Robustness in Queueing Systems

ECQT 2014

Stavros Lopatatzidis

Ghent University, SYSTeMS

5th September 2014

A few words about us

SYSTeMS

Gert de Cooman
Jasper De Bock
Stavros Lopatatzidis

SMACS

Joris Walraevens

Industrial Management and Operational Research

Stijn De Vuyst

Robustness

Robustness

What do we mean?

Robustness

What do we mean? ⇒ Bounds of the system

Robustness

What do we mean? $\Rightarrow$ Bounds of the system

Why robustness?

Robustness

What do we mean? ⇒ Bounds of the system

Why robustness? ⇒ Model uncertainty

Robustness

What do we mean? ⇒ Bounds of the system

Why robustness? ⇒ Model uncertainty

Our Purpose?

Robustness

What do we mean? ⇒ Bounds of the system

Why robustness? ⇒ Model uncertainty

Our Purpose? ⇒ Minimum & Maximum values of various performance measures

Robustness

What do we mean? $\Rightarrow$ Bounds of the system

Why robustness? $\Rightarrow$ Model uncertainty

Our Purpose? $\Rightarrow$ Minimum & Maximum values of various performance measures

Which performance measures?

Robustness

What do we mean? ⇒ Bounds of the system

Why robustness? ⇒ Model uncertainty

Our Purpose? ⇒ Minimum & Maximum values of various performance measures

Which performance measures?

- Expected queue length
- Probability of a certain length
- Turning on the server (probability having 1 given 0)
- Averages

Our Queueing System

Our model → *Geo/Geo/1/L*

Our Queueing System

Our model → *Geo/Geo/1/L*

Probability of arrival *a* and probability of departure *d* (independent at each time point!)

Discrete Time, Single-server (1) queue with finite capacity (*L*)

Our Queueing System

Our model → *Geo/Geo/1/L*

Probability of arrival a and probability of departure d
(independent at each time point!)

Discrete Time, Single-server (1) queue with finite capacity (L)

Other assumptions

Our Queueing System

Our model → *Geo/Geo/1/L*

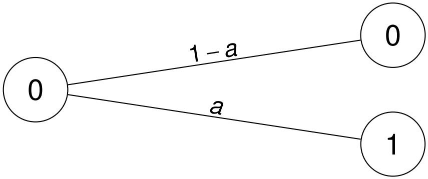
Probability of arrival a and probability of departure d (independent at each time point!)

Discrete Time, Single-server (1) queue with finite capacity (L)

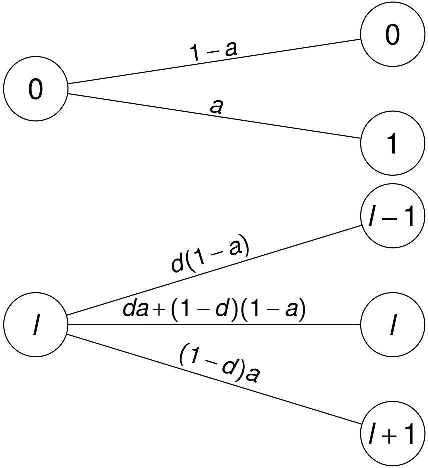
Other assumptions

- A departure occurs prior to an arrival
- Service obeys the *FCFS* principle
- Item stays till served!

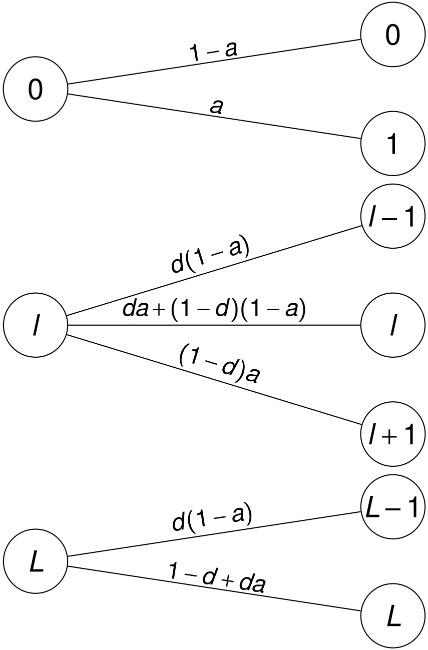
Our Queueing System



Our Queueing System



Our Queueing System



Expectations (Preliminaries)

State space $\Rightarrow \mathscr{X} = \{0, \ldots, L\}$

For any function h on $\mathscr{X}$ we have $E(h) = \sum_{x \in \mathscr{X}} h(x) P[X = x]$

$$P[X = 0] = \frac{d - a}{d - \frac{(1-d)^L a^{L+1}}{(d(1-a))^L}}$$

$$P[X = l] = \frac{(1-d)^{l-1} a^l}{(d(1-a))^l} P[X = 0]$$

Expectations (Notation)

local/conditional probability ⇒ $p(\cdot|x_n, a, d)$ with $x_n \in \mathscr{X}$ at any time point n

probability mass functions

$$p(x_1)\prod_{i=1}^{n-1} p(x_{i+1}|x_{1:i}) = p(x_1)\prod_{i=1}^{n-1} p(x_{i+1}|x_i, a, d)$$

denoted by $p_{1:n,a,d}$

Expectations

Let f be a function on $\mathscr{X}^n := \underbrace{\mathscr{X} \times \cdots \times \mathscr{X}}_{n}$ then,

$$E(f) = \sum_{x_{1:n} \in \mathscr{X}^n} f(x_{1:n}) p(x_{1:n}) = \sum_{x_{1:n} \in \mathscr{X}^n} f(x_{1:n}) p(x_1) \prod_{i=1}^{n-1} p(x_{i+1} | x_{1:i})$$

Expectations

Let *f* be a function on $\mathscr{X}^n := \underbrace{\mathscr{X} \times \cdots \times \mathscr{X}}_{n}$ then,

$$E(f) = \sum_{x_{1:n} \in \mathscr{X}^n} f(x_{1:n}) p(x_{1:n}) = \sum_{x_{1:n} \in \mathscr{X}^n} f(x_{1:n}) p(x_1) \prod_{i=1}^{n-1} p(x_{i+1} | x_{1:i})$$

$$= \sum_{x_{1:n} \in \mathscr{X}^n} f(x_{1:n}) p(x_1) \prod_{i=1}^{n-1} p(x_{i+1} | x_i, a, d)$$

