Calculating Bounds on Expected Return and First Passage Times in Finite-State Imprecise Birth-Death Chains

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Abstract

We provide simple methods for computing exact bounds on expected return and first passage times in finite-state birth-death chains, when the transition probabilities are imprecise, in the sense that they are only known to belong to convex closed sets of probability mass functions. These so-called imprecise birth-death chains are special types of time-homogeneous imprecise Markov chains. We also present numerical results and discuss the special case where the local models are linear-vacuous mixtures, for which our methods simplify even more.

Keywords. Birth-death chain, Markov chain, imprecise, return time, first passage time, credal set.

1 Introduction

A birth-death chain [11, Section 9.4] is a special type of time-homogeneous Markov chain that is used in various scientific fields, including evolutionary biology and queueing theory. We consider the generalised case of an imprecise birth-death chain, where the transition probabilities are imprecise, in the sense that they are only known to belong to convex closed sets of probability mass functions—credal sets. This may be the case because the transition probabilities are based on partial expert knowledge or limited data, or for the purposes of conducting a sensitivity analysis. Similar models were already considered in Reference [2], which presented results on limiting conditional distributions for imprecise birth-death chains with one absorbing state. Imprecise birth-death chains are themselves a special case of so-called (time-homogeneous) imprecise Markov chains, which were studied in—amongst others—References [5, 7, 9].

This paper focusses on—upward and downward—first passage times and return times.[1] For precise birth-death chains, these have been studied in, for example, Reference [8]. For the more general case of imprecise birth-death chains, we are not aware of any results. Our most important contribution are simple methods for computing lower and upper—exact bounds for—expected values of first passage times and return times in finite-state imprecise birth-death chains. We also present numerical results and discuss the special case where the local models are linear-vacuous mixtures, for which our methods simplify even more.

We start in section 2 by discussing the notion of a precise birth-death chain and then introduce our imprecise version of it in Section 3. Section 4 defines return and—upward and downward—first passage times and their lower and upper expected values. In Sections 5 and 6, we provide our methods for computing lower and upper expected upward and downward first passage times. We use these methods in section 7 to calculate lower and upper expected return times. Section 8 discusses the special case where the local models are linear-vacuous mixtures and Section 9 presents numerical results. We conclude the paper in Section 10.

2 Birth-Death Chains

Finite-state birth-death chains are special cases of time-homogeneous finite-state Markov chains. Their state space, denoted by $\mathcal{X}$, is finite and can be linearly ordered by an integer. Without loss of generality, we may assume that $\mathcal{X} = \{0, \dots, L\}$, with $L \in \mathbb{N}$.[2] At any time point $n \in \mathbb{N}$, the state of the chain is represented by a random variable, denoted by X_n, which takes values in the state space $\mathcal{X}$. For every $n \in \mathbb{N}$, the sequence of variables $X_1, \dots, X_n$ is denoted by $X_{1:n}$ and takes values $x_{1:n} := x_1, \dots, x_n$ in $\mathcal{X}^n$. Similarly, we use $X_{1:\infty}$ as a shorthand notation for the infinite sequence $X_1, \dots, X_n, \dots$ Also, for every $k \in \mathbb{N}$ such that $k \leq n$, we let $X_{k:n}$ and $X_{k:\infty}$ be the sequences of states from time point k to n or infinity, respectively.

[1] These are often called recurrence times as well.

[2] We do not consider zero to be a natural number.

Since finite-state birth-death chains are special cases of (time-homogeneous) Markov chains, they satisfy the Markov condition, which requires that

$$E_{n+1}(\cdot|x_{1:n}) = E_{n+1}(\cdot|x_n) \text{ for all } x_{1:n} \in \mathcal{X}^n, \quad (1)$$

where $E_{n+1}(\cdot|x_n)$ is the expectation operator that corresponds to the probability mass function $p(X_{n+1}|x_n)$ for X_{n+1}, conditional on $X_n = x_n$, and similarly for $E_{n+1}(\cdot|x_{1:n})$. If the Markov chain is furthermore time-homogeneous, then $p(X_{n+1}|x_n)$—and therefore also $E_{n+1}(\cdot|x_n)$—does not depend on n, which implies that all the transition probabilities can be summarised by means of a single stochastic matrix P of dimension $L+1$, by letting $P_{ij} := p(j|i)$ for all $i,j \in \mathcal{X}$. In the special case of a birth-death chain, this stochastic matrix is tridiagonal, which expresses that transitions are only possible between adjacent states. Hence, P is of the form

$$P = \begin{pmatrix} r_0 & p_0 & 0 & \cdots & \cdots & 0 \\ q_1 & r_1 & p_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & q_{L-1} & r_{L-1} & p_{L-1} \\ 0 & \cdots & \cdots & 0 & q_L & r_L \end{pmatrix}$$

where the elements of each row sum to 1. For any $i \in \mathcal{X}/\{0,L\}$, we will assume that the probabilities p_i, q_i and r_i are positive, and similarly for r_0, p_0, q_L, r_L. Figure 1 depicts a graphical representation of a finite-state birth-death chain.

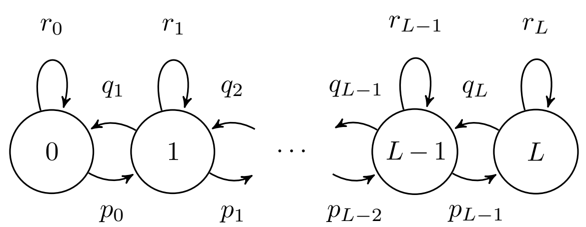

Figure 1: A birth-death chain with $\mathcal{X} = \{0, \ldots, L\}$

3 Imprecise Birth-Death Chains

Imprecise birth-death chains are similar to precise birth death chains. The main difference is that the probability mass functions that make up the matrix P do not need to be specified exactly. They are only known to belong to convex closed sets of probability mass functions, called credal sets. Formally, for every finite set $\mathcal{Y}$, a credal set on $\mathcal{Y}$ is a closed and convex subset of the set

$$\Sigma_{\mathcal{Y}} := \left\{ \pi \in \mathbb{R}^{\mathcal{Y}} \colon \sum_{y \in \mathcal{Y}} \pi(y) = 1, (\forall y \in \mathcal{Y})\, \pi(y) \geq 0 \right\}$$

of all probability mass functions on $\mathcal{Y}$.

For every $i \in \mathcal{X} \setminus \{0, L\}$, we consider a credal set $\mathcal{Q}_i$ on $\mathcal{X}_m := \{\ell, e, u\}$, where—for reasons that should become clear soon—m stands for middle and ℓ, e and u stand for lower, equal and upper, respectively. For the individual probability mass functions $\pi_i \in Q_i$, we will make frequent use of the notational convention that

$$(p_i, r_i, q_i) = \big(\pi_i(\ell), \pi_i(e), \pi_i(u)\big),$$

thereby establishing an intuitive connection with the matrix P that characterises a precise birth-death chain. Similarly, $\mathcal{Q}_0$ and $\mathcal{Q}_L$ are taken to be credal sets on $\mathcal{X}_0 := \{e, u\}$ and $\mathcal{X}_L := \{\ell, e\}$, respectively. For their elements $\pi_0 \in \mathcal{Q}_0$ and $\pi_L \in \mathcal{Q}_L$, we adopt the following notational conventions:

$$(r_0, p_0) = \big(\pi_0(e), \pi_0(u)\big)$$

and

$$(q_L, r_L) = \big(\pi_L(\ell), \pi_L(e)\big).$$

Since $\mathcal{X}_0$ is binary, $\mathcal{Q}_0$ is fully determined by the minimum and maximum value of p_0, as π_0 ranges over the elements of $\mathcal{Q}_0$. We denote this minimum and maximum by $\underline{p}_0$ and $\overline{p}_0$, respectively. Similarly, $\mathcal{Q}_L$ is fully determined by $\underline{q}_L$ and $\overline{q}_L$.

