

# **Credal Networks under Epistemic Irrelevance: Theory and Algorithms**

Jasper De Bock

13 May 2015, Ghent, Belgium

For the second part of the proof, we start by considering the following collection of gambles on  $\mathcal{X}_K$ :

$$\mathcal{A}_{K \rfloor x_{P(K)}}^* := \left\{ \mathbb{I}_{\{z_{PN(s) \cap K_1}\}} f_{s, z_{PN(s)}} : s \in K, z_{PN(s)} \in \mathcal{X}_{PN(s)}, \right. \\ \left. z_{P(s) \setminus P_K(s)} = x_{P(s) \setminus P_K(s)}, P(s) \cap K \subseteq K_1 \subseteq K, \right. \\ \left. f_{s, z_{PN(s)}} \neq 0 \right\},$$

which is a finite subset of  $\mathcal{D}_{K \rfloor x_{P(K)}}^{\text{irr}} := \text{posi}(\mathcal{A}_{K \rfloor x_{P(K)}}^{\text{irr}})$ . To see why, first notice that because  $PN_K(s) = PN(s) \cap K$  due to Lemma 79(iii)<sub>184</sub>,  $\mathbb{I}_{\{z_{PN(s) \cap K_1}\}}$  is clearly the (finite) sum of all indicators  $\mathbb{I}_{\{y_{PN_K(s)}\}}$  such that  $y_{PN_K(s)} \in \mathcal{X}_{PN_K(s)}$  and  $y_{PN(s) \cap K_1} = z_{PN(s) \cap K_1}$ . By definition of the posi operator, we are now left to show that for any  $y_{PN_K(s)} \in \mathcal{X}_{PN_K(s)}$  such that  $y_{PN(s) \cap K_1} = z_{PN(s) \cap K_1}$ , we have  $\mathbb{I}_{\{y_{PN_K(s)}\}} f_{s, z_{PN(s)}} \in \mathcal{A}_{K \rfloor x_{P(K)}}^{\text{irr}}$ . By construction of  $\mathcal{A}_{K \rfloor x_{P(K)}}^*$ , we know that  $z_{P(s) \setminus P_K(s)} = x_{P(s) \setminus P_K(s)}$ , and it therefore suffices to show that  $y_{P_K(s)} = z_{P_K(s)}$ . To see why this last equality holds, first notice that  $P_K(s) = P(s) \cap K$  due to Lemma 76<sub>181</sub>. Also,  $P(s) \cap K \subseteq PN(s) \cap K_1$  because  $P(s) \cap K \subseteq K_1$  by construction of  $\mathcal{A}_{K \rfloor x_{P(K)}}^*$  and  $P(s) \cap K \subseteq PN(s)$  by definition of  $PN(s)$ . Therefore, we find that  $P_K(s) \subseteq PN(s) \cap K_1$ , implying that  $y_{P_K(s)} = z_{P_K(s)}$  is a direct consequence of  $y_{PN(s) \cap K_1} = z_{PN(s) \cap K_1}$ .

**Credal Networks** under  
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**Bayesian**

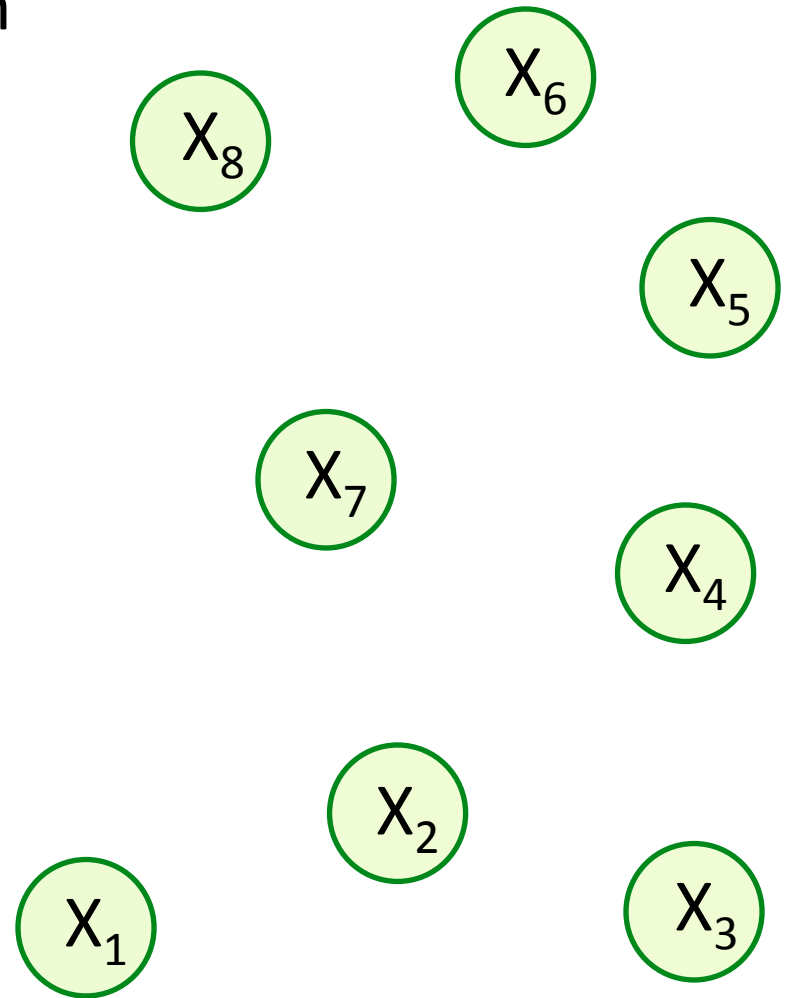
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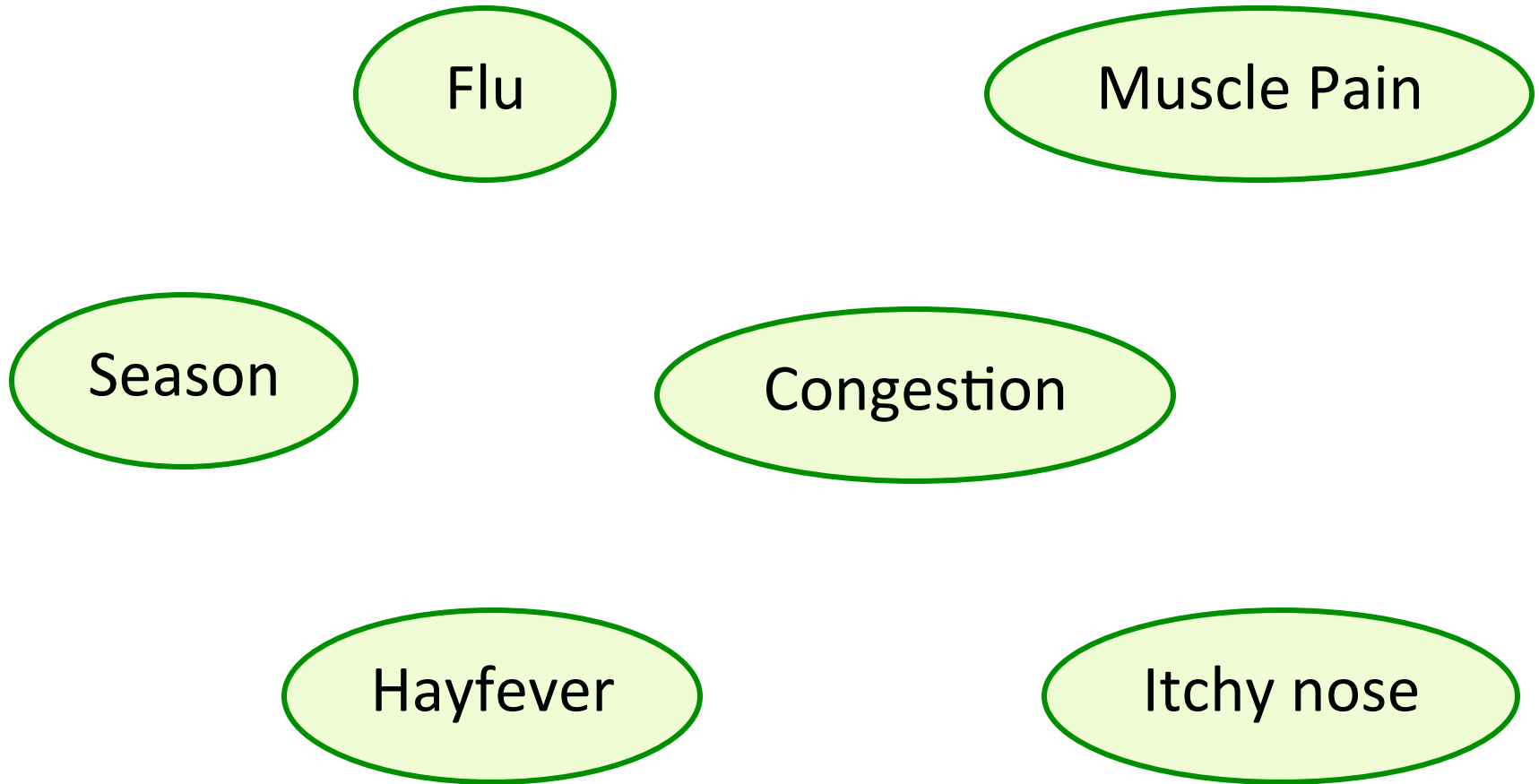
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# Bayesian networks: variables

- Variables  $X_s$  take values  $x_s$  in a finite non-empty set  $\mathcal{X}_s$