Expectations

Let f be a function on $\mathscr{X}^n := \underbrace{\mathscr{X} \times \cdots \times \mathscr{X}}_{n}$ then,

$$E(f) = \sum_{x_{1:n} \in \mathscr{X}^n} f(x_{1:n}) p(x_{1:n}) = \sum_{x_{1:n} \in \mathscr{X}^n} f(x_{1:n}) p(x_1) \prod_{i=1}^{n-1} p(x_{i+1} | x_{1:i})$$

$$= \sum_{x_{1:n} \in \mathscr{X}^n} f(x_{1:n}) p(x_1) \prod_{i=1}^{n-1} p(x_{i+1} | x_i, a, d)$$

$$= \sum_{x_1 \in \mathscr{X}} p(x_1) \sum_{x_2 \in \mathscr{X}} p(x_2 | x_1, a, d) \cdots \sum_{x_n \in \mathscr{X}} f(x_1, \ldots, x_n) p(x_n | x_{n-1}, a, d)$$

Expectations

Let f be a function on $\mathscr{X}^n := \underbrace{\mathscr{X} \times \cdots \times \mathscr{X}}_{n}$ then,

$$
\begin{aligned}
E(f) &= \sum_{x_{1:n} \in \mathscr{X}^n} f(x_{1:n}) p(x_{1:n}) = \sum_{x_{1:n} \in \mathscr{X}^n} f(x_{1:n}) p(x_1) \prod_{i=1}^{n-1} p(x_{i+1} | x_{1:i}) \\
&= \sum_{x_{1:n} \in \mathscr{X}^n} f(x_{1:n}) p(x_1) \prod_{i=1}^{n-1} p(x_{i+1} | x_i, a, d) \\
&= \sum_{x_1 \in \mathscr{X}} p(x_1) \sum_{x_2 \in \mathscr{X}} p(x_2 | x_1, a, d) \cdots \sum_{x_n \in \mathscr{X}} f(x_1, \ldots, x_n) p(x_n | x_{n-1}, a, d)
\end{aligned}
$$

which is the Law of Iterated Expectation (LIE)

$$E_{1:n}(f) = E(f) = E(E(\ldots E(f | X_{1:n-1}) \ldots | X_1) | \square)$$

with $\square$ being the initial state

Expectations

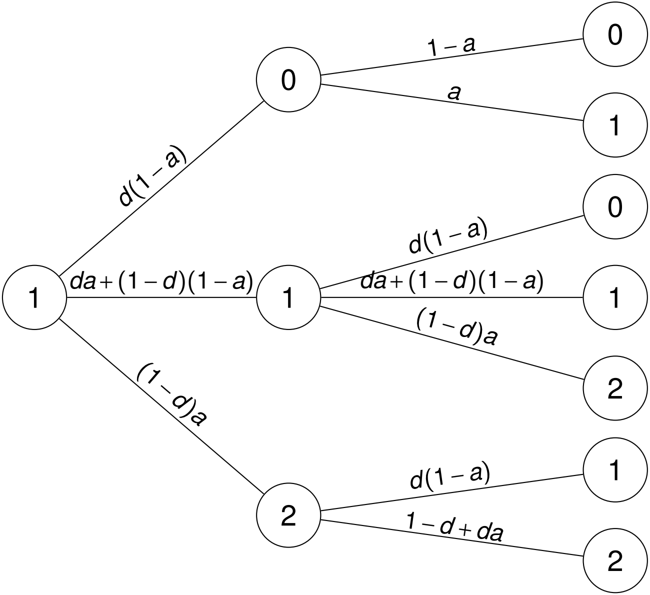
Functions on $\mathscr{X}$ will be denoted by h

$$E_n(h) = \sum_{x_n \in \mathscr{X}} h(x_n) p(x_1) \prod_{i=1}^{n-1} p(x_{i+1}|x_i) =$$

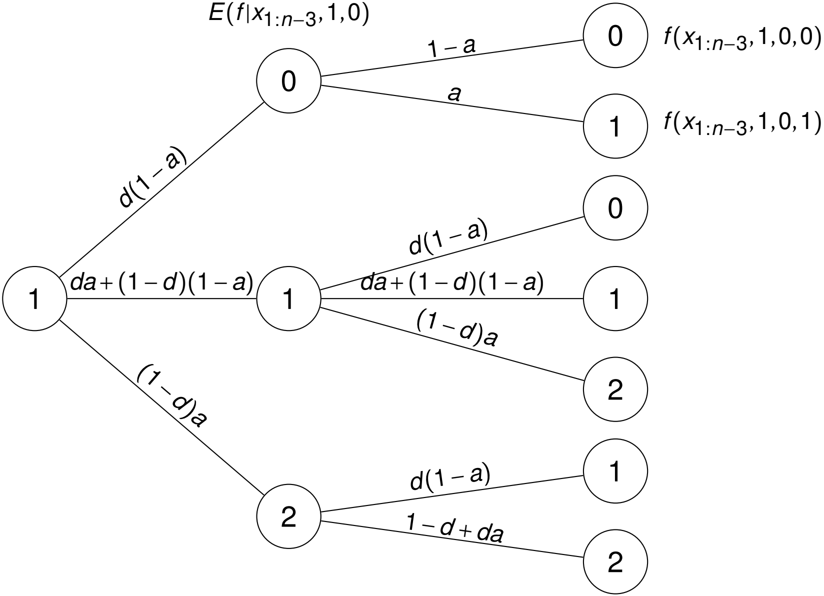
$$E(f) = E(E(\ldots E(f|X_{n-1})\ldots|X_1)|\Box)$$

For probabilities we use *indicator functions*
i.e. $\mathbb{I}_A$ assigns 1 when A happens else 0