For reasons of mathematical convenience, we adopt the following positivity assumption.

Assumption 1 (Positivity assumption)**.** The local credal sets $\mathcal{Q}_i$, $i \in \mathcal{X}$, consist of strictly positive probability mass functions.

This assumption implies—amongst many other useful properties, such as Theorem 1—that the lower probabilities $\underline{p}_0$ and $\underline{q}_L$ are strictly positive.

We now use the credal sets $\mathcal{Q}_i$ to define corresponding credal sets $\mathcal{M}_i$ on $\mathcal{X}$. For all $i \in \mathcal{X} \setminus \{0, L\}$, a probability mass function $\phi_i \in \Sigma_{\mathcal{X}}$ belongs to $\mathcal{M}_i$ if and only if there is some $\pi_i \in \mathcal{Q}_i$ such that

$$\phi_i(j) = \begin{cases} q_i & \text{if } j = i-1 \\ r_i & \text{if } j = i \\ p_i & \text{if } j = i+1 \\ 0 & \text{otherwise} \end{cases} \quad \text{for all } j \in \mathcal{X}.$$

Similarly, ϕ_0 belongs to $\mathcal{M}_0$ if and only if there is some $\pi_0 \in \mathcal{Q}_0$ such that

$$\phi_0(j) = \begin{cases} r_0 & \text{if } j = 0 \\ p_0 & \text{if } j = 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{for all } j \in \mathcal{X}$$

and ϕ_L belongs to $\mathcal{M}_L$ if and only if there is some $\pi_L \in \mathcal{Q}_L$ such that

$$\phi_L(j) = \begin{cases} q_L & \text{if } j = L-1 \\ r_L & \text{if } j = L \\ 0 & \text{otherwise} \end{cases} \qquad \text{for all } j \in \mathcal{X}.$$

For any real-valued function f on $\mathcal{X}$ and any state i in $\mathcal{X}$, we now consider the corresponding lower and upper expectation of f, defined by

$$\underline{E}(f|i) := \min_{\phi_i \in \mathcal{M}_i} E_{\phi_i}(f) = \min_{\phi_i \in \mathcal{M}_i} \left\{ \sum_{j \in \mathcal{X}} \phi_i(j) f(j) \right\}$$

and

$$\overline{E}(f|i) := \max_{\phi_i \in \mathcal{M}_i} E_{\phi_i}(f) = \max_{\phi_i \in \mathcal{M}_i} \left\{ \sum_{j \in \mathcal{X}} \phi_i(j) f(j) \right\},$$

where $E_{\phi_i}(f) := \sum_{j \in \mathcal{X}} \phi_i(j) f(j)$. The resulting lower and upper expectation operators are connected by conjugacy: $\overline{E}(f|i) = -\underline{E}(-f|i)$. For that reason, without loss of generality, we can focus on the lower expectation operators $\underline{E}(\cdot|i)$, $i \in \mathcal{X}$.

An imprecise birth-death chain is now simply a time-homogeneous imprecise Markov chain [5] that has these lower previsions $\underline{E}(\cdot|i)$—or equivalently, the credal sets $\mathcal{M}_i$—as its local transition models. The corresponding global uncertainty models are derived from the conditional lower expectation operators $\underline{E}_{n+1}(\cdot|x_{1:n})$, defined for all $n \in \mathbb{N}$ and $x_{1:n} \in \mathcal{X}^n$ by[3]

$$\underline{E}_{n+1}(\cdot|x_{1:n}) = \underline{E}_{n+1}(\cdot|x_n) := \underline{E}(\cdot|x_n), \tag{2}$$

where the first equality follows from the so-called imprecise Markov condition and the second equality follows from time-homogeneity.

We want to stress here that the imprecise Markov condition that is imposed by Equation (2) is *not* equivalent to an element-wise application of the (precise) Markov condition in Equation (1). We do *not* require $E_{n+1}(\cdot|x_{1:n})$ and $E_{n+1}(\cdot|x_n)$ to be equal; we only require the bounds on these expectations to be equal. Imposing Equation (1) element-wise would be equivalent to considering a set of precise birth-death chains, each of which is required to satisfy the usual precise Markov assumption. Our approach imposes less stringent constraints. Using imprecise-probabilistic terminology: we impose epistemic irrelevance rather than strong independence; more information can be found in Reference [1].

From the local assessments that are provided by Equation (2), we now derive global uncertainty models for our imprecise Markov chain. For any $i \in \mathcal{X}$ and $n' \in \mathbb{N}$ such that $n' > n$, the global uncertainty model for the variables $X_{n+1:n'}$, conditional on $X_n = i$, is a lower expectation operator $\underline{E}_{n+1:n'}(\cdot|i)$ that takes real-valued functions on $\mathcal{X}^{n'-n}$ as its argument. It is given by the natural extension [10] of the models that were defined in Equation (2); see Reference [4] for more details and alternative interpretations. For the purposes of this paper, we need global uncertainty models that are even more general. In particular, for every $i \in \mathcal{X}$ and $n \in \mathbb{N}$, we need an uncertainty model for the infinite sequence of variables $X_{n+1:\infty}$, conditional on $X_n = i$, in the form of a lower expectation operator $\underline{E}_{n+1:\infty}(\cdot|i)$, defined for all extended real-valued functions on $\mathcal{X}^{\mathbb{N}}$.

These more general global models can be defined in multiple ways, and typically require some additional technical continuity argument; see Reference [3] for a definition in terms of submartingales, which is the one that we will adopt here. However, for our present purposes, the exact definition is only relevant for Theorem 1, which—due to the page limit constraint—is stated without proof. Therefore, and in order to avoid having to introduce the technical concept of a submartingale, we choose not to provide a definition for the global models $\underline{E}_{n+1:\infty}(\cdot|i)$. All that is important for the developments in this paper is that these global models are time-homogeneous and satisfy—a specific version of—the law of iterated expectation. For every $n \in \mathbb{N}$ and every extended real-valued function g on $\mathcal{X}^{\mathbb{N}}$, it holds that

$$\underline{E}_{n+1:\infty}(g(X_{n+1:\infty})|i) = \underline{E}_{n+2:\infty}(g(X_{n+2:\infty})|i). \tag{3}$$

Furthermore, if we define the—possibly extended—real-valued function f' on $\mathcal{X}$ by

$$f'(i') := \underline{E}_{n+2:\infty}(g(i', X_{n+2:\infty})|i') \text{ for all } i' \in \mathcal{X},$$

then, if f' is real-valued, we have that

$$\underline{E}_{n+1:\infty}(g(X_{n+1:\infty})|i) = \underline{E}_{n+1}(f'|i) = \underline{E}(f'|i), \tag{4}$$

where the second equality follows from Equation (2).