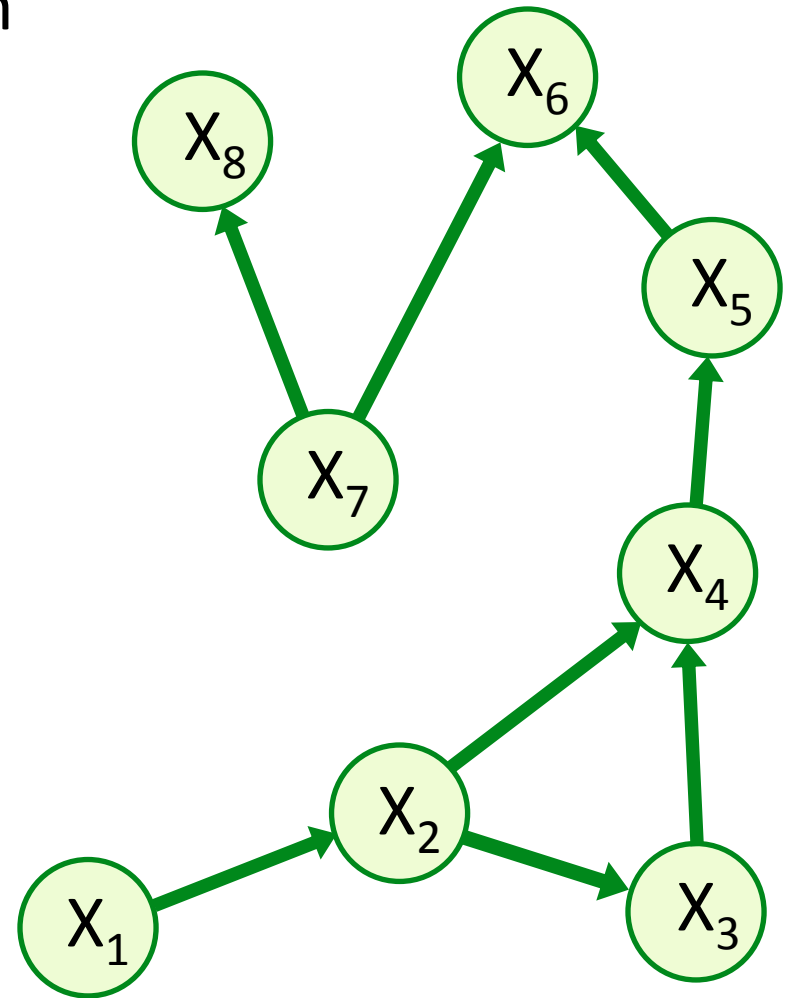


# Bayesian networks: variables

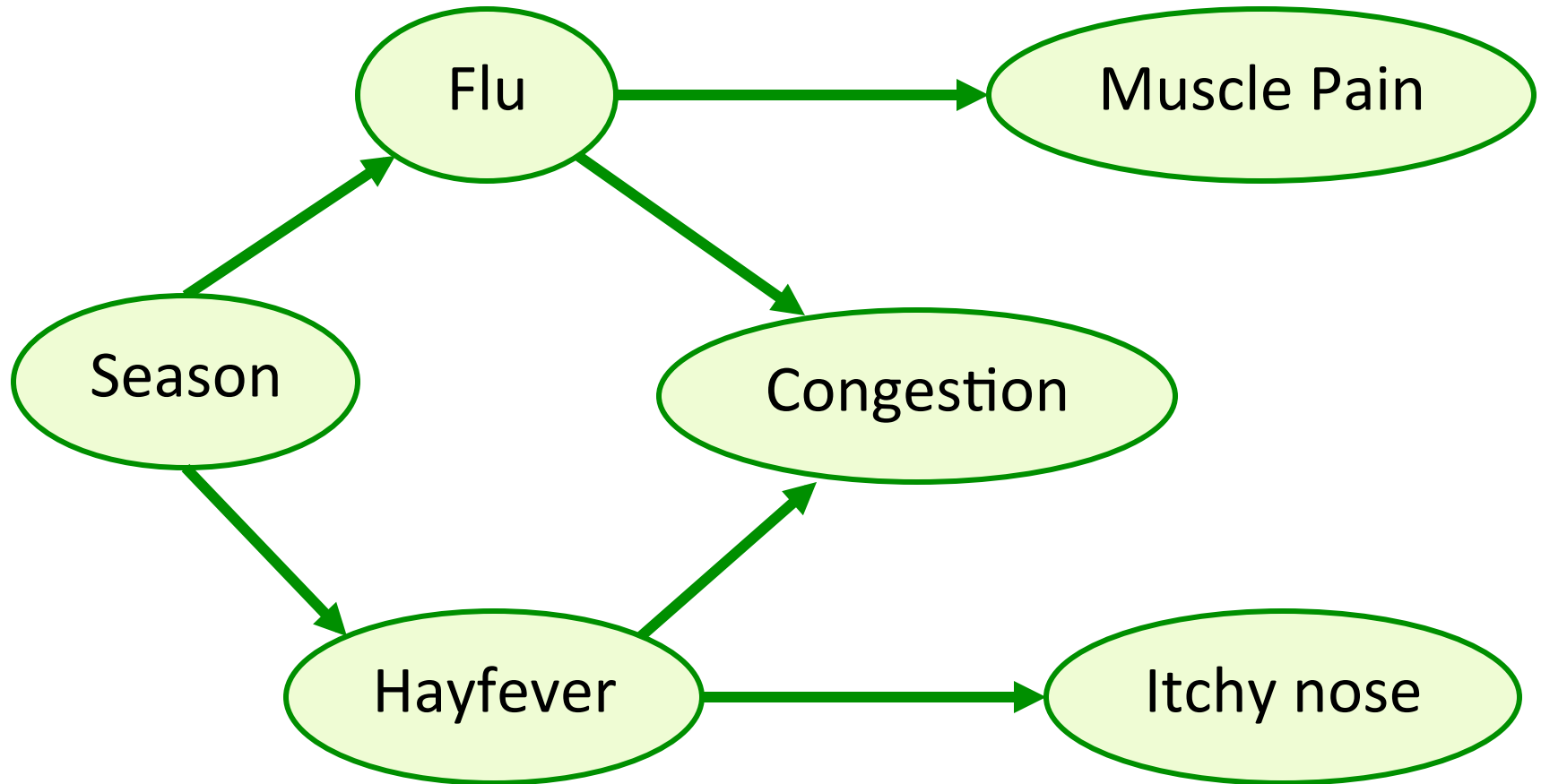


# Bayesian networks: graphical structure

- Variables  $X_s$  take values  $x_s$  in a finite non-empty set  $\mathcal{X}_s$
- Graphical structure: DAG



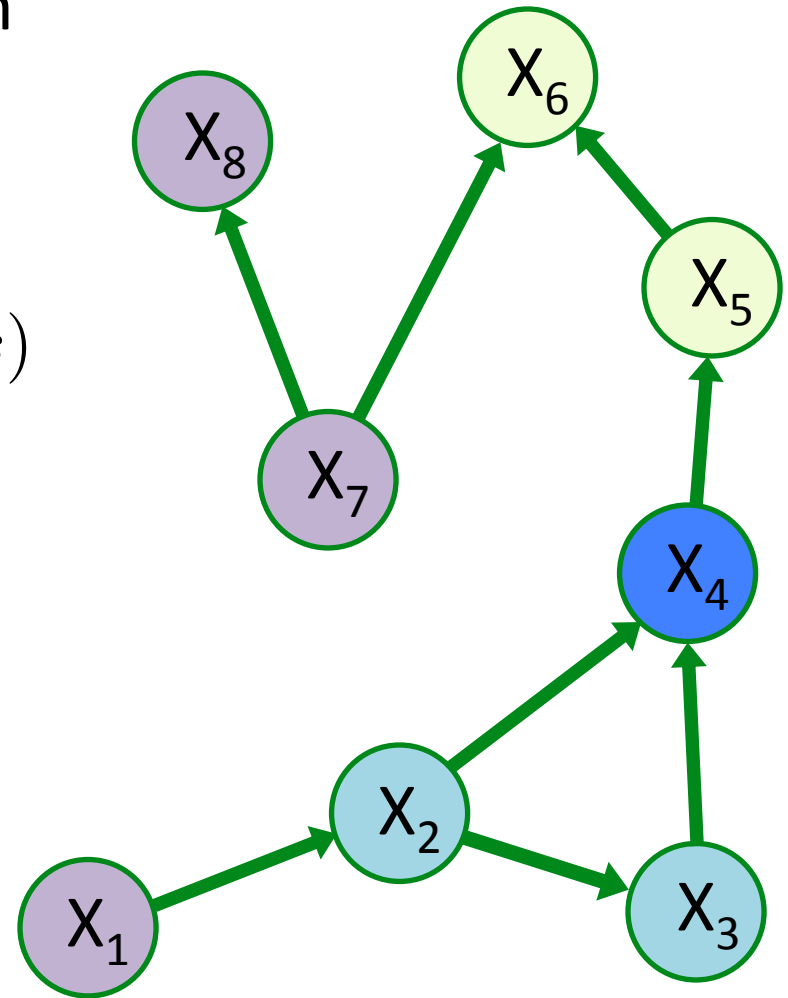
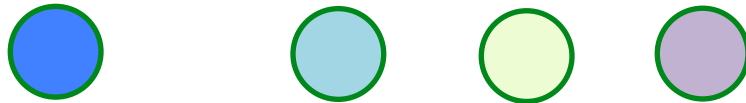
# Bayesian networks: graphical structure