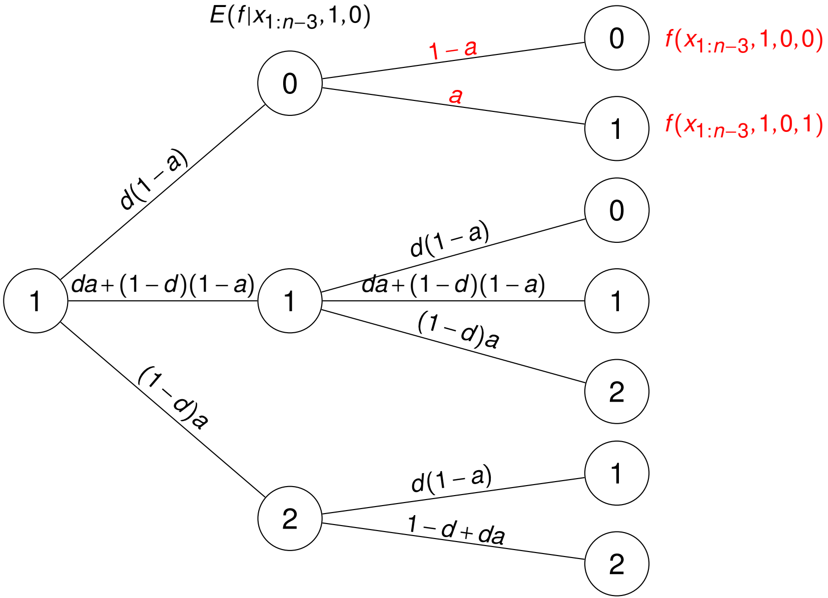
Calculating with Backwards Recursion



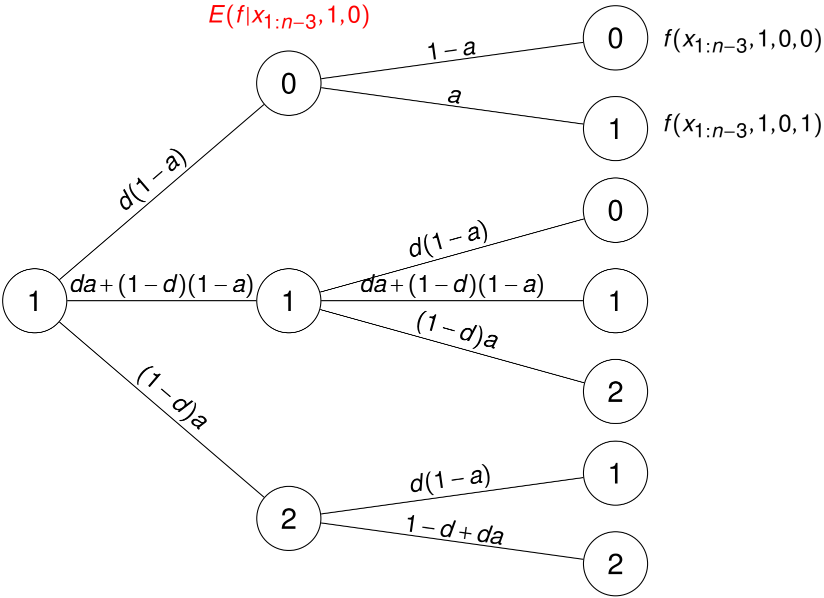
Calculating with Backwards Recursion



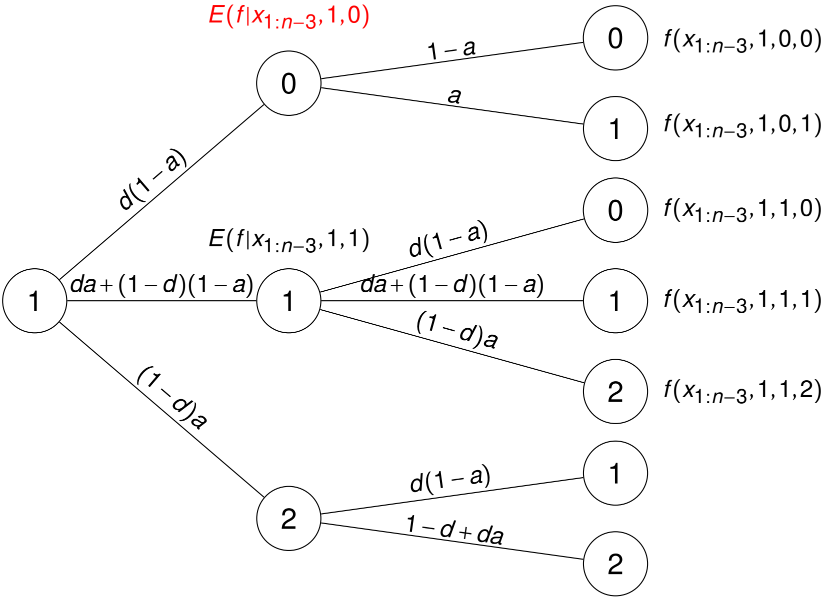
Calculating with Backwards Recursion



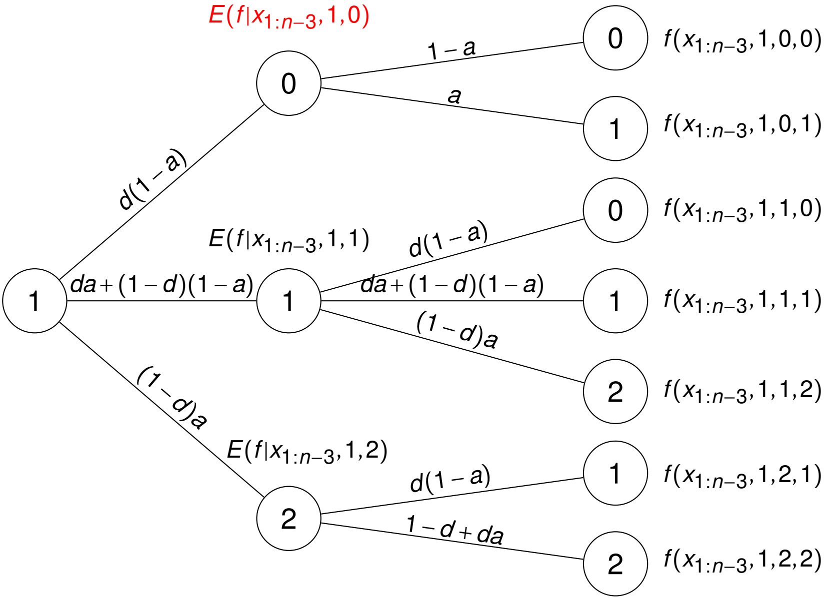
Calculating with Backwards Recursion



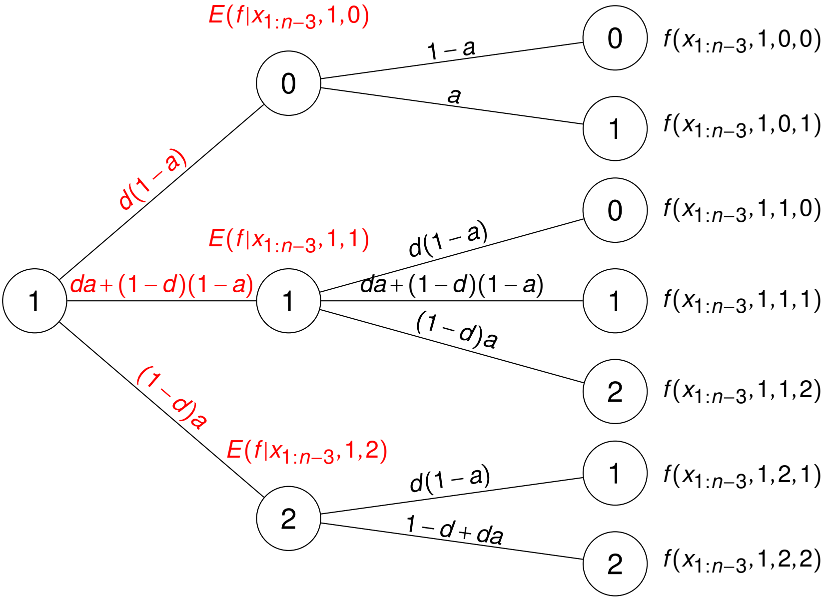
Calculating with Backwards Recursion



Calculating with Backwards Recursion



Calculating with Backwards Recursion



Uncertainty

Uncertainty in the parameters of the model

Calculate bounds (Lower & Upper Expectations)

$$\underline{E}^{\mathscr{P}}(g) := \min\left\{E^P(g) : P \in \mathscr{P}\right\} \quad \text{and} \quad \overline{E}^{\mathscr{P}}(g) := \max\left\{E^P(g) : P \in \mathscr{P}\right\}$$

Combining with our notation $\Rightarrow$ $\underline{E}_n$, $\overline{E}_n$ & $\underline{E}_{1:n}$, $\overline{E}_{1:n}$

Uncertainty

Uncertainty in the parameters of the model

Calculate bounds (Lower & Upper Expectations)

$\underline{E}^{\mathscr{P}}(g) := \min\left\{E^P(g) : P \in \mathscr{P}\right\}$ and $\overline{E}^{\mathscr{P}}(g) := \max\left\{E^P(g) : P \in \mathscr{P}\right\}$

Combining with our notation $\Rightarrow$ $\underline{E}_n$, $\overline{E}_n$ & $\underline{E}_{1:n}$, $\overline{E}_{1:n}$

Geo/Geo/1/L $\Rightarrow$ *interval probabilities* $a \rightarrow [\underline{a}, \overline{a}]$ & $d \rightarrow [\underline{d}, \overline{d}]$

where each P has form

$$p(x_1) \prod_{i=1}^{n-1} p(x_{i+1} | x_{1:i}, a_{x_{1:i}}, d_{x_{1:i}}) \text{ with } a_{x_{1:i}} \in [\underline{a}, \overline{a}],\ d_{x_{1:i}} \in [\underline{d}, \overline{d}]\ (p_{1:n,A,D})$$

Two approaches to deal with uncertainty

1st Approach

Related to typical sensitivity analysis

Tree corresponding to lower (or upper) expectation consists of time-homogeneous/stationary probabilities of arrival and departure

