \\ 0 & 0 & 0.3 & 0.5 & 0.2 \\ 0 & 0 & 0 & 0.6 & 0.4 \end{pmatrix}$$

then, for all $i \in \mathscr{X} \setminus \{0, L\}$



we calculate lower and upper expected return times

i	$\underline{\tau}_{i\to i}$	$\overline{\tau}_{i\to i}$
0	1.584	91.41
1	1.526	24.956
2	1.678	17.845
3	1.656	79.71
4	2.037	503.724

General example

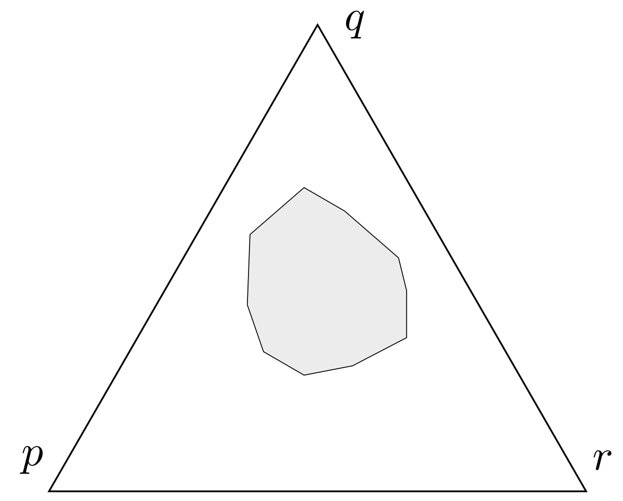
Consider state space $\mathscr{X} := \{0, \ldots, 4\}$

$\mathscr{M}_0$ is determined by $p_0 \in [0.15, 0.4]$ and $\mathscr{M}_L$ by $q_L \in [0.2, 0.6]$

For all $i \in \mathscr{X} \setminus \{0, L\}$, $\mathscr{M}_i$ is characterised by triplets of the form (q_i, r_i, p_i)

$$(0.65, 0.15, 0.2), (0.6, 0.25, 0.15), (0.5, 0.4, 0.1),$$
$$(0.43, 0.45, 0.12), (0.33, 0.5, 0.17), (0.27, 0.43, 0.3),$$
$$(0.25, 0.35, 0.4), (0.3, 0.25, 0.45), (0.4, 0.17, 0.43),$$
$$(0.55, 0.1, 0.35)$$

$\Rightarrow$



lower and upper expected upward and downward first passage times

$\underline{\tau}_{0\to1}$	2.5	$\underline{\tau}_{4\to3}$	1.666
$\underline{\tau}_{1\to2}$	3.889	$\underline{\tau}_{3\to2}$	2.051
$\underline{\tau}_{2\to3}$	4.814	$\underline{\tau}_{2\to1}$	2.169
$\underline{\tau}_{3\to4}$	5.432	$\underline{\tau}_{1\to0}$	2.206
$\overline{\tau}_{0\to1}$	6.666	$\overline{\tau}_{4\to3}$	5
$\overline{\tau}_{1\to2}$	43.333	$\overline{\tau}_{3\to2}$	12
$\overline{\tau}_{2\to3}$	226.666	$\overline{\tau}_{2\to1}$	23.2
$\overline{\tau}_{3\to4}$	1143.333	$\overline{\tau}_{1\to0}$	41.12

Conclusions and future work

- Simple methods for computing lower and upper expected first passage and return times

- Applying similar methods to other type of chains, e.g. Bonus-Malus systems

- Applying similar methods to continuous time systems