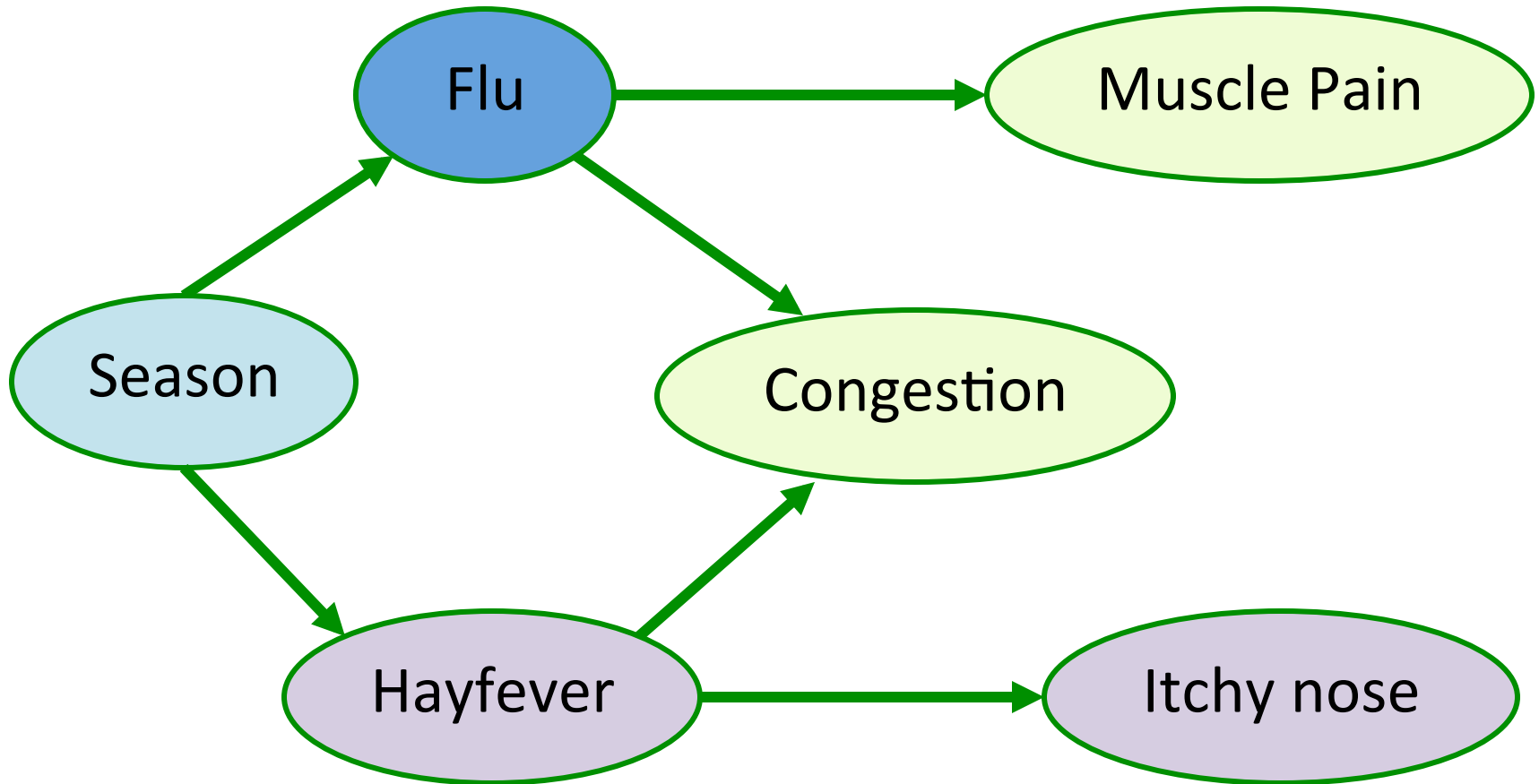


# Bayesian networks: graphical structure

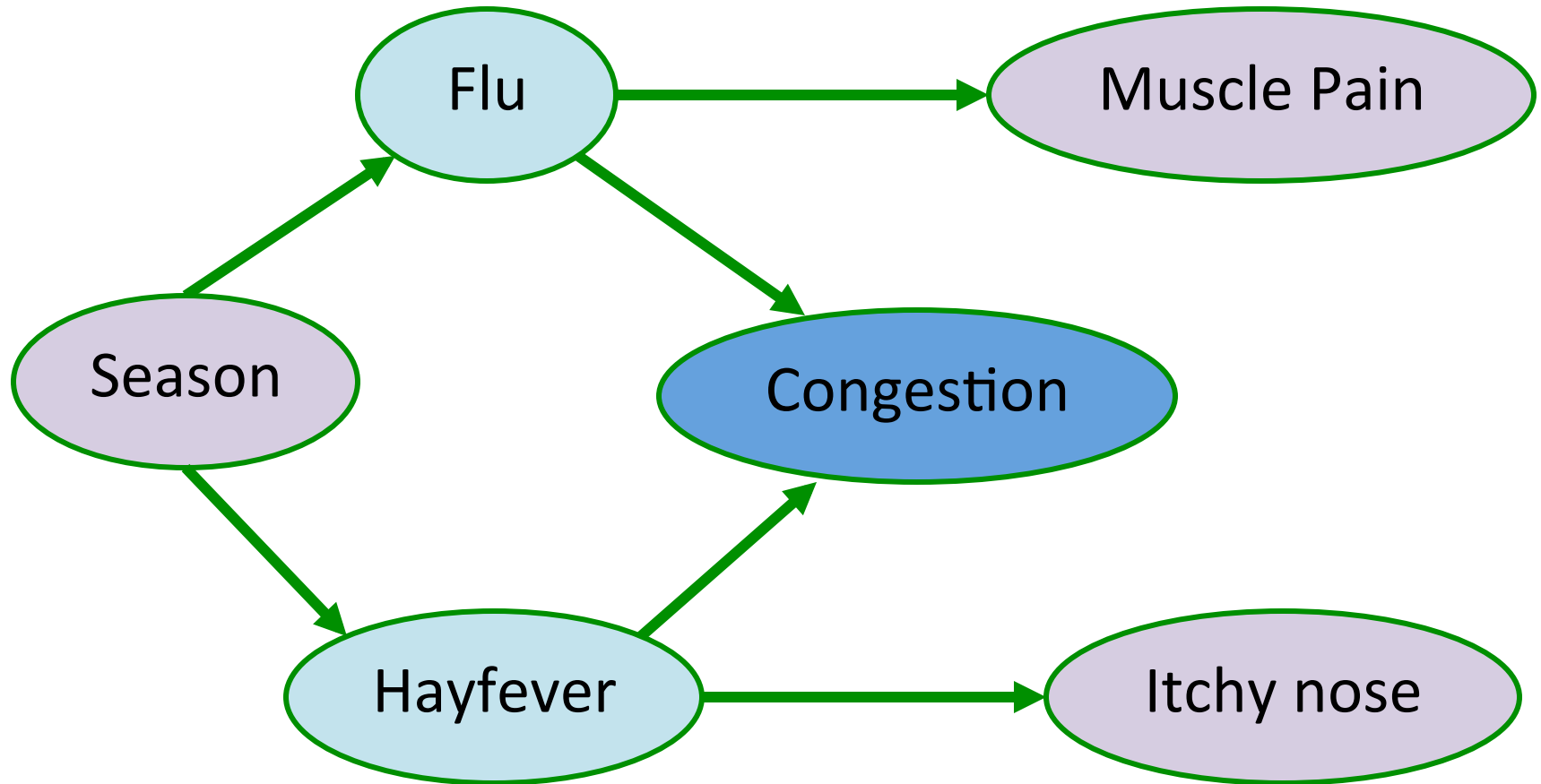
- Variables  $X_s$  take values  $x_s$  in a finite non-empty set  $\mathcal{X}_s$
- Graphical structure: DAG  
 $\Rightarrow \forall s \in G: P(s), D(s), N(s)$



# Bayesian networks: graphical structure

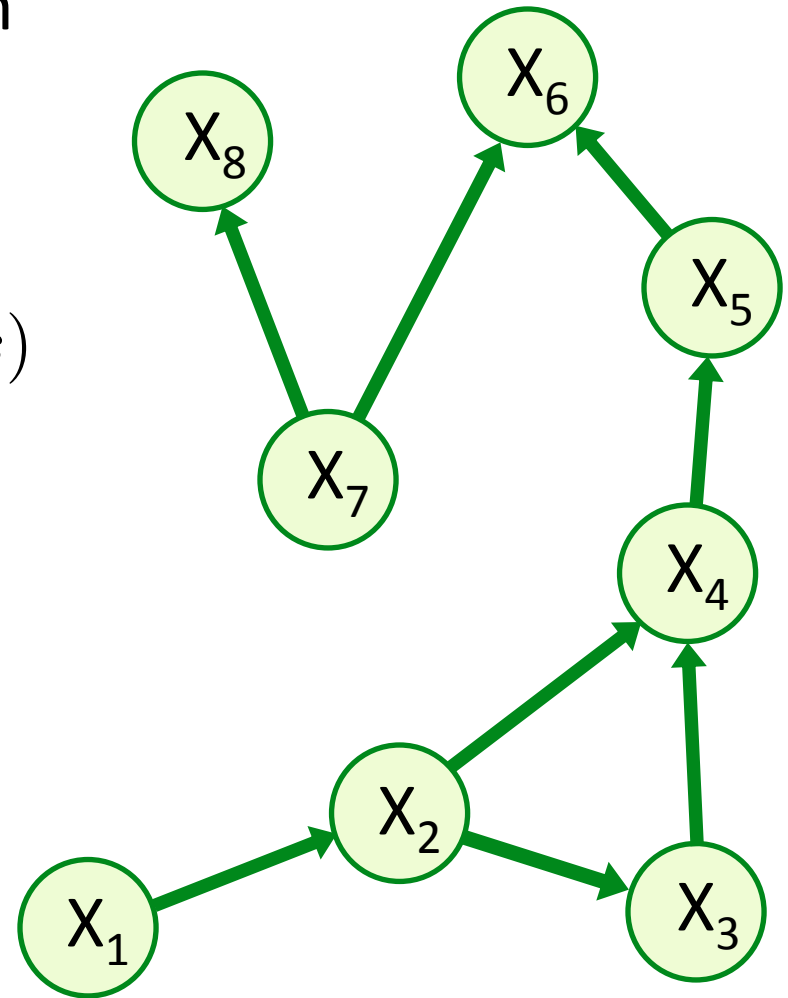


# Bayesian networks: graphical structure



# Bayesian networks: local models

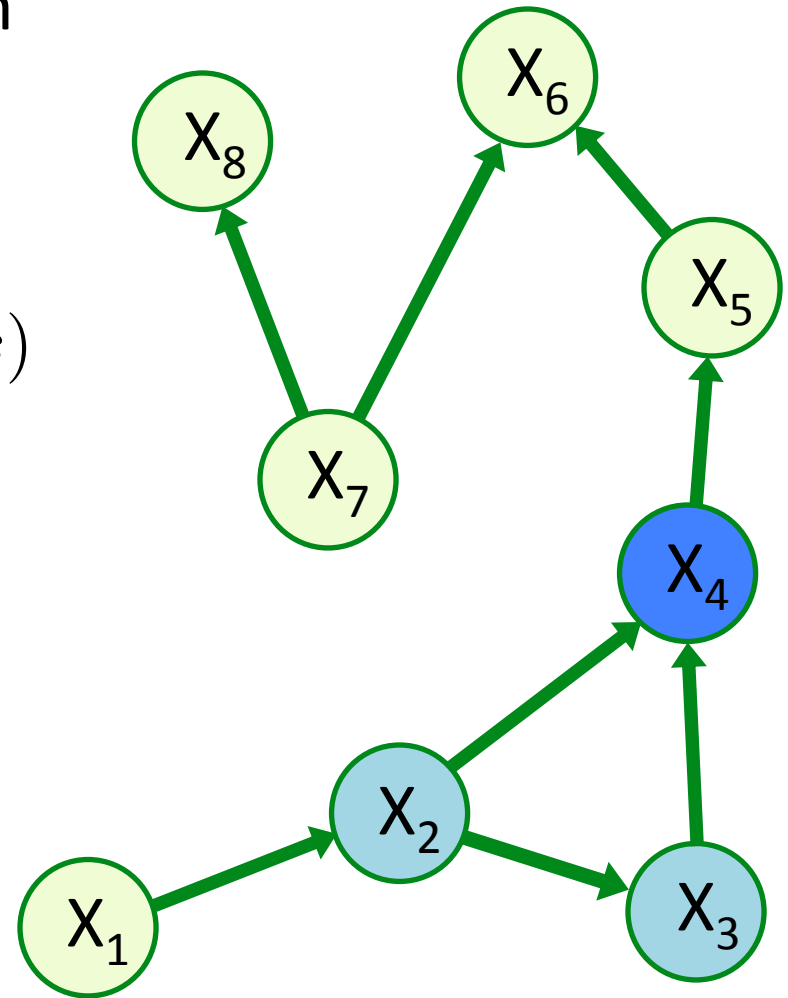
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- Local uncertainty models