1st Approach

Related to typical sensitivity analysis

Tree corresponding to lower (or upper) expectation consists of time-homogeneous/stationary probabilities of arrival and departure

$$\underline{E}^{s}_{1:n}(f) = \min\left\{E^{p_{1:n,a,d}}(f) : a \in [\underline{a}, \overline{a}], d \in [\underline{d}, \overline{d}]\right\}$$

$$\overline{E}^{s}_{1:n}(f) = \max\left\{E^{p_{1:n,a,d}}(f) : a \in [\underline{a}, \overline{a}], d \in [\underline{d}, \overline{d}]\right\}$$

We are mainly interested in $n \to \infty$

Calculations under the 1st Approach

Given a function h on $\mathscr{X}$, w.r.t to lower expectation in the limit

$$\lim_{n\to\infty} \underline{E}_n^s(h) = \min\left\{\sum_{x\in\mathscr{X}} h(x)P[X_n = x] : a \in [\underline{a},\overline{a}], d \in [\underline{d},\overline{d}]\right\} \quad (1)$$

$$P[X=0] = \frac{d-a}{d - \frac{(1-d)^L a^{L+1}}{(d(1-a))^L}}. \quad (2)$$

$$P[X=l] = \frac{(1-d)^{l-1} a^l}{(d(1-a))^l} P[X=0] \quad (3)$$

We solve (1), where the parameters of (2) and (3) vary in $[\underline{a},\overline{a}]$ and $[\underline{d},\overline{d}]$

Calculations under the 1st Approach

For functions on $\mathscr{X}^n$ which represent averages of a function h on $\mathscr{X}$ $\Rightarrow$ the lower expectation in the limit approaches the value of (1)

For general f on $\mathscr{X}^n$ it is difficult to formulate and solve a similar to (1) optimization problem

We approximate lower and upper expectations by

- choosing a number of probabilities from $[\underline{a}, \overline{a}]$ and $[\underline{d}, \overline{d}]$
- and calculating for all combinations using LIE in backwards recursion in combination with formulas (2) and (3)

2nd Approach

We drop stationarity

The tree corresponding to lower (or upper) expectation can have any probability of arrival and departure, from the respective sets, at any time point and given any sequence of queue lengths