4 Return and First Passage Times

Consider a timepoint $n \in \mathbb{N}$ and two—possibly identical—states i and j in $\mathcal{X}$. If the variable X_n has i as its value, then the corresponding first passage time to j—the number of time-steps required to reach j—is a function $\tau_{i \to j}(i, X_{n+1:\infty})$ of the infinite sequence of variables $X_{n+1:\infty}$, defined by the following

[3] In general, an initial model $\underline{E}_1(\cdot)$ is required as well. However, for our present purposes, it is not necessary to specify one.

recursion equation:

$$
\begin{aligned}
&\tau_{i\to j}(i, X_{n+1:\infty})\\
&\quad:= \begin{cases} 1 & \text{if } X_{n+1} = j \\ 1 + \tau_{X_{n+1}\to j}(X_{n+1}, X_{n+2:\infty}) & \text{if } X_{n+1} \neq j \end{cases}\\
&\quad= 1 + \mathbb{I}_{j^c}(X_{n+1})\tau_{X_{n+1}\to j}(X_{n+1}, X_{n+2:\infty}) \qquad (5)
\end{aligned}
$$

where $\mathbb{I}_{j^c}$ is the indicator of $j^c := \mathcal{X} \setminus \{j\}$, defined by

$$\mathbb{I}_{j^c}(x) := \begin{cases} 0 & \text{if } x = j \\ 1 & \text{if } x \neq j \end{cases} \quad \text{for all } x \in \mathcal{X}.$$

If $i = j$, the corresponding first passage time is referred to as the return time to i. The so-called upward and downward first passage times correspond to the cases $i < j$ and $i > j$, respectively.

Due to Equation (3), we know that the lower expected value

$$\underline{\tau}_{i\to j,n} := \underline{E}_{n+1:\infty}(\tau_{i\to j}(i, X_{n+1:\infty})|i)$$

and upper expected value

$$
\begin{aligned}
\overline{\tau}_{i\to j,n} &:= \overline{E}_{n+1:\infty}(\tau_{i\to j}(i, X_{n+1:\infty})|i)\\
&:= -\underline{E}_{n+1:\infty}(-\tau_{i\to j}(i, X_{n+1:\infty})|i)
\end{aligned}
$$

of the first passage time from i to j do not depend on the specific timepoint $n \in \mathbb{N}$ that is chosen. For that reason, we can simply denote them by $\underline{\tau}_{i\to j}$ and $\overline{\tau}_{i\to j}$, respectively.

Theorem 1. *If Assumption 1 is satisfied, then for all $i, j \in \mathcal{X}$, the lower and upper first passage times $\underline{\tau}_{i\to j}$ and $\overline{\tau}_{i\to j}$ are real-valued and strictly positive.*

By combining Equations (4) and (5) with Theorem 1, we find that

$$\underline{\tau}_{i\to j} = 1 + \underline{E}(\mathbb{I}_{j^c}\underline{\tau}_{\bullet\to j}|i) \qquad (6)$$

and

$$\overline{\tau}_{i\to j} = 1 + \overline{E}(\mathbb{I}_{j^c}\overline{\tau}_{\bullet\to j}|i), \qquad (7)$$

where $\underline{\tau}_{\bullet\to j}$ and $\overline{\tau}_{\bullet\to j}$ are functions on $\mathcal{X}$, defined for all $x \in \mathcal{X}$ by

$$\underline{\tau}_{\bullet\to j}(x) := \underline{\tau}_{x\to j} \text{ and } \overline{\tau}_{\bullet\to j}(x) := \overline{\tau}_{x\to j}.$$

Taking into account our definition for $\underline{E}(\cdot|i)$, Equation (6) results in the following system of non-linear equalities: for all $j \in \mathcal{X}$, we have that

$$\underline{\tau}_{0\to j} = 1 + \min_{\pi_0\in\mathcal{Q}_0} \left\{ r_0\mathbb{I}_{j^c}(0)\underline{\tau}_{0\to j} + p_0\mathbb{I}_{j^c}(1)\underline{\tau}_{1\to j} \right\}, \quad (8)$$

$$
\begin{aligned}
\underline{\tau}_{L\to j} = 1 + \min_{\pi_L\in\mathcal{Q}_L} \big\{ &q_L\mathbb{I}_{j^c}(L-1)\underline{\tau}_{L-1\to j}\\
&+ r_L\mathbb{I}_{j^c}(L)\underline{\tau}_{L\to j} \big\}
\end{aligned}
$$

and, for all $i \in \mathcal{X}/\{0, L\}$, that

$$
\begin{aligned}
\underline{\tau}_{i\to j} = 1 + \min_{\pi_i\in\mathcal{Q}_i} \big\{ &q_i\mathbb{I}_{j^c}(i-1)\underline{\tau}_{i-1\to j} + r_i\mathbb{I}_{j^c}(i)\underline{\tau}_{i\to j}\\
&+ p_i\mathbb{I}_{j^c}(i+1)\underline{\tau}_{i+1\to j} \big\}. \qquad (9)
\end{aligned}
$$

A similar system of non-linear equalities can be derived from Equation (7) as well. In the remainder of this paper, we will solve these non-linear systems, leading to simple expressions that can be used to compute $\underline{\tau}_{i\to j}$ and $\overline{\tau}_{i\to j}$, for any $i, j \in \mathcal{X}$.

5 Lower and Upper Expected Upward First Passage Times

We start by computing lower expected values of upward first passage times, that is, for any $i, j \in \mathcal{X}$ such that $i < j$, we will compute $\underline{\tau}_{i\to j}$. We initially focus on calculating $\underline{\tau}_{i\to i+1}$, for $i \in \mathcal{X} \setminus \{L\}$, and then show that any lower expected upward first passage time can be obtained as a sum of such terms. Similar results are obtained for upper expected upward first passage times.

Finding $\underline{\tau}_{0\to 1}$ is easy. It follows from Equation (8), with $j = 1$, that

$$
\begin{aligned}
\underline{\tau}_{0\to 1} &= 1 + \min_{\pi_0\in\mathcal{Q}_0} \{r_0\underline{\tau}_{0\to 1}\}\\
&= 1 + \min_{\pi_0\in\mathcal{Q}_0} \{(1-p_0)\underline{\tau}_{0\to 1}\}\\
&= 1 + \underline{\tau}_{0\to 1} - \max_{\pi_0\in\mathcal{Q}_0} \{p_0\underline{\tau}_{0\to 1}\}\\
&= 1 + \underline{\tau}_{0\to 1} - \overline{p}_0\underline{\tau}_{0\to 1}, \qquad (10)
\end{aligned}
$$

where the second equality holds because π_0 is a probability mass function and the last equality holds because we know from Theorem 1 that $\underline{\tau}_{0\to 1}$ is real-valued and therefore finite. Since Theorem 1 also tells us that $\underline{\tau}_{0\to 1}$ is strictly positive, we infer from Equation (10) that

$$\underline{\tau}_{0\to 1} = \frac{1}{\overline{p}_0}. \qquad (11)$$

Finding $\underline{\tau}_{0\to j}$, for $j \in \{2, \ldots, L\}$, is more involved. We start by establishing a connection with $\underline{\tau}_{1\to j}$. By applying Equation (8), we find that

$$
\begin{aligned}
\underline{\tau}_{0\to j} &= 1 + \min_{\pi_0\in\mathcal{Q}_0} \{r_0\underline{\tau}_{0\to j} + p_0\underline{\tau}_{1\to j}\}\\
&= 1 + \min_{\pi_0\in\mathcal{Q}_0} \{(1-p_0)\underline{\tau}_{0\to j} + p_0\underline{\tau}_{1\to j}\}\\
&= 1 + \underline{\tau}_{0\to j} + \min_{\pi_0\in\mathcal{Q}_0} \{p_0(\underline{\tau}_{1\to j} - \underline{\tau}_{0\to j})\},
\end{aligned}
$$

which implies, due to Theorem 1, that

$$\min_{\pi_0 \in \mathcal{Q}_0} \{p_0(\underline{\tau}_{1\to j} - \underline{\tau}_{0\to j})\} = -1. \tag{12}$$