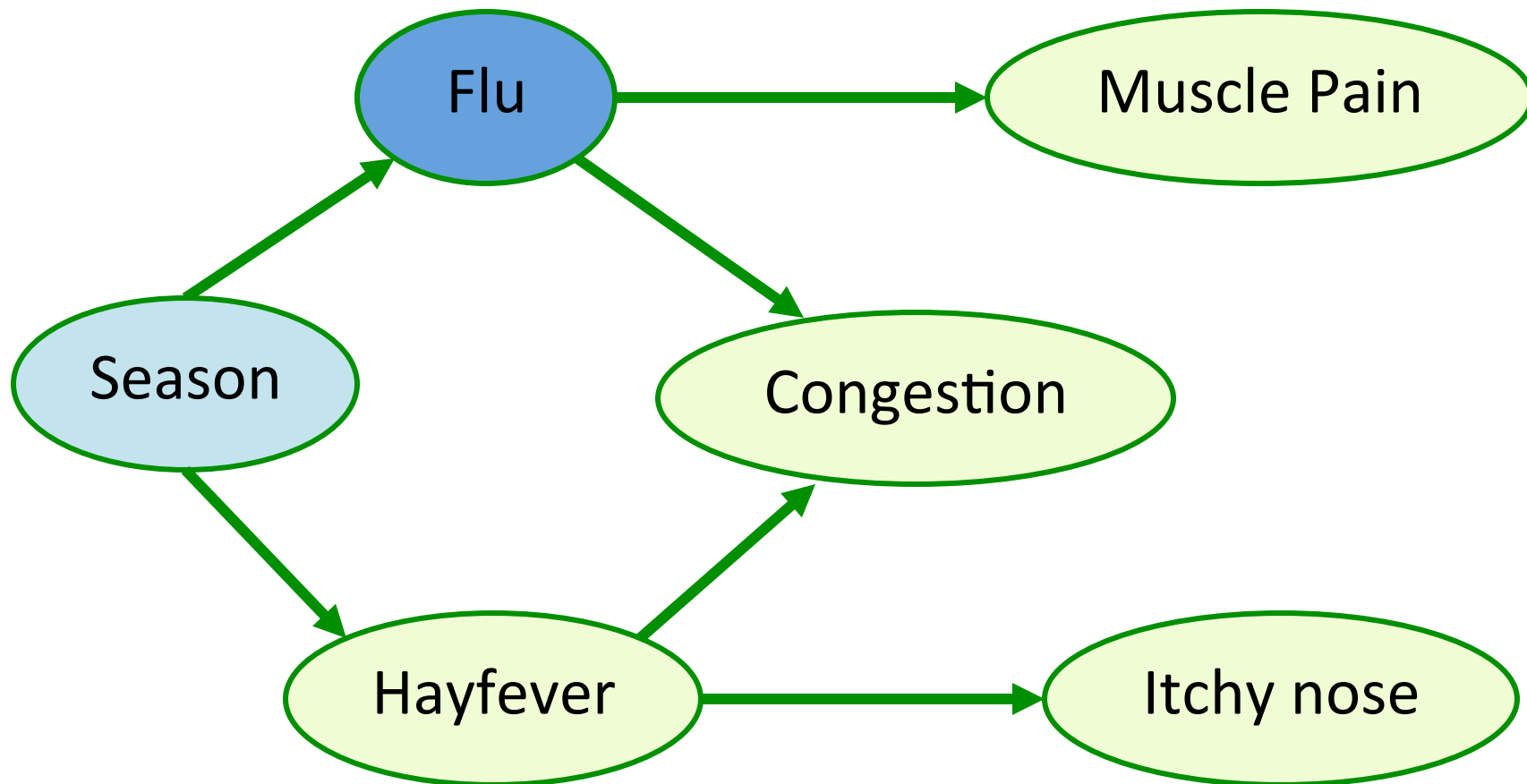
# Bayesian networks: local models

- Variables  $X_s$  take values  $x_s$  in a finite non-empty set  $\mathcal{X}_s$
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- Local uncertainty models:  
mass functions  $p_{s|x_{P(s)}}$