2nd Approach

We drop stationarity

The tree corresponding to lower (or upper) expectation can have any probability of arrival and departure, from the respective sets, at any time point and given any sequence of queue lengths

$$\underline{E}^{ei}_{1:n}(f) = \min\left\{E^{p_{1:n,A,D}}(f) : (\forall i \leq n)(\forall x_{1:i} \in \mathscr{X}^i) a_{x_{1:i}} \in [\underline{a}, \overline{a}], d_{x_{1:i}} \in [\underline{d}, \overline{d}]\right\}$$

$$\overline{E}^{ei}_{1:n}(f) = \max\left\{E^{p_{1:n,A,D}}(f) : (\forall i \leq n)(\forall x_{1:i} \in \mathscr{X}^i) a_{x_{1:i}} \in [\underline{a}, \overline{a}], d_{x_{1:i}} \in [\underline{d}, \overline{d}]\right\}$$

Calculations under the 2nd Approach

What is important...

- for any n (approaching or not infinity)
- for any function (on $\mathscr{X}$ or $\mathscr{X}^n$)

we can always use LIE for calculating efficiently lower and upper expectations

Calculations under the 2nd Approach

What is important...

- for any n (approaching or not infinity)
- for any function (on $\mathscr{X}$ or $\mathscr{X}^n$)

we can always use LIE for calculating efficiently lower and upper expectations

Proposition

For any real-valued function f on $\mathscr{X}^n$, with $n \in \mathbb{N}_0$

$$\underline{E}^{ei}_{1:n}(f) = \underline{E}_1(\underline{E}_2(\ldots \underline{E}_n(f|X_{1:n-1})\ldots|X_1)|\Box)$$

- Linear complexity in the number of steps n
- In each iteration we can calculate conditional expectations by using only the extreme points $(\underline{a}, \overline{a}, \underline{d}, \overline{d})$

2nd Approach vs 1st Approach

Comparing to the 1st approach...

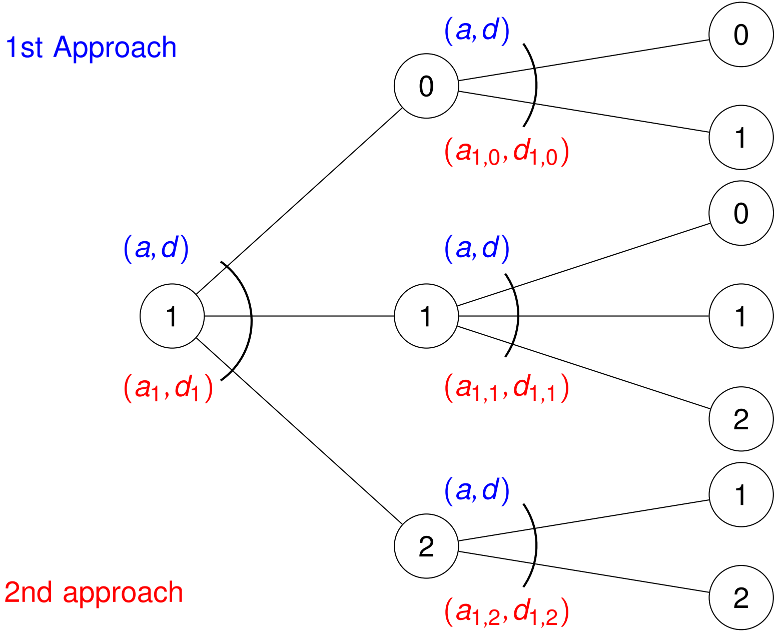
Lemma

For any real-valued map f on $\mathscr{X}^n$, with $n \in \mathbb{N}_0$, and any $x_{1:i} \in \mathscr{X}^i$ with $i \in \{1, \ldots, n\}$, it holds that

$$\underline{E}^{ei}_{i:n}(f|x_{1:i}) \leq \underline{E}^{s}_{i:n}(f|x_{1:i}) \text{ and } \overline{E}^{ei}_{i:n}(f|x_{1:i}) \geq \overline{E}^{s}_{i:n}(f|x_{1:i}).$$

The second approach is associated with all the possible probability trees, whereas the first one only with the stationary ones

Probability trees under both approaches



Experiments and Discussion

Some useful properties

Interested in $n \to \infty$

1st Approach

For functions on $\mathscr{X}$ we have convergence independent of the initial model

Functions on $\mathscr{X}^n$ convergence to a value affected by the initial model

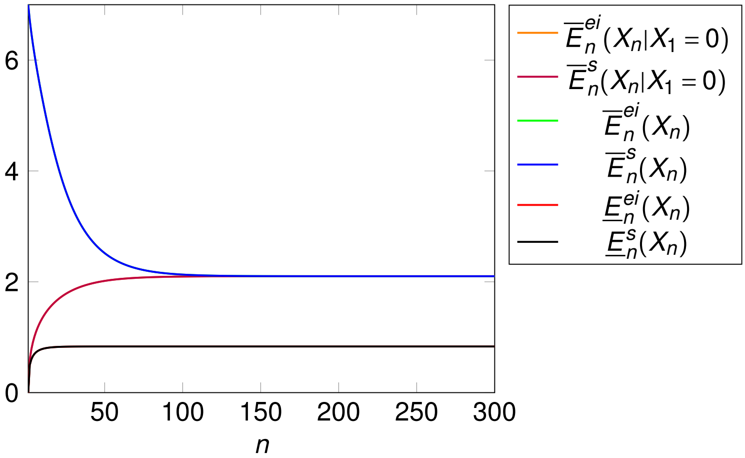
2nd Approach

The same convergence properties hold, but for the bounds

Our setting

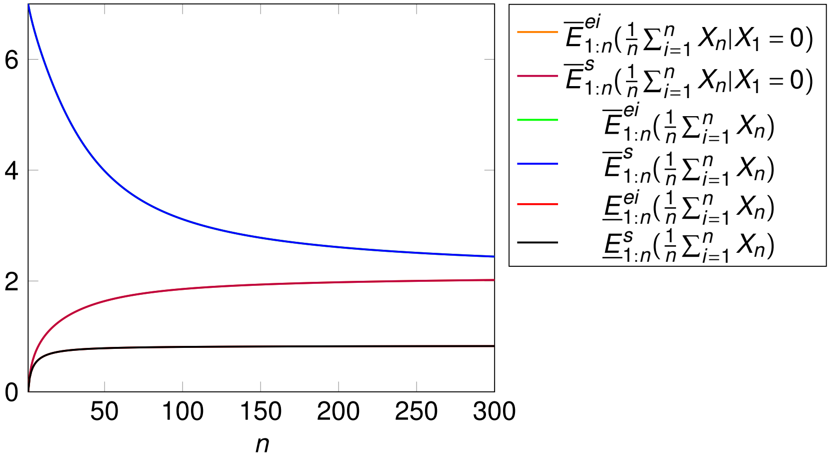
- queue length = 7
- arrival $\in [0.5, 0.6]$
- departure $\in [0.7, 0.8]$