Since the minimum in Equation (12) is negative and p_0 is a probability and therefore non-negative, it must be that $\underline{\tau}_{1\to j} - \underline{\tau}_{0\to j} < 0$. Therefore, Equation (12) is minimised for $p_0 = \overline{p}_0$ and we find that

$$\underline{\tau}_{0\to j} = \frac{1}{\overline{p}_0} + \underline{\tau}_{1\to j}. \tag{13}$$

By combining Equations (11) and (13), we see that

$$\underline{\tau}_{0\to j} = \underline{\tau}_{0\to 1} + \underline{\tau}_{1\to j} \text{ for all } j \in \{2, \dots, L\}. \tag{14}$$

Since we already know $\underline{\tau}_{0\to 1}$—see Equation (11)—we are now left to find $\underline{\tau}_{1\to j}$.

We first consider the case $j = 2$. It follows from Equation (9), with $i = 1$ and $j = 2$, that

$$\begin{aligned}
\underline{\tau}_{1\to 2} &= 1 + \min_{\pi_1 \in \mathcal{Q}_1} \{q_1 \underline{\tau}_{0\to 2} + r_1 \underline{\tau}_{1\to 2}\} \\
&= 1 + \min_{\pi_1 \in \mathcal{Q}_1} \{q_1 \underline{\tau}_{0\to 2} + (1 - q_1 - p_1)\underline{\tau}_{1\to 2}\} \\
&= 1 + \underline{\tau}_{1\to 2} + \min_{\pi_1 \in \mathcal{Q}_1} \{q_1(\underline{\tau}_{0\to 2} - \underline{\tau}_{1\to 2}) - p_1 \underline{\tau}_{1\to 2}\},
\end{aligned}$$

which implies, due to Theorem 1, that

$$\min_{\pi_1 \in \mathcal{Q}_1} \{q_1(\underline{\tau}_{0\to 2} - \underline{\tau}_{1\to 2}) - p_1 \underline{\tau}_{1\to 2}\} = -1.$$

By applying Equation (14), for $j = 2$, we find that

$$\min_{\pi_1 \in \mathcal{Q}_1} \{q_1 \underline{\tau}_{0\to 1} - p_1 \underline{\tau}_{1\to 2}\} = -1. \tag{15}$$

Since we already know $\underline{\tau}_{0\to 1}$, it follows from Assumption 1 and the following lemma that $\underline{\tau}_{1\to 2}$ is the unique solution to Equation (15).

Lemma 2. *Consider a credal set $\mathcal{Q}$ on $\mathcal{X}_m$ that consists of strictly positive probability mass functions and let c be a real constant. Then*

$$\min_{\pi \in \mathcal{Q}} \{qc - p\mu\}$$

is a strictly decreasing function of μ.

This unique solution $\underline{\tau}_{1\to 2}$ is furthermore easy to compute. It follows from Lemma 2 that a simple bisection method suffices.

Next, we consider the case $j \in \{3, \dots, L\}$. By applying Equation (9), for such a j and with $i = 1$, we find that

$$\begin{aligned}
\underline{\tau}_{1\to j} &= 1 + \min_{\pi_1 \in \mathcal{Q}_1} \{q_1 \underline{\tau}_{0\to j} + r_1 \underline{\tau}_{1\to j} + p_1 \underline{\tau}_{2\to j}\} \\
&= 1 + \min_{\pi_1 \in \mathcal{Q}_1} \{q_1 \underline{\tau}_{0\to j} + (1 - q_1 - p_1)\underline{\tau}_{1\to j} + p_1 \underline{\tau}_{2\to j}\} \\
&= 1 + \underline{\tau}_{1\to j} + \min_{\pi_1 \in \mathcal{Q}_1} \{q_1(\underline{\tau}_{0\to j} - \underline{\tau}_{1\to j}) \\
&\qquad + p_1(\underline{\tau}_{2\to j} - \underline{\tau}_{1\to j})\},
\end{aligned}$$

which implies, due to Theorem 1, that

$$\min_{\pi_1 \in \mathcal{Q}_1} \{q_1(\underline{\tau}_{0\to j} - \underline{\tau}_{1\to j}) + p_1(\underline{\tau}_{2\to j} - \underline{\tau}_{1\to j})\} = -1.$$

In combination with Equation (14), this results in

$$\min_{\pi_1 \in \mathcal{Q}_1} \{q_1 \underline{\tau}_{0\to 1} + p_1(\underline{\tau}_{2\to j} - \underline{\tau}_{1\to j})\} = -1. \tag{16}$$

Since we know from Assumption 1 and Lemma 2 that the equation

$$\min_{\pi_1 \in \mathcal{Q}_1} \{q_1 \underline{\tau}_{0\to 1} + p_1 \mu\} = -1$$

has a unique solution μ, it follows directly from Equations (15) and (16) that

$$\underline{\tau}_{1\to j} = \underline{\tau}_{1\to 2} + \underline{\tau}_{2\to j} \text{ for all } j \in \{3, \dots, L\}. \tag{17}$$

At this point, we already know how to compute $\underline{\tau}_{0\to 1}$ and $\underline{\tau}_{1\to 2}$ and we have also established the following additivity property:

$$\underline{\tau}_{i\to j} = \underline{\tau}_{i\to i+1} + \underline{\tau}_{i+1\to j}$$

for all $i \in \{0, 1\}$ and $j \in \{i+2, \dots, L\}$. Continuing in a similar way, we now derive an expression for computing $\underline{\tau}_{2\to 3}$ and prove that the above additivity property holds for $i = 2$ as well. By applying Equation (9), for $i = 2$ and $j = 3$, we find that

$$\begin{aligned}
\underline{\tau}_{2\to 3} &= 1 + \min_{\pi_2 \in \mathcal{Q}_2} \{q_2 \underline{\tau}_{1\to 3} + r_2 \underline{\tau}_{2\to 3}\} \\
&= 1 + \min_{\pi_2 \in \mathcal{Q}_2} \{q_2 \underline{\tau}_{1\to 3} + (1 - q_2 - p_2)\underline{\tau}_{2\to 3}\} \\
&= 1 + \underline{\tau}_{2\to 3} + \min_{\pi_2 \in \mathcal{Q}_2} \{q_2(\underline{\tau}_{1\to 3} - \underline{\tau}_{2\to 3}) - p_2 \underline{\tau}_{2\to 3}\},
\end{aligned}$$

which implies, due to Theorem 1, that

$$\min_{\pi_2 \in \mathcal{Q}_2} \{q_2(\underline{\tau}_{1\to 3} - \underline{\tau}_{2\to 3}) - p_2 \underline{\tau}_{2\to 3}\} = -1.$$

By applying Equation (17), for $j = 3$, we find that

$$\min_{\pi_2 \in \mathcal{Q}_2} \{q_2 \underline{\tau}_{1\to 2} - p_2 \underline{\tau}_{2\to 3}\} = -1. \tag{18}$$

Since we have already computed $\underline{\tau}_{1\to 2}$, it follows from Assumption 1 and Lemma 2 that $\underline{\tau}_{2\to 3}$ is the unique solution to Equation (18) and that this unique solution can furthermore easily be computed by means of a bisection method.