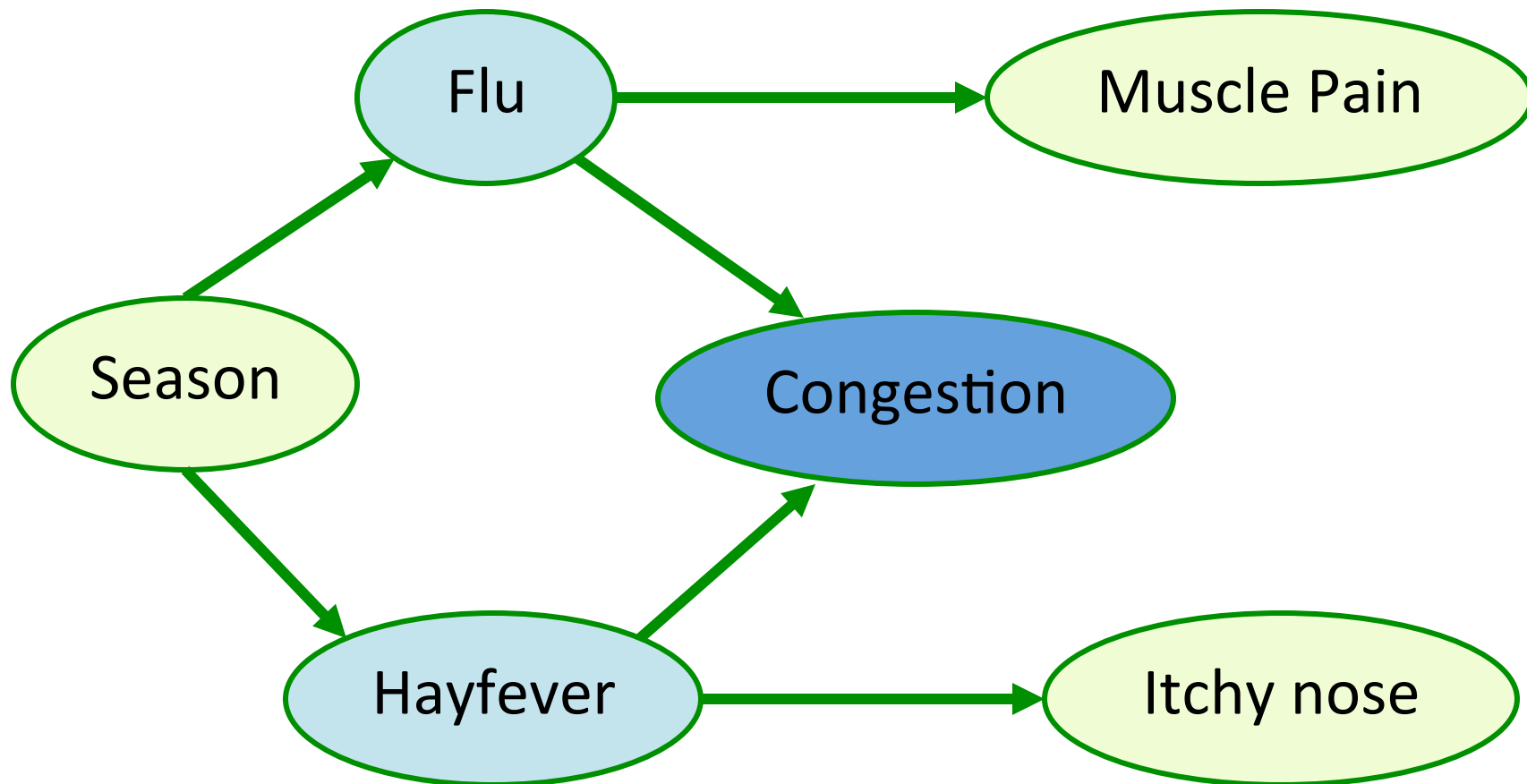
Example:  $p_{4|x_{\{2,3\}}}$



# Bayesian networks: local models

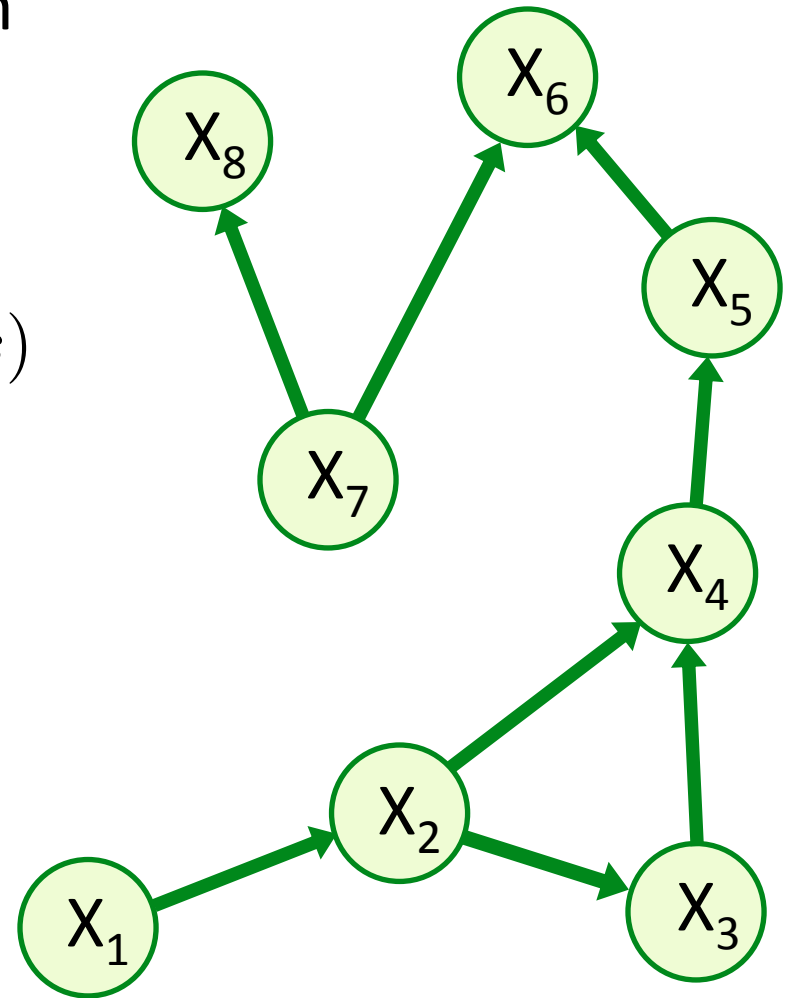


# Bayesian networks: local models



# Bayesian networks: independence

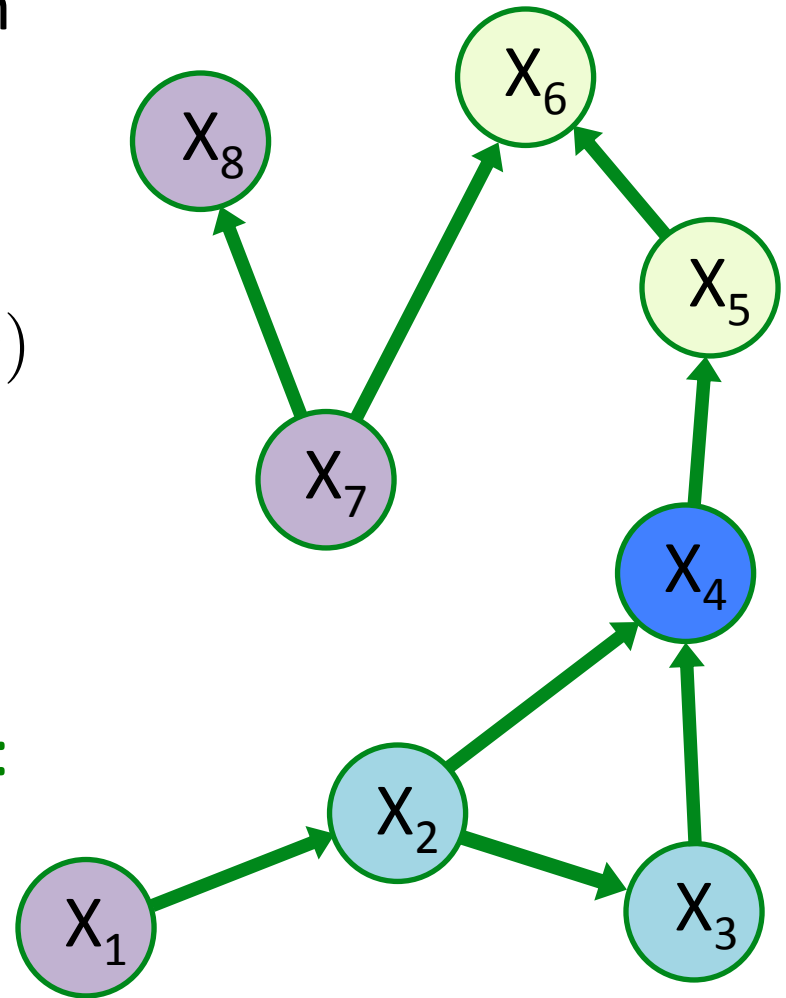
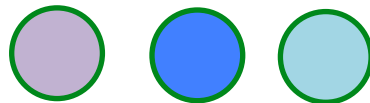
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- Graphical structure: DAG  
 $\Rightarrow \forall s \in G: P(s), D(s), N(s)$
- Local uncertainty models:  
mass functions  $p_{s|x_{P(s)}}$
- Independence assumptions



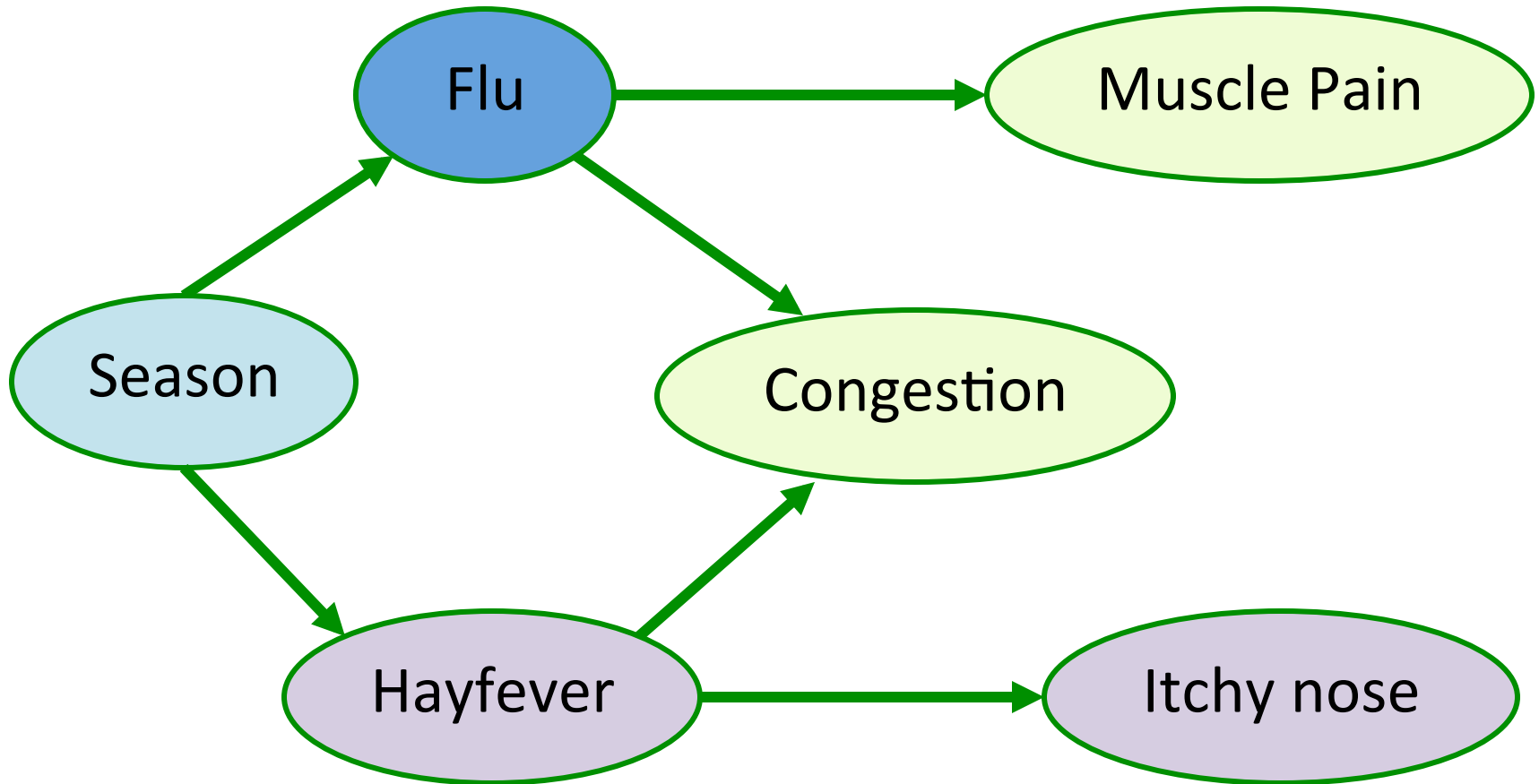


# Bayesian networks: independence

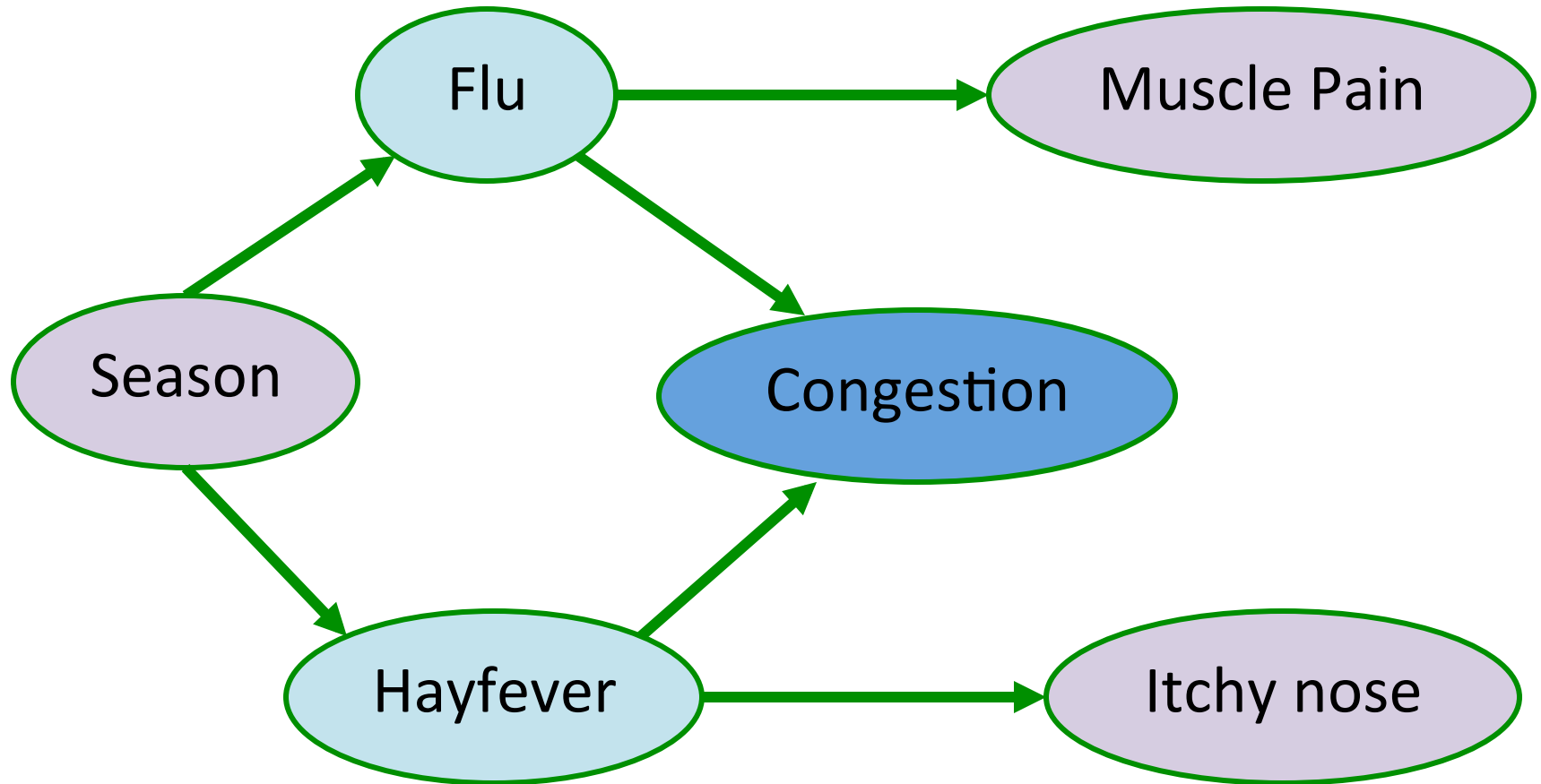
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 $\Rightarrow \forall s \in G: P(s), D(s), N(s)$
- Local uncertainty models:  
mass functions  $p_{s|x} P(s)$
- Independence assumptions:  
 $\forall s \in G: I(N(s), s | P(s))$