Expected (Average) Queue Length



Lower and upper expected queue length

Expected (Average) Queue Length



Lower and upper expected average queue length

Expected (Average) Queue Length

Both approaches lead to the same corresponding tree

- For lower expected (average) queue length largest departure rate, lowest arrival rate
- For upper expected (average) queue length lowest departure rate, largest arrival rate

Due to the monotonicity of the function

(Average) Probability of queue length

k	0	1	2	7
$\underline{E}_n^{ei}(\mathbb{I}_k(X_n))$	0.148638	0.290906	0.108098	0.000114
$\underline{E}_{1:n}^{ei}(\frac{1}{n}\sum_{i=1}^{n}\mathbb{I}_k(X_i))$	0.148301	0.308642	0.109882	0.000114
$\overline{E}_n^{ei}(\mathbb{I}_k(X_n))$	0.375014	0.534395	0.268357	0.022481
$\overline{E}_{1:n}^{ei}(\frac{1}{n}\sum_{i=1}^{n}\mathbb{I}_k(X_i))$	0.375084	0.517315	0.257773	0.022987
$\underline{E}_n^{s}(\mathbb{I}_k(X_n))$	0.148638 (0.6,0.7)	0.31815 (0.6,0.7)	0.117192 (0.5,0.8)	0.000114 (0.5,0.8)
$\underline{E}_{1:n}^{s}(\frac{1}{n}\sum_{i=1}^{n}\mathbb{I}_k(X_i))$	0.148301 (0.6,0.7)	0.317824 (0.6,0.7)	0.117162 (0.5,0.8)	0.000114 (0.5,0.8)
$\overline{E}_n^{s}(\mathbb{I}_k(X_n))$	0.375014 (0.5,0.8)	0.477512 (0.55,0.8)	0.206501 (0.6,0.72)	0.022481 (0.6,0.7)
$\overline{E}_{1:n}^{s}(\frac{1}{n}\sum_{i=1}^{n}\mathbb{I}_k(X_i))$	0.375084 (0.5,0.8)	0.477569 (0.55,0.8)	0.206624 (0.6,0.72)	0.022987 (0.6,0.7)

(Average) Probability of queue length

k	0	1	2	7
$\underline{E}^{ei}_n(\mathbb{I}_k(X_n))$	0.148638	0.290906	0.108098	0.000114
$\underline{E}^{ei}_{1:n}(\frac{1}{n}\sum_{i=1}^n \mathbb{I}_k(X_i))$	0.148301	0.308642	0.109882	0.000114
$\overline{E}^{ei}_n(\mathbb{I}_k(X_n))$	0.375014	0.534395	0.268357	0.022481
$\overline{E}^{ei}_{1:n}(\frac{1}{n}\sum_{i=1}^n \mathbb{I}_k(X_i))$	0.375084	0.517315	0.257773	0.022987
$\underline{E}^{s}_n(\mathbb{I}_k(X_n))$	0.148638 (0.6,0.7)	0.31815 (0.6,0.7)	0.117192 (0.5,0.8)	0.000114 (0.5,0.8)
$\underline{E}^{s}_{1:n}(\frac{1}{n}\sum_{i=1}^n \mathbb{I}_k(X_i))$	0.148301 (0.6,0.7)	0.317824 (0.6,0.7)	0.117162 (0.5,0.8)	0.000114 (0.5,0.8)
$\overline{E}^{s}_n(\mathbb{I}_k(X_n))$	0.375014 (0.5,0.8)	0.477512 (0.55,0.8)	0.206501 (0.6,0.72)	0.022481 (0.6,0.7)
$\overline{E}^{s}_{1:n}(\frac{1}{n}\sum_{i=1}^n \mathbb{I}_k(X_i))$	0.375084 (0.5,0.8)	0.477569 (0.55,0.8)	0.206624 (0.6,0.72)	0.022987 (0.6,0.7)

(Average) Probability of queue length

k	0	1	2	7
$\underline{E}_n^{ei}(\mathbb{I}_k(X_n))$	0.148638	0.290906	0.108098	0.000114
$\underline{E}_{1:n}^{ei}(\frac{1}{n}\sum_{i=1}^{n}\mathbb{I}_k(X_i))$	0.148301	0.308642	0.109882	0.000114
$\overline{E}_n^{ei}(\mathbb{I}_k(X_n))$	0.375014	0.534395	0.268357	0.022481
$\overline{E}_{1:n}^{ei}(\frac{1}{n}\sum_{i=1}^{n}\mathbb{I}_k(X_i))$	0.375084	0.517315	0.257773	0.022987
$\underline{E}_n^{s}(\mathbb{I}_k(X_n))$	0.148638 (0.6,0.7)	0.31815 (0.6,0.7)	0.117192 (0.5,0.8)	0.000114 (0.5,0.8)
$\underline{E}_{1:n}^{s}(\frac{1}{n}\sum_{i=1}^{n}\mathbb{I}_k(X_i))$	0.148301 (0.6,0.7)	0.317824 (0.6,0.7)	0.117162 (0.5,0.8)	0.000114 (0.5,0.8)
$\overline{E}_n^{s}(\mathbb{I}_k(X_n))$	0.375014 (0.5,0.8)	0.477512 (0.55,0.8)	0.206501 (0.6,0.72)	0.022481 (0.6,0.7)
$\overline{E}_{1:n}^{s}(\frac{1}{n}\sum_{i=1}^{n}\mathbb{I}_k(X_i))$	0.375084 (0.5,0.8)	0.477569 (0.55,0.8)	0.206624 (0.6,0.72)	0.022987 (0.6,0.7)