Next, by applying Equation (9), for $i = 2$ and j in $\{4, \dots, L\}$, we find that

$$\begin{aligned}
\underline{\tau}_{2\to j} &= 1 + \min_{\pi_2 \in \mathcal{Q}_2} \{q_2 \underline{\tau}_{1\to j} + r_2 \underline{\tau}_{2\to j} + p_2 \underline{\tau}_{3\to j}\} \\
&= 1 + \min_{\pi_2 \in \mathcal{Q}_2} \{q_2 \underline{\tau}_{1\to j} + (1 - q_2 - p_2)\underline{\tau}_{2\to j} + p_2 \underline{\tau}_{3\to j}\} \\
&= 1 + \underline{\tau}_{2\to j} + \min_{\pi_2 \in \mathcal{Q}_2} \{q_2(\underline{\tau}_{1\to j} - \underline{\tau}_{2\to j}) \\
&\qquad + p_2(\underline{\tau}_{3\to j} - \underline{\tau}_{2\to j})\},
\end{aligned}$$

which implies, due to Theorem 1, that

$$\min_{\pi_2 \in \mathcal{Q}_2} \{q_2(\underline{\tau}_{1\to j} - \underline{\tau}_{2\to j}) + p_2(\underline{\tau}_{3\to j} - \underline{\tau}_{2\to j})\} = -1.$$

In combination with Equation (17), this results in

$$\min_{\pi_2 \in \mathcal{Q}_2} \{q_2\underline{\tau}_{1\to 2} + p_2(\underline{\tau}_{3\to j} - \underline{\tau}_{2\to j})\} = -1. \tag{19}$$

It now follows from Equations (18) and (19), Assumption 1 and Lemma 2, that

$$\underline{\tau}_{2\to j} = \underline{\tau}_{2\to 3} + \underline{\tau}_{3\to j} \text{ for all } j \in \{4, \ldots, L\}.$$

At this point, it should be clear that, by continuing in this way, we obtain the following two results.

Proposition 3. *For all $i \in \mathcal{X} \setminus \{0, L\}$, we have that*

$$\min_{\pi_i \in \mathcal{Q}_i} \{q_i\underline{\tau}_{i-1\to i} - p_i\underline{\tau}_{i\to i+1}\} = -1. \tag{20}$$

Proposition 4. *For all $i, j \in \mathcal{X}$ such that $i + 1 < j$, we have that*

$$\underline{\tau}_{i\to j} = \underline{\tau}_{i\to i+1} + \underline{\tau}_{i+1\to j}.$$

For any $i \in \mathcal{X} \setminus \{L\}$, the value of $\underline{\tau}_{i\to i+1}$ can therefore be computed recursively. For $i = 0$, we simply apply Equation (11). For any other $i \in \mathcal{X} \setminus \{0, L\}$, it follows from Assumption 1, Lemma 2 and Proposition 3 that $\underline{\tau}_{i\to i+1}$ is the unique solution to Equation (20), which can be obtained by means of a bisection method. In this equation, the value of $\underline{\tau}_{i-1\to i}$ has already been computed earlier on in this recursive procedure.

The following additivity result is a direct consequence of Proposition 4.

Corollary 5. *For any $i, j \in \mathcal{X}$ such that $i < j$, we have that*

$$\underline{\tau}_{i\to j} = \sum_{k=i}^{j-1} \underline{\tau}_{k\to k+1}.$$

It implies that the recursive techniques that we developed in this section can be used to compute any lower expected upward first passage time.

Similar results can be proved for upper expected values of upward first passage times. We only provide the final expressions; the derivation is completely analogous. In this case, the starting point is that

$$\overline{\tau}_{0\to 1} = \frac{1}{\underline{p}_0} \tag{21}$$

For any $i \in \mathcal{X} \setminus \{0, L\}$, the value of $\overline{\tau}_{i\to i+1}$ can then be computed recursively, due to Assumption 1 and the following two results.

Proposition 6. *For all $i \in \mathcal{X} \setminus \{0, L\}$, we have that*

$$\max_{\pi_i \in \mathcal{Q}_i} \{q_i\overline{\tau}_{i-1\to i} - p_i\overline{\tau}_{i\to i+1}\} = -1.$$

Corollary 7. *Consider a credal set $\mathcal{Q}$ on $\mathcal{X}_m$ that consists of strictly positive probability mass functions and let c be a real constant. Then*

$$\max_{\pi \in \mathcal{Q}} \{qc - p\mu\}$$

is a strictly decreasing function of μ.

Due to the next result, this recursive technique allows us to compute arbitrary upper expected upward first passage times.

Proposition 8. *For any $i, j \in \mathcal{X}$ such that $i < j$, we have that*

$$\overline{\tau}_{i\to j} = \sum_{k=i}^{j-1} \overline{\tau}_{k\to k+1}$$

6 Lower and Upper Expected Downward First Passage Times

Lower and upper expected values of downward first passage times can be computed in more or less the same way. The main difference is that the recursive expressions now start from the other side, that is, from $i = L$. We find that

$$\underline{\tau}_{L\to L-1} = \frac{1}{\overline{q}_L} \tag{22}$$

and

$$\overline{\tau}_{L\to L-1} = \frac{1}{\underline{q}_L} \tag{23}$$

For any $i \in \mathcal{X} \setminus \{0, L\}$, due to Assumption 1, the values of $\underline{\tau}_{i\to i-1}$ and $\overline{\tau}_{i\to i-1}$ can now be computed recursively, using the following two results.

Proposition 9. *For all $i \in \mathcal{X} \setminus \{0, L\}$, we have that*

$$\min_{\pi_i \in \mathcal{Q}_i} \{-q_i\underline{\tau}_{i\to i-1} + p_i\underline{\tau}_{i+1\to i}\} = -1$$

and

$$\max_{\pi_i \in \mathcal{Q}_i} \{-q_i\overline{\tau}_{i\to i-1} + p_i\overline{\tau}_{i+1\to i}\} = -1$$

Corollary 10. *Consider a credal set $\mathcal{Q}$ on $\mathcal{X}_m$ that consists of strictly positive probability mass functions and let c be a real constant. Then*

$$\min_{\pi \in \mathcal{Q}} \{-q\mu + pc\} \text{ and } \max_{\pi \in \mathcal{Q}} \{-q\mu + pc\}$$

are strictly decreasing functions of μ.