# Bayesian networks: independence

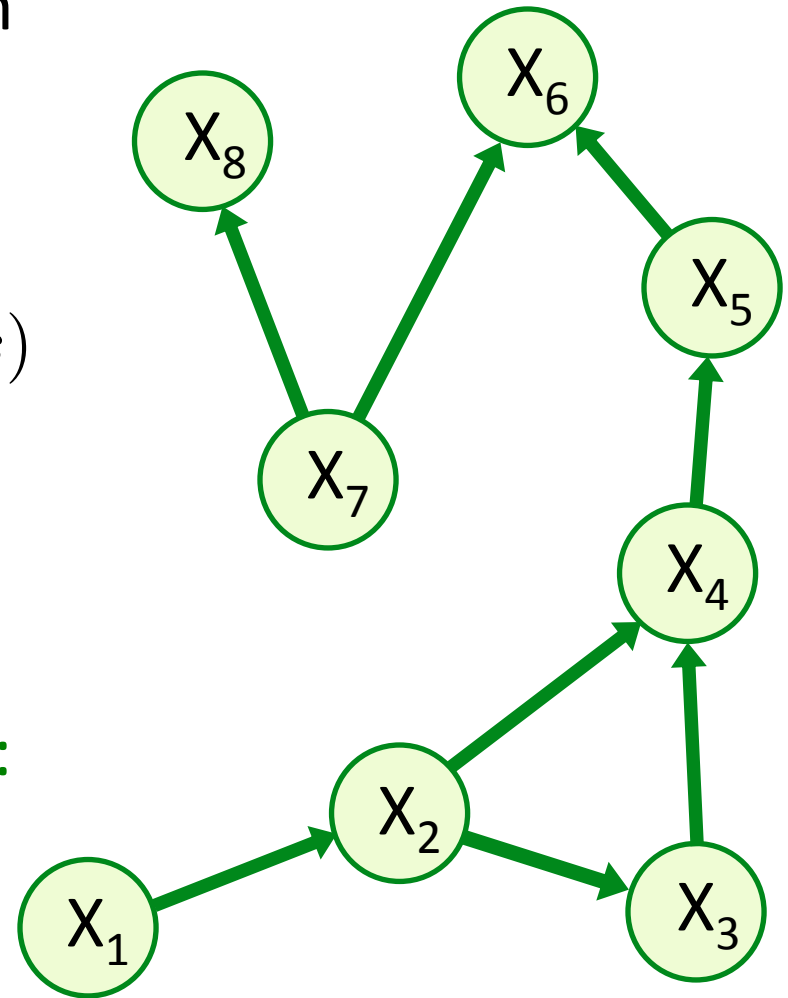


# Bayesian networks: independence



# Bayesian networks: basic setup

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- Graphical structure: DAG  
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- Local uncertainty models:  
mass functions  $p_{s|x} P(s)$
- Independence assumptions:  
 $\forall s \in G: I(N(s), s | P(s))$



# Bayesian networks: the global model

$$p(x_s | x_{P(s)}, x_{N(s)}) = p(x_s | x_{P(s)})$$

Independence assumptions:

$$\forall s \in G: I(N(s), s | P(s))$$

# Bayesian networks: the global model

$$p(x_s | x_{P(s)}, x_{N(s)}) = p(x_s | x_{P(s)}) = p_{s | x_{P(s)}}(x_s)$$

Local uncertainty models:  
mass functions  $p_{s | x_{P(s)}}$

Independence assumptions:

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Local uncertainty models:

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Independence assumptions:

$\forall s \in G: I(N(s), s | P(s))$

# Bayesian networks: the global model

$$p(x_s | x_{P(s)}, x_{N(s)}) = p(x_s | x_{P(s)}) = p_{s | x_{P(s)}}(x_s)$$



$$p(x_G) = \prod_{s \in G} p_{s | x_{P(s)}}(x_s)$$



# Bayesian networks: the global model

$$p(x_G) = \prod_{s \in G} p_{s | x_{P(s)}}(x_s)$$

# Bayesian networks: inference

$$p(x_S) = \sum_{x_{G \setminus S}} p(x_S, x_{G \setminus S})$$

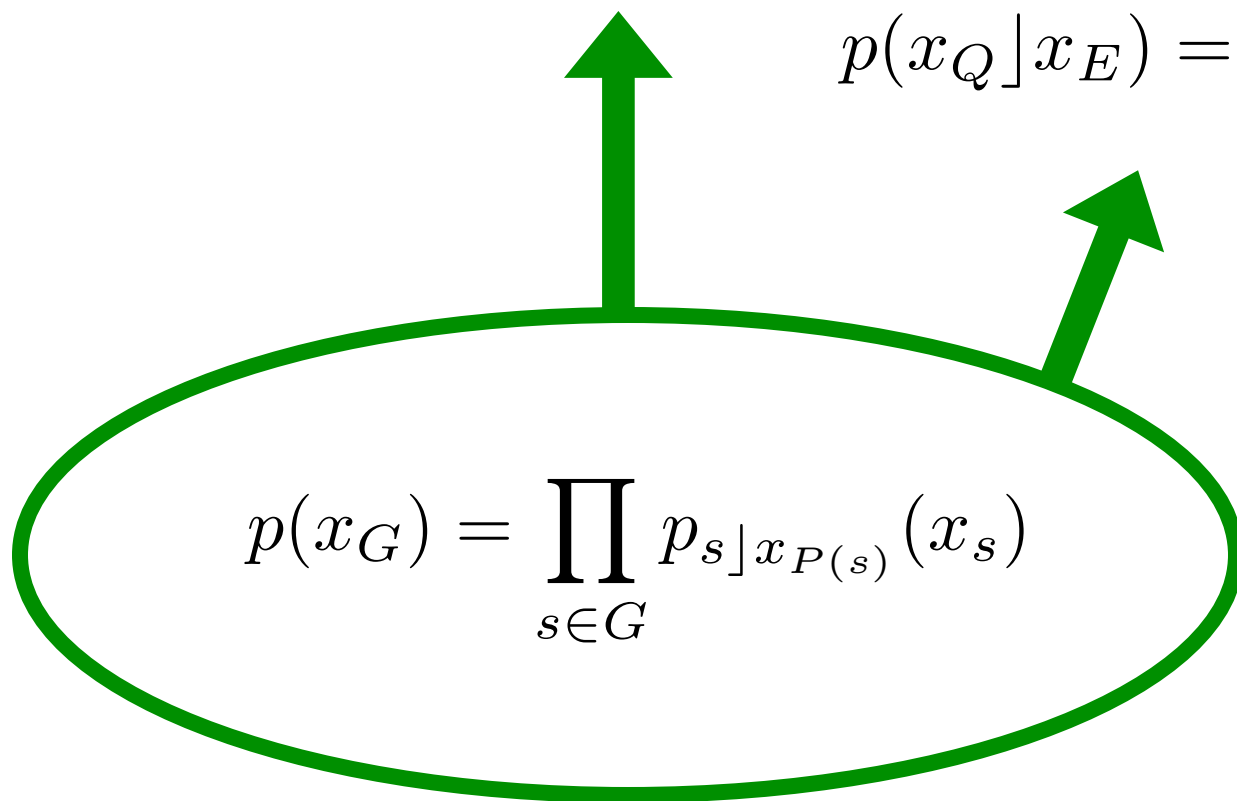


$$p(x_G) = \prod_{s \in G} p_{s | x_{P(s)}}(x_s)$$

# Bayesian networks: inference

$$p(x_S) = \sum_{x_{G \setminus S}} p(x_S, x_{G \setminus S})$$

$$p(x_Q | x_E) = \frac{p(x_{Q \cup E})}{p(x_E)}$$

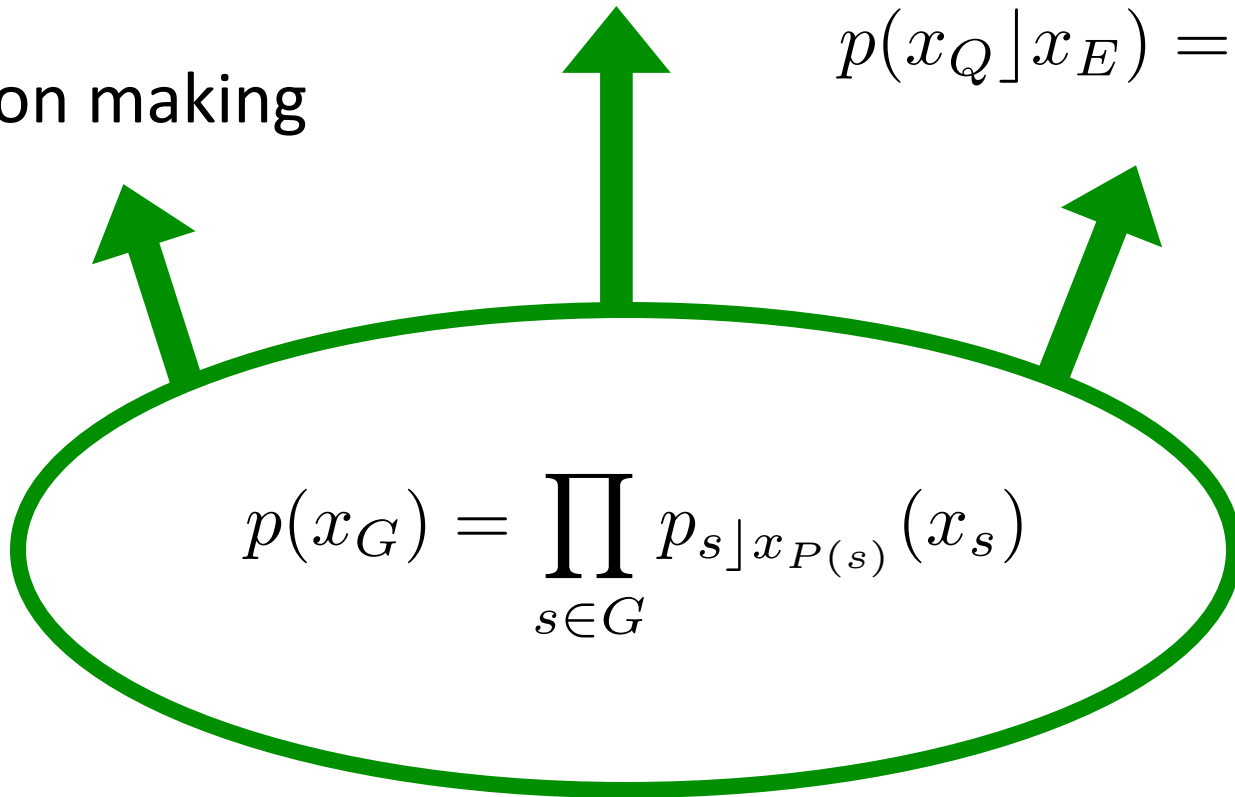


# Bayesian networks: inference

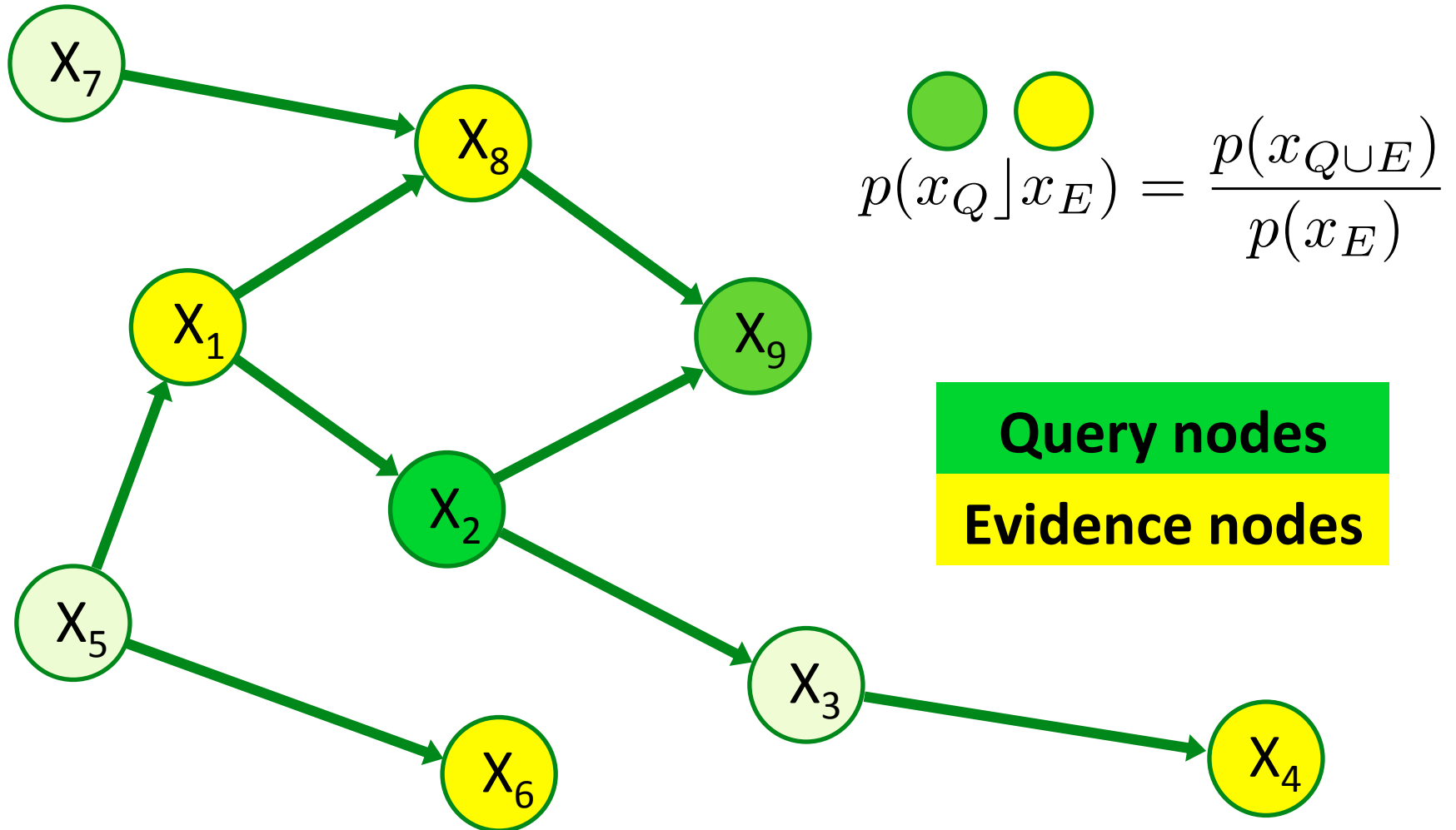
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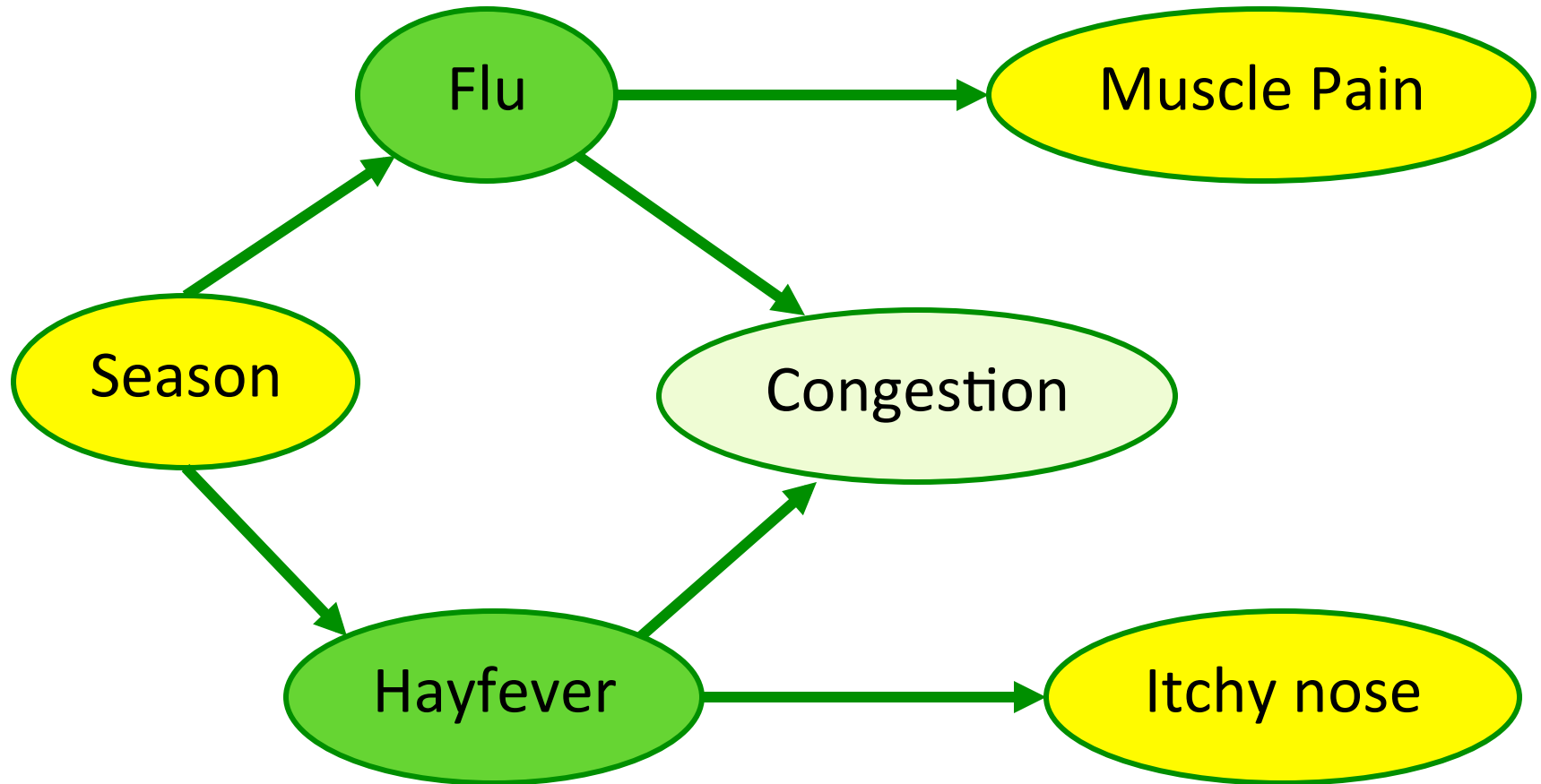
Decision making