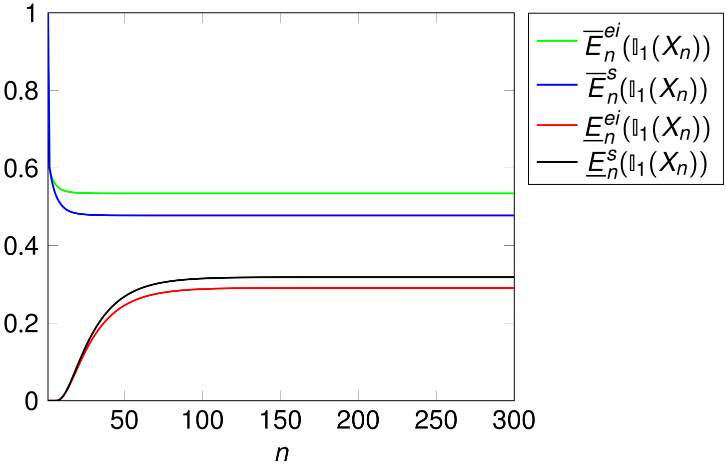
(Average) Probability of queue length

k	0	1	2	7
$\underline{E}_n^{ei}(\mathbb{I}_k(X_n))$	0.148638	0.290906	0.108098	0.000114
$\underline{E}_{1:n}^{ei}(\frac{1}{n}\sum_{i=1}^{n}\mathbb{I}_k(X_i))$	0.148301	0.308642	0.109882	0.000114
$\overline{E}_n^{ei}(\mathbb{I}_k(X_n))$	0.375014	0.534395	0.268357	0.022481
$\overline{E}_{1:n}^{ei}(\frac{1}{n}\sum_{i=1}^{n}\mathbb{I}_k(X_i))$	0.375084	0.517315	0.257773	0.022987
$\underline{E}_n^{s}(\mathbb{I}_k(X_n))$	0.148638 (0.6,0.7)	0.31815 (0.6,0.7)	0.117192 (0.5,0.8)	0.000114 (0.5,0.8)
$\underline{E}_{1:n}^{s}(\frac{1}{n}\sum_{i=1}^{n}\mathbb{I}_k(X_i))$	0.148301 (0.6,0.7)	0.317824 (0.6,0.7)	0.117162 (0.5,0.8)	0.000114 (0.5,0.8)
$\overline{E}_n^{s}(\mathbb{I}_k(X_n))$	0.375014 (0.5,0.8)	0.477512 (0.55,0.8)	0.206501 (0.6,0.72)	0.022481 (0.6,0.7)
$\overline{E}_{1:n}^{s}(\frac{1}{n}\sum_{i=1}^{n}\mathbb{I}_k(X_i))$	0.375084 (0.5,0.8)	0.477569 (0.55,0.8)	0.206624 (0.6,0.72)	0.022987 (0.6,0.7)

(Average) Probability of queue length

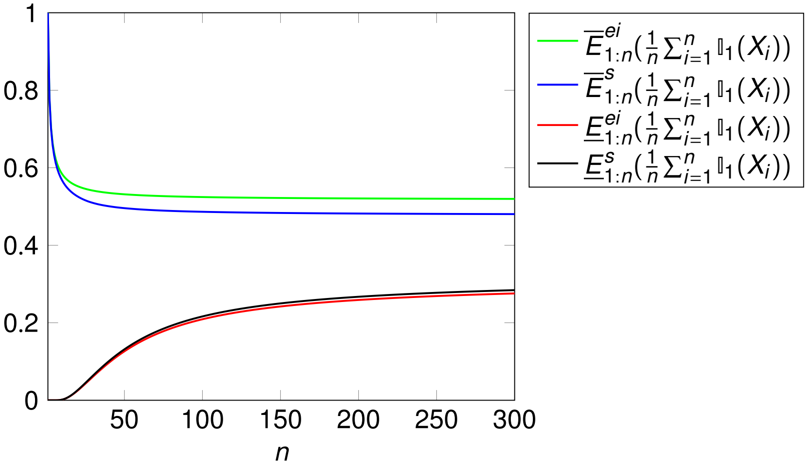
k	0	1	2	7
$\underline{E}_n^{ei}(\mathbb{I}_k(X_n))$	0.148638	0.290906	0.108098	0.000114
$\underline{E}_{1:n}^{ei}(\frac{1}{n}\sum_{i=1}^{n}\mathbb{I}_k(X_i))$	0.148301	0.308642	0.109882	0.000114
$\overline{E}_n^{ei}(\mathbb{I}_k(X_n))$	0.375014	0.534395	0.268357	0.022481
$\overline{E}_{1:n}^{ei}(\frac{1}{n}\sum_{i=1}^{n}\mathbb{I}_k(X_i))$	0.375084	0.517315	0.257773	0.022987
$\underline{E}_n^{s}(\mathbb{I}_k(X_n))$	0.148638 (0.6,0.7)	0.31815 (0.6,0.7)	0.117192 (0.5,0.8)	0.000114 (0.5,0.8)
$\underline{E}_{1:n}^{s}(\frac{1}{n}\sum_{i=1}^{n}\mathbb{I}_k(X_i))$	0.148301 (0.6,0.7)	0.317824 (0.6,0.7)	0.117162 (0.5,0.8)	0.000114 (0.5,0.8)
$\overline{E}_n^{s}(\mathbb{I}_k(X_n))$	0.375014 (0.5,0.8)	0.477512 (0.55,0.8)	0.206501 (0.6,0.72)	0.022481 (0.6,0.7)
$\overline{E}_{1:n}^{s}(\frac{1}{n}\sum_{i=1}^{n}\mathbb{I}_k(X_i))$	0.375084 (0.5,0.8)	0.477569 (0.55,0.8)	0.206624 (0.6,0.72)	0.022987 (0.6,0.7)