Once we have computed $\underline{\tau}_{i\to i-1}$ and $\overline{\tau}_{i\to i-1}$ for all $i \in \mathcal{X} \setminus \{L\}$, the following result enables us to easily obtain all other lower and upper expected downward first passage times.

Proposition 11. *For any $i, j \in \mathcal{X}$ such that $i > j$, we have that*

$$\underline{\tau}_{i\to j} = \sum_{k=j}^{i-1} \underline{\tau}_{k+1\to k} \text{ and } \overline{\tau}_{i\to j} = \sum_{k=j}^{i-1} \overline{\tau}_{k+1\to k}$$

7 Lower and Upper Expected Return Times

Lower and upper expected return times can now be computed very easily. By applying Equations (8)–(9), with j equal to 0, L and i, respectively, we find that

$$\underline{\tau}_{0\to 0} = 1 + \min_{\pi_0 \in \mathcal{Q}_0} \{p_0 \underline{\tau}_{1\to 0}\} = 1 + \underline{p}_0 \underline{\tau}_{1\to 0}, \quad (24)$$

$$\underline{\tau}_{L\to L} = 1 + \min_{\pi_L \in \mathcal{Q}_L} \{q_L \underline{\tau}_{L-1\to L}\} = 1 + \underline{q}_L \underline{\tau}_{L-1\to L} \quad (25)$$

and, for all $i \in \mathcal{X} \setminus \{0, L\}$, that

$$\underline{\tau}_{i\to i} = 1 + \min_{\pi_i \in \mathcal{Q}_i} \{q_i \underline{\tau}_{i-1\to i} + p_i \underline{\tau}_{i+1\to i}\} \quad (26)$$

In these expressions, the lower expected first passage times $\underline{\tau}_{1\to 0}$, $\underline{\tau}_{L-1\to L}$, $\underline{\tau}_{i-1\to i}$ and $\underline{\tau}_{i+1\to i}$ can be computed using the recursive techniques that we developed in the previous two sections. Similarly, for the upper case, we find that

$$\overline{\tau}_{0\to 0} = 1 + \max_{\pi_0 \in \mathcal{Q}_0} \{p_0 \overline{\tau}_{1\to 0}\} = 1 + \overline{p}_0 \overline{\tau}_{1\to 0}, \quad (27)$$

$$\overline{\tau}_{L\to L} = 1 + \max_{\pi_L \in \mathcal{Q}_L} \{q_L \overline{\tau}_{L-1\to L}\} = 1 + \overline{q}_L \overline{\tau}_{L-1\to L} \quad (28)$$

and, for all $i \in \mathcal{X} \setminus \{0, L\}$, that

$$\overline{\tau}_{i\to i} = 1 + \max_{\pi_i \in \mathcal{Q}_i} \{q_i \overline{\tau}_{i-1\to i} + p_i \overline{\tau}_{i+1\to i}\}. \quad (29)$$

Again, the upper expected first passage times $\overline{\tau}_{1\to 0}$, $\overline{\tau}_{L-1\to L}$, $\overline{\tau}_{i-1\to i}$ and $\overline{\tau}_{i+1\to i}$ that appear in these expressions can be computed with the recursive techniques that were introduced above.

8 Linear-Vacuous Mixtures

We now apply our results to the special case where all the local models are linear-vacuous mixtures. In that case, the computation of lower and upper expected first passage and return times becomes even simpler.

We start from given strictly positive probability mass functions $\pi_0^* = (r_0^*, p_0^*) \in \Sigma_{\mathcal{X}_0}$, $\pi_L^* = (q_L^*, r_L^*) \in \Sigma_{\mathcal{X}_L}$ and, for all $i \in \mathcal{X} \setminus \{0, L\}$, $\pi_i^* = (q_i^*, r_i^*, p_i^*) \in \Sigma_{\mathcal{X}_m}$. Furthermore, for all $i \in \mathcal{X}$, we consider some real-valued $\varepsilon_i \in [0, 1)$. We use these parameters to define the following so-called linear-vacuous [10, Section 2.9.2] local credal sets:

$$\mathcal{Q}_0 = \mathcal{Q}_{\pi_0^*}^{\varepsilon_0} := \{(1-\varepsilon_0)\pi_0^* + \varepsilon_0 \pi_0' : \pi_0' \in \Sigma_{\mathcal{X}_0}\},$$

$$\mathcal{Q}_L = \mathcal{Q}_{\pi_L^*}^{\varepsilon_L} := \{(1-\varepsilon_L)\pi_L^* + \varepsilon_L \pi_L' : \pi_L' \in \Sigma_{\mathcal{X}_L}\}$$

and, for all $i \in \mathcal{X} \setminus \{0, L\}$,

$$\mathcal{Q}_i = \mathcal{Q}_{\pi_i^*}^{\varepsilon_i} := \{(1-\varepsilon_i)\pi_i^* + \varepsilon_i \pi_i' : \pi_i' \in \Sigma_{\mathcal{X}_m}\},$$

which can be regarded as neighbourhood models for the probability mass functions π_i^*, $i \in \mathcal{X}$. Furthermore, for all $i \in \mathcal{X} \setminus \{0\}$, we define

$$\underline{q}_i := (1-\varepsilon_i) q_i^* \text{ and } \overline{q}_i := (1-\varepsilon_i) q_i^* + \varepsilon_i$$

and, for all $i \in \mathcal{X} \setminus \{L\}$,

$$\underline{p}_i := (1-\varepsilon_i) p_i^* \text{ and } \overline{p}_i := (1-\varepsilon_i) p_i^* + \varepsilon_i,$$

which are the minimum and maximum values of q_i and p_i, for $\pi_i \in \mathcal{Q}_i$, respectively.

In this special case, Equation (20) can be solved analytically. For all $i \in \mathcal{X} \setminus \{0, L\}$, we find that

$$\begin{aligned}
&\min_{\pi_i \in \mathcal{Q}_i} \{q_i \underline{\tau}_{i-1\to i} - p_i \underline{\tau}_{i\to i+1}\} \\
&= \min_{\pi' \in \Sigma_{\mathcal{X}_m}} \big\{[(1-\varepsilon_i) q_i^* + \varepsilon_i q_i'] \underline{\tau}_{i-1\to i} \\
&\qquad - [(1-\varepsilon_i) p_i^* + \varepsilon_i p_i'] \underline{\tau}_{i\to i+1}\big\} \\
&= (1-\varepsilon_i)(q_i^* \underline{\tau}_{i-1\to i} - p_i^* \underline{\tau}_{i\to i+1}) \\
&\qquad + \varepsilon_i \min_{\pi_i' \in \Sigma_{\mathcal{X}_m}} \{q_i' \underline{\tau}_{i-1\to i} - p_i' \underline{\tau}_{i\to i+1}\} \\
&= (1-\varepsilon_i)(q_i^* \underline{\tau}_{i-1\to i} - p_i^* \underline{\tau}_{i\to i+1}) - \varepsilon_i \underline{\tau}_{i\to i+1} \\
&= \underline{q}_i \underline{\tau}_{i-1\to i} - \overline{p}_i \underline{\tau}_{i\to i+1},
\end{aligned}$$

where the third equation holds because we know from Theorem 1 that $\underline{\tau}_{i-1\to i}$ and $\underline{\tau}_{i\to i+1}$ are real-valued and positive. Therefore, for all $i \in \mathcal{X} \setminus \{0, L\}$, it follows directly from Equation (20) that

$$\underline{\tau}_{i\to i+1} = \frac{1}{\overline{p}_i} + \frac{\underline{q}_i}{\overline{p}_i} \underline{\tau}_{i-1\to i}.$$

By combining this recursive expression with Equation (11), we can derive explicit expressions. For all $i \in \mathcal{X} \setminus \{L\}$, we find that:

$$\underline{\tau}_{i\to i+1} = \sum_{k=0}^{i} \frac{\prod_{\ell=k+1}^{i} \underline{q}_\ell}{\prod_{m=k}^{i} \overline{p}_m}. \quad (30)$$

In combination with Corollary 5, this equation allows us to easily compute all lower expected upward first passage times for the linear-vacuous case.

Similar results can be obtained for upper expected upward first passage times and for lower and upper expected downward first passage times. For all $i \in \mathcal{X} \setminus \{0, L\}$, we find that

$$\overline{\tau}_{i\to i+1} = \frac{1}{\underline{p}_i} + \frac{\overline{q}_i}{\underline{p}_i}\overline{\tau}_{i-1\to i},$$

$$\underline{\tau}_{i\to i-1} = \frac{1}{\overline{q}_i} + \frac{\underline{p}_i}{\overline{q}_i}\underline{\tau}_{i+1\to i}$$

and

$$\overline{\tau}_{i\to i-1} = \frac{1}{\underline{q}_i} + \frac{\overline{p}_i}{\underline{q}_i}\overline{\tau}_{i+1\to i}.$$

By combining these recursive equations with Equations (21), (22) and (23), respectively, we can obtain explicit expressions. For all $i \in \mathcal{X} \setminus \{L\}$, we find that

$$\overline{\tau}_{i\to i+1} = \sum_{k=0}^{i} \frac{\prod_{\ell=k+1}^{i} \overline{q}_\ell}{\prod_{m=k}^{i} \underline{p}_m}$$

and, for all $i \in \mathcal{X} \setminus \{0\}$, we find that

$$\underline{\tau}_{i\to i-1} = \sum_{k=i}^{L} \frac{\prod_{\ell=i}^{k-1} \underline{p}_\ell}{\prod_{m=i}^{k} \overline{q}_m} \tag{31}$$

and

$$\overline{\tau}_{i\to i-1} = \sum_{k=i}^{L} \frac{\prod_{\ell=i}^{k-1} \overline{p}_\ell}{\prod_{m=i}^{k} \underline{q}_m}.$$

In combination with Proposition 8 and 11, these equations allow us to easily compute all upper expected upward first passage times and all lower and upper expected downward first passage times for the linear-vacuous case.

For the lower and upper return times, we still use Equations (24) and (25) if $i = 0$ and Equations (27) and (28) if $i = L$. If $i \in \mathcal{X} \setminus \{0, L\}$, then, for this linear-vacuous case, Equations (26) and (29) can be simplified. We find that

$$\begin{aligned}
\underline{\tau}_{i\to i} &= 1 + \min_{\pi_i \in \mathcal{Q}_i} \{q_i \underline{\tau}_{i-1\to i} + p_i \underline{\tau}_{i+1\to i}\} \\
&= 1 + \min_{\pi_i' \in \Sigma_{\mathcal{X}_m}} \big\{[(1-\varepsilon_i)q_i^* + \varepsilon_i q_i']\underline{\tau}_{i-1\to i} \\
&\qquad\qquad + [(1-\varepsilon_i)p_i^* + \varepsilon_i p_i']\underline{\tau}_{i+1\to i}\big\} \\
&= 1 + (1-\varepsilon_i)(q_i^* \underline{\tau}_{i-1\to i} + p_i^* \underline{\tau}_{i+1\to i}) \\
&= 1 + \underline{q}_i \underline{\tau}_{i-1\to i} + \underline{p}_i \underline{\tau}_{i+1\to i}.
\end{aligned} \tag{32}$$

and that

$$\begin{aligned}
\overline{\tau}_{i\to i} &= 1 + (1-\varepsilon_i)(q_i^* \overline{\tau}_{i-1\to i} + p_i^* \overline{\tau}_{i+1\to i}) \\
&\qquad + \varepsilon_i \max\{\overline{\tau}_{i-1\to i}, \overline{\tau}_{i+1\to i}\} \\
&= 1 + \max\{\overline{q}_i \overline{\tau}_{i-1\to i} + \underline{p}_i \overline{\tau}_{i+1\to i}, \\
&\qquad\qquad \underline{q}_i \overline{\tau}_{i-1\to i} + \overline{p}_i \overline{\tau}_{i+1\to i}\}.
\end{aligned}$$

9 Numerical Results

We end by computing lower and upper expected first passage and return times for two examples of imprecise birth-death chains. The first is a general example of an imprecise birth-death chain and the second one is an imprecise birth-death chain with linear-vacuous local models. In both examples, we take $\mathcal{Q}_i$ to be identical for all $i \in \mathcal{X} \setminus \{0, L\}$, and simply denote it by $\mathcal{Q}$, which is a credal set on $\mathcal{X}_m$. Some of the lower and upper expected values that we compute have many decimal points; we present them up to the third decimal point.

General Example

Consider an imprecise birth-death chain with state space $\mathcal{X} = \{0, 1, 2, 3, 4\}$, that is, $L = 4$. Let $\mathcal{Q}_0$ be determined by $\underline{p}_0 = 0.15$ and $\overline{p}_0 = 0.4$ and let $\mathcal{Q}_L$ be determined by $\underline{q}_L = 0.2$ and $\overline{q}_L = 0.6$. The credal set $\mathcal{Q}$ is taken to be the convex hull of the following 10 extreme points, which are of the form $\pi = (q, r, p)$.

$$\begin{aligned}
&(0.65, 0.15, 0.2), (0.6, 0.25, 0.15), (0.5, 0.4, 0.1), \\
&(0.43, 0.45, 0.12), (0.33, 0.5, 0.17), (0.27, 0.43, 0.3), \\
&(0.25, 0.35, 0.4), (0.3, 0.25, 0.45), (0.4, 0.17, 0.43), \\
&(0.55, 0.1, 0.35)
\end{aligned}$$

Figure 2 provides a graphical representation of this credal set $\mathcal{Q}$.[4]

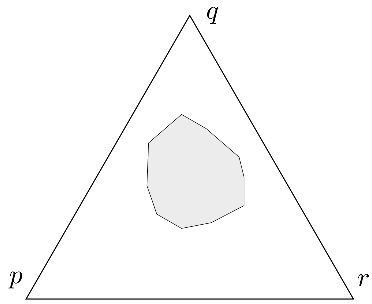

Figure 2: The grey zone depicts the credal set $\mathcal{Q}$ from the birth-death chain in the general example.

For this particular example, we now compute $\underline{\tau}_{0\to 4}$, $\overline{\tau}_{0\to 4}$, $\underline{\tau}_{4\to 0}$ and $\overline{\tau}_{4\to 0}$.