Probability of queue length 1



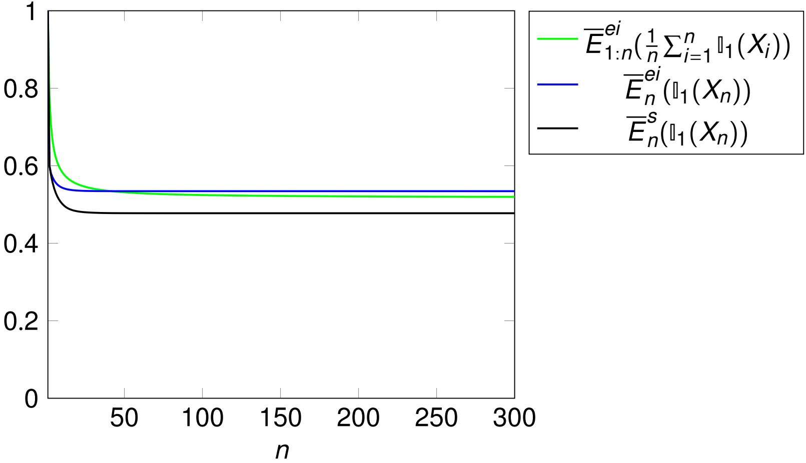
Lower and upper probability of queue length 1

Average Probability of queue length 1



Lower and upper average probability of queue length 1

(Average) Probability of queue length 1



Upper (average) probability of queue length 1

A useful theorem

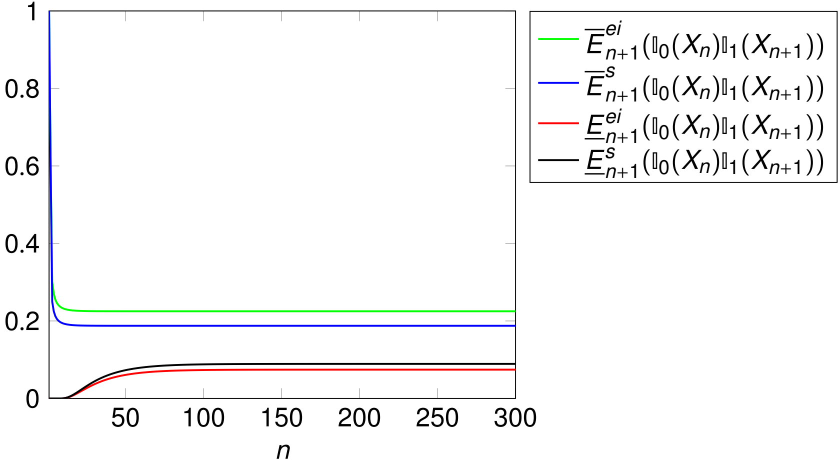
Theorem

Let $L \in \mathbb{N}_0$. Then, for all $k \in \{1, \ldots, L-1\}$ it holds that

$$\lim_{n\to\infty} \underline{E}_n^{ei}(\mathbb{I}_k(X_n)) \leq \lim_{n\to\infty} \underline{E}_{1:n}^{ei}(\frac{1}{n}\sum_{i=1}^{n} \mathbb{I}_k(X_i)) \text{ and}$$

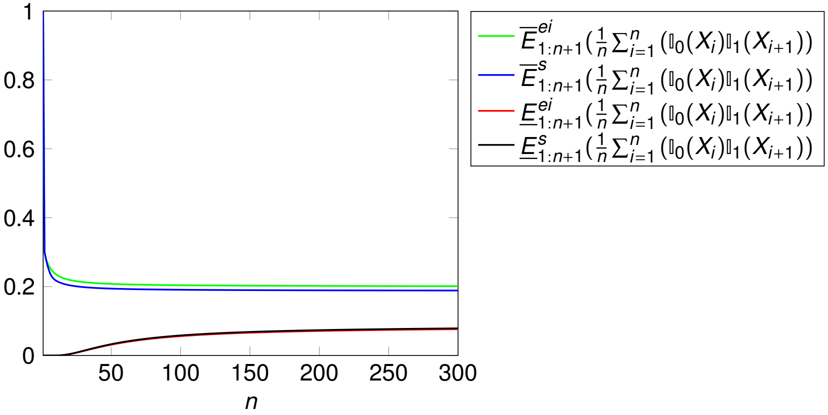
$$\lim_{n\to\infty} \overline{E}_n^{ei}(\mathbb{I}_k(X_n)) \geq \lim_{n\to\infty} \overline{E}_{1:n}^{ei}(\frac{1}{n}\sum_{i=1}^{n} \mathbb{I}_k(X_i))$$

Turning on the server



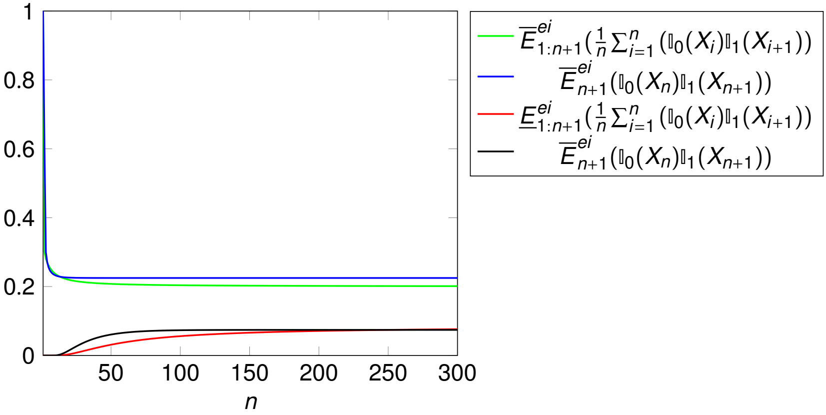
Lower and upper probability of turning on the server

Turning on the server



Lower and upper average probability of turning on the server

Turning on the server



Lower and upper (average) probability of turning on the server

Conclusions & Future work

The 2nd approach provides wider bounds

When we are uncertain about the model, an average might not represent the actual situation

Formulas for calculating lower and upper probabilities under the second approach

Compare the approaches with the state dependent model

Thank you for your attention!!!