Due to Corollary 5, we know that

$$\underline{\tau}_{0\to 4} = \underline{\tau}_{0\to 1} + \underline{\tau}_{1\to 2} + \underline{\tau}_{2\to 3} + \underline{\tau}_{3\to 4}, \tag{33}$$

[4]We represent $\Sigma_{\mathcal{X}_m}$ by means of a equilateral triangle of height one. The elements $\pi = (q, r, p)$ of $\Sigma_{\mathcal{X}_m}$ correspond to points in this triangle. For every such π, the value of q, r, p is equal to the perpendicular distance from that point to the edge that opposes the corresponding corner.

$\underline{\tau}_{0\to4}$	16.635
$\overline{\tau}_{0\to4}$	1420
$\underline{\tau}_{4\to0}$	8.093
$\overline{\tau}_{4\to0}$	81.32

Table 1: Final results for the general example.

$\underline{\tau}_{0\to1}$	2.5	$\underline{\tau}_{4\to3}$	1.666
$\underline{\tau}_{1\to2}$	3.889	$\underline{\tau}_{3\to2}$	2.051
$\underline{\tau}_{2\to3}$	4.814	$\underline{\tau}_{2\to1}$	2.169
$\underline{\tau}_{3\to4}$	5.432	$\underline{\tau}_{1\to0}$	2.206
$\overline{\tau}_{0\to1}$	6.666	$\overline{\tau}_{4\to3}$	5
$\overline{\tau}_{1\to2}$	43.333	$\overline{\tau}_{3\to2}$	12
$\overline{\tau}_{2\to3}$	226.666	$\overline{\tau}_{2\to1}$	23.2
$\overline{\tau}_{3\to4}$	1143.333	$\overline{\tau}_{1\to0}$	41.12

Table 2: Intermediate results for the general example.

where, using Equation (11),

$$\underline{\tau}_{0\to1} = 1/\overline{p}_0 = 2.5.$$

By plugging this value for $\underline{\tau}_{0\to1}$ in Equation (20), for $i = 1$, we find that

$$\min_{\pi_1\in\mathcal{Q}}\{2.5q_1 - p_1\underline{\tau}_{1\to2}\} = -1$$

As we know from Lemma 2, this equality has a unique solution that can for example be obtained by means of a bisection method. We find that $\underline{\tau}_{1\to2} = 3.889$. Similarly, in a recursive fashion, we find that $\underline{\tau}_{2\to3} = 4.814$ and $\underline{\tau}_{3\to4} = 5.432$. A final application of Equation (33) tells us that $\underline{\tau}_{0\to4} = 16.635$. $\overline{\tau}_{0\to4}$, $\underline{\tau}_{4\to0}$ and $\overline{\tau}_{4\to0}$ can be computed analogously; the results are given in Table 1. Intermediate results can be found in Table 2.

Linear-Vacuous Example

Consider a precise birth-death chain with state space $\mathcal{X} = \{0, 1, 2, 3, 4\}$—$L = 4$—and the following probability matrix:

$$P^* = \begin{pmatrix} 0.55 & 0.45 & 0 & 0 & 0 \\ 0.3 & 0.5 & 0.2 & 0 & 0 \\ 0 & 0.3 & 0.5 & 0.2 & 0 \\ 0 & 0 & 0.3 & 0.5 & 0.2 \\ 0 & 0 & 0 & 0.6 & 0.4, \end{pmatrix}$$

which is completely characterised by the probability mass functions $\pi_0^* = (0.55, 0.45)$, $\pi_L^* = (0.6, 0.4)$ and, for all $i \in \mathcal{X} \setminus \{0, L\}$, $\pi_i^* = \pi^* = (0.3, 0.5, 0.2)$.

We now let $\varepsilon_i = \varepsilon = 0.4$ for all $i \in \mathcal{X}$ and consider the imprecise birth-death chain that has the corresponding linear-vacuous credal sets as its local models. In this way, we obtain the following lower and upper probabilities:

$$\underline{p}_0 = 0.27,\ \overline{p}_0 = 0.67,\ \underline{q}_L = 0.36,\ \overline{q}_L = 0.76$$

and, for all $i \in \mathcal{X} \setminus \{0, L\}$:

$$\underline{q}_i = 0.18,\ \overline{q}_i = 0.58,\ \underline{p}_i = 0.12,\ \overline{p}_i = 0.52.$$

For all $i \in \mathcal{X} \setminus \{0, L\}$, the credal set $\mathcal{Q}_i$ is equal to $\mathcal{Q}^{\varepsilon}_{\pi^*}$, which is the convex hull of the following three extreme points:

$$(0.58, 0.3, 0.12), (0.18, 0.7, 0.12), (0.18, 0.3, 0.52).$$

Figure 3 provides a graphical representation of this credal set $\mathcal{Q}^{\varepsilon}_{\pi^*}$.

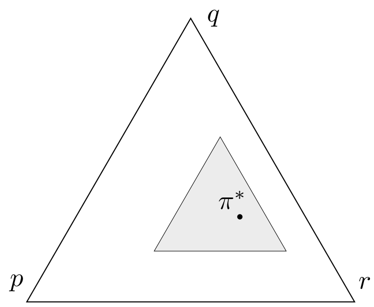

Figure 3: The grey zone depicts the credal set $\mathcal{Q}^{\varepsilon}_{\pi^*}$ from the birth-death chain in the the linear-vacuous example.

The lower and upper expected return times that correspond to this particular example can be found in Table 3. For the sake of this example, we compute $\underline{\tau}_{1\to1}$ explicitly.

We start by applying Equation (32) for $i = 1$, which tells us that

$$\begin{aligned}\underline{\tau}_{1\to1} &= 1 + \underline{q}_1\underline{\tau}_{0\to1} + \underline{p}_1\underline{\tau}_{2\to1} \\ &= 1 + 0.18\underline{\tau}_{0\to1} + 0.12\underline{\tau}_{2\to1}.\end{aligned}$$

Therefore, since we know from Equations (30) and (31) that

$$\underline{\tau}_{0\to1} = \frac{1}{\overline{p}_0} = 1.492$$

and

$$\underline{\tau}_{2\to1} = \frac{1}{\overline{q}_2} + \frac{\underline{p}_2}{\overline{q}_2\overline{q}_3} + \frac{\underline{p}_2\underline{p}_3}{\overline{q}_2\overline{q}_3\overline{q}_4} = 2.154,$$

we find that $\underline{\tau}_{1\to1} = 1.526$.

i	$\underline{\tau}_{i\to i}$	$\overline{\tau}_{i\to i}$
0	1.584	91.41
1	1.526	24.956
2	1.678	17.845
3	1.656	79.71
4	2.037	503.724

Table 3: Lower and upper expected return times for the birth-death chain in the linear-vacuous mixture example.

10 Summary and Future Work

We have presented a simple method for computing lower and upper expected—upward and downward—first passage times and return times in imprecise birth-death chains, have presented numerical results, and have discussed a special case for which our method simplifies even more.

In future research, we plan to try and apply similar methods to (a) other simple types of imprecise Markov chains—different from birth-death chains—such as, for example, the Bonus-Malus systems that are described in Reference [6] and (b) continuous—rather than discrete—time models.

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