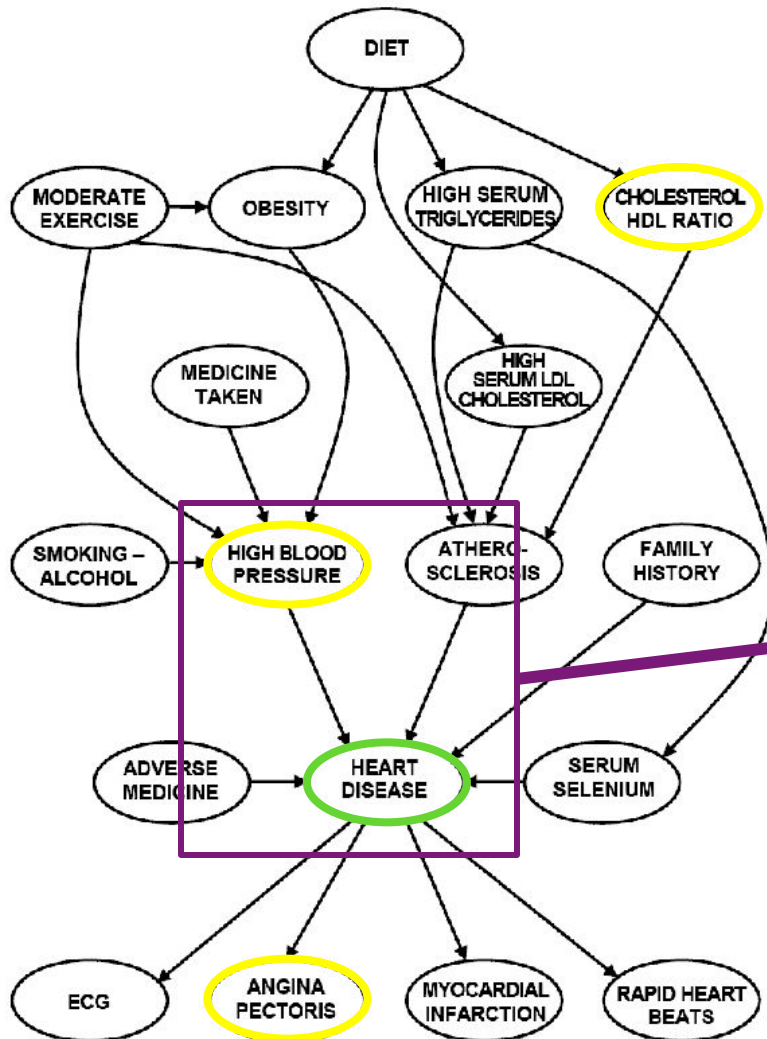
# Bayesian networks: inference



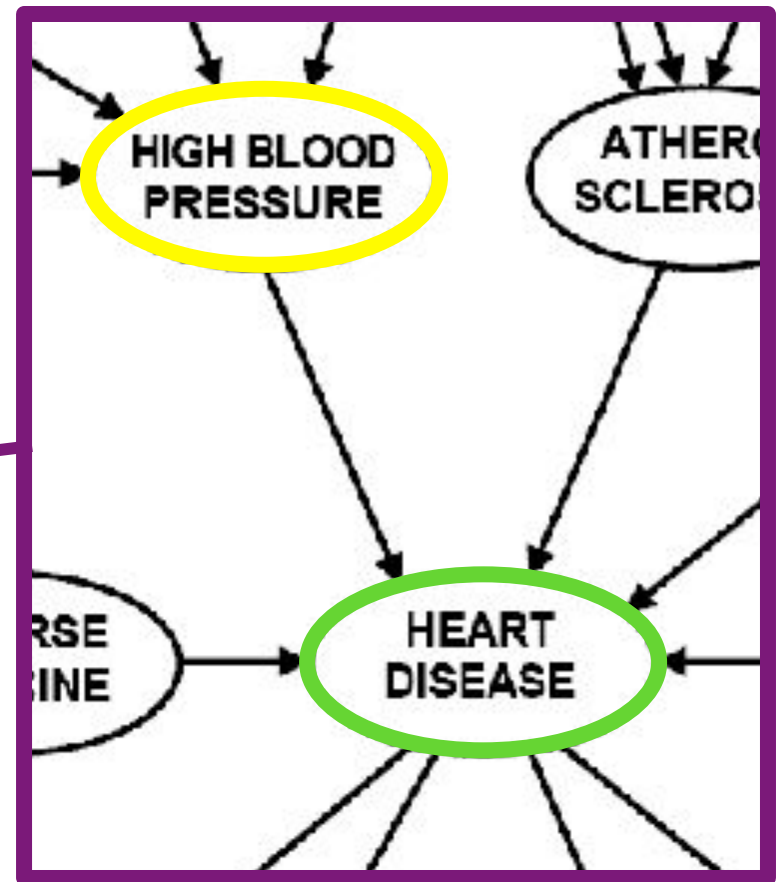
# Bayesian networks: inference



# Bayesian networks: examples

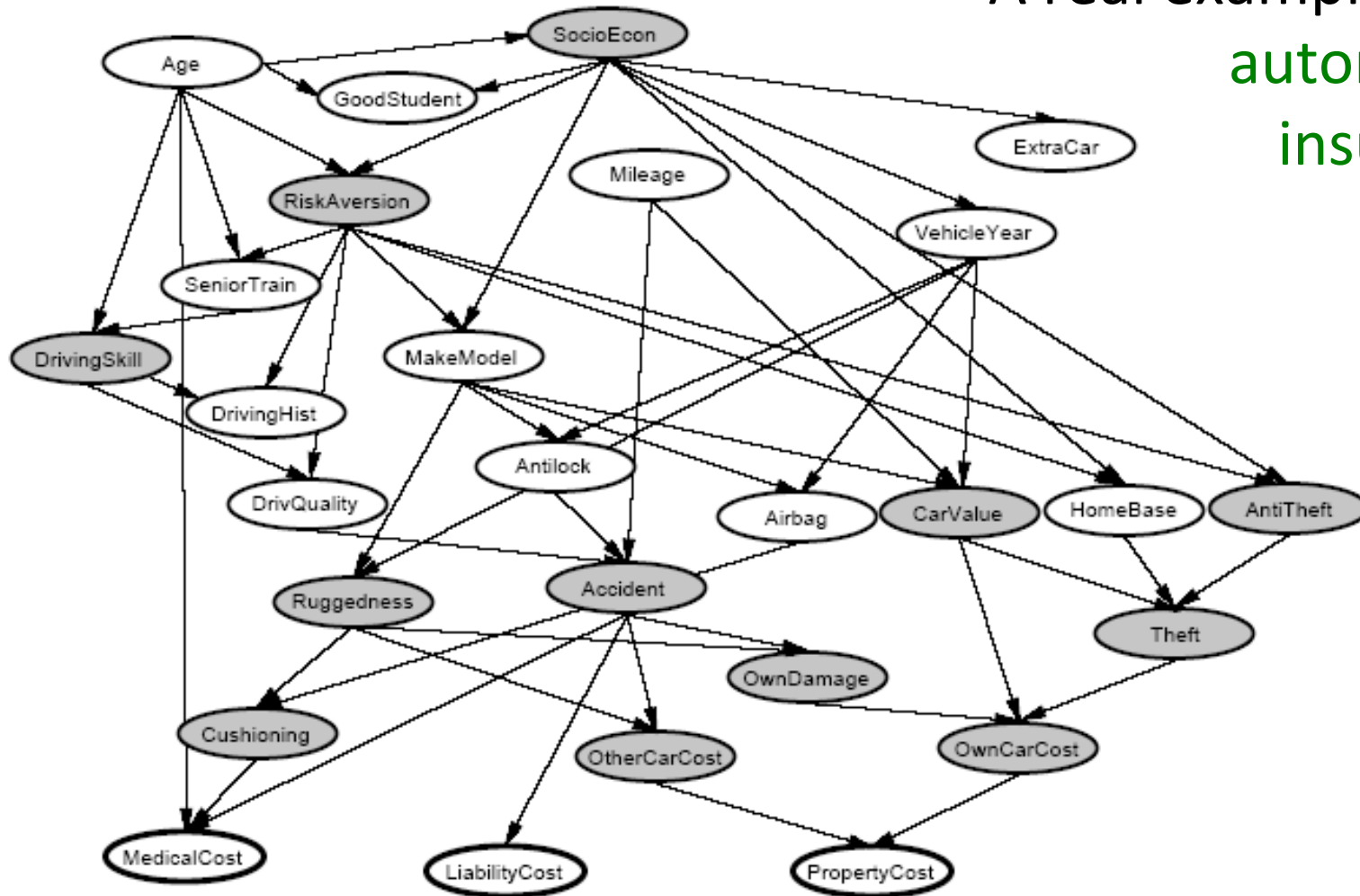


A real **medical** example



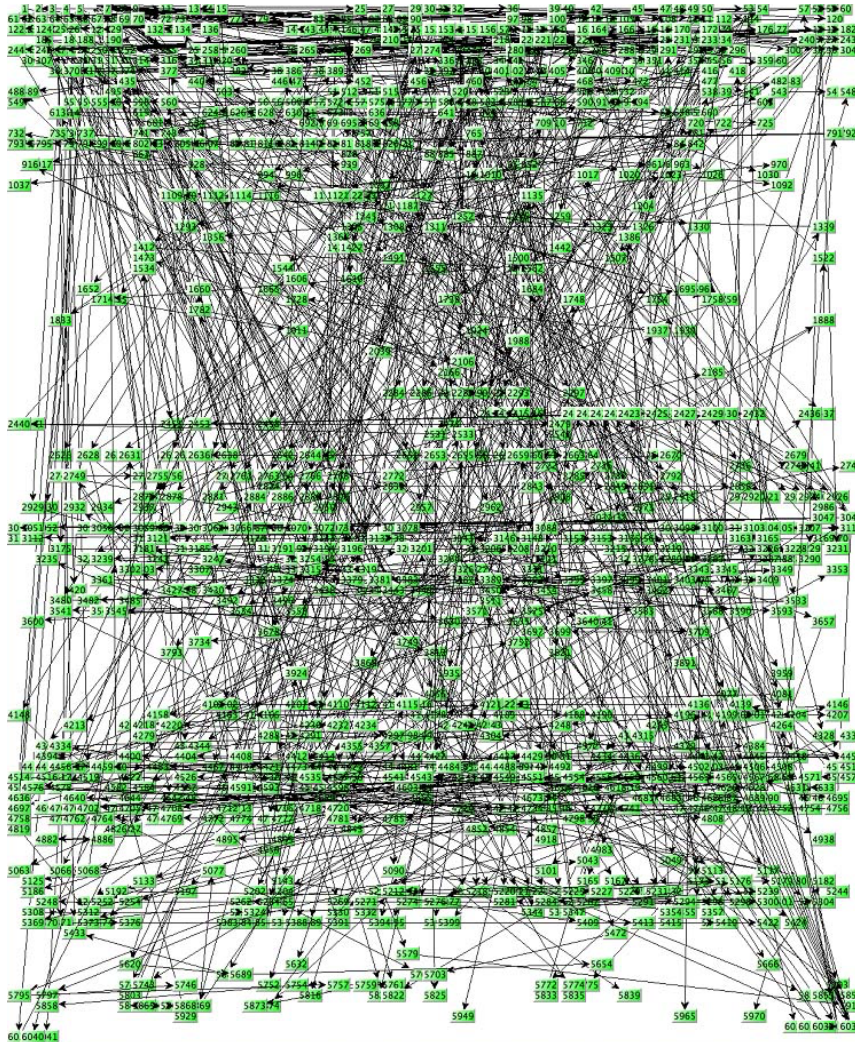
# Bayesian networks: examples

A real example from  
automobile  
insurance





# Bayesian networks: examples

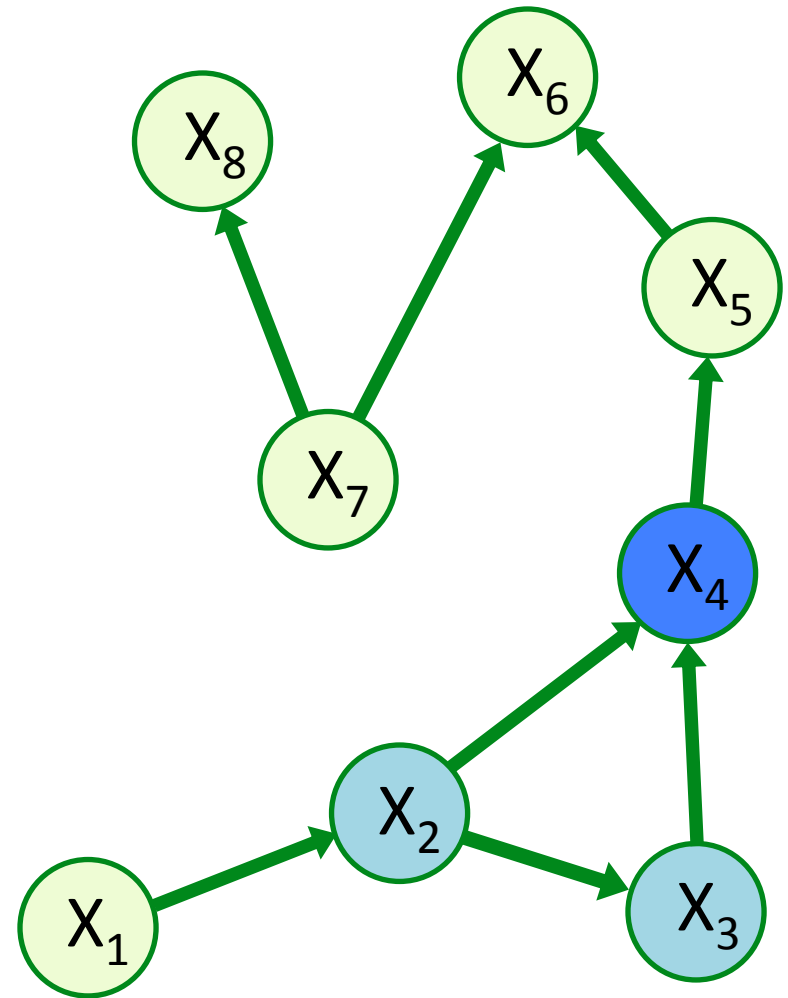


A real example in advertising, used to optimise and individualise the advertisements that are shown on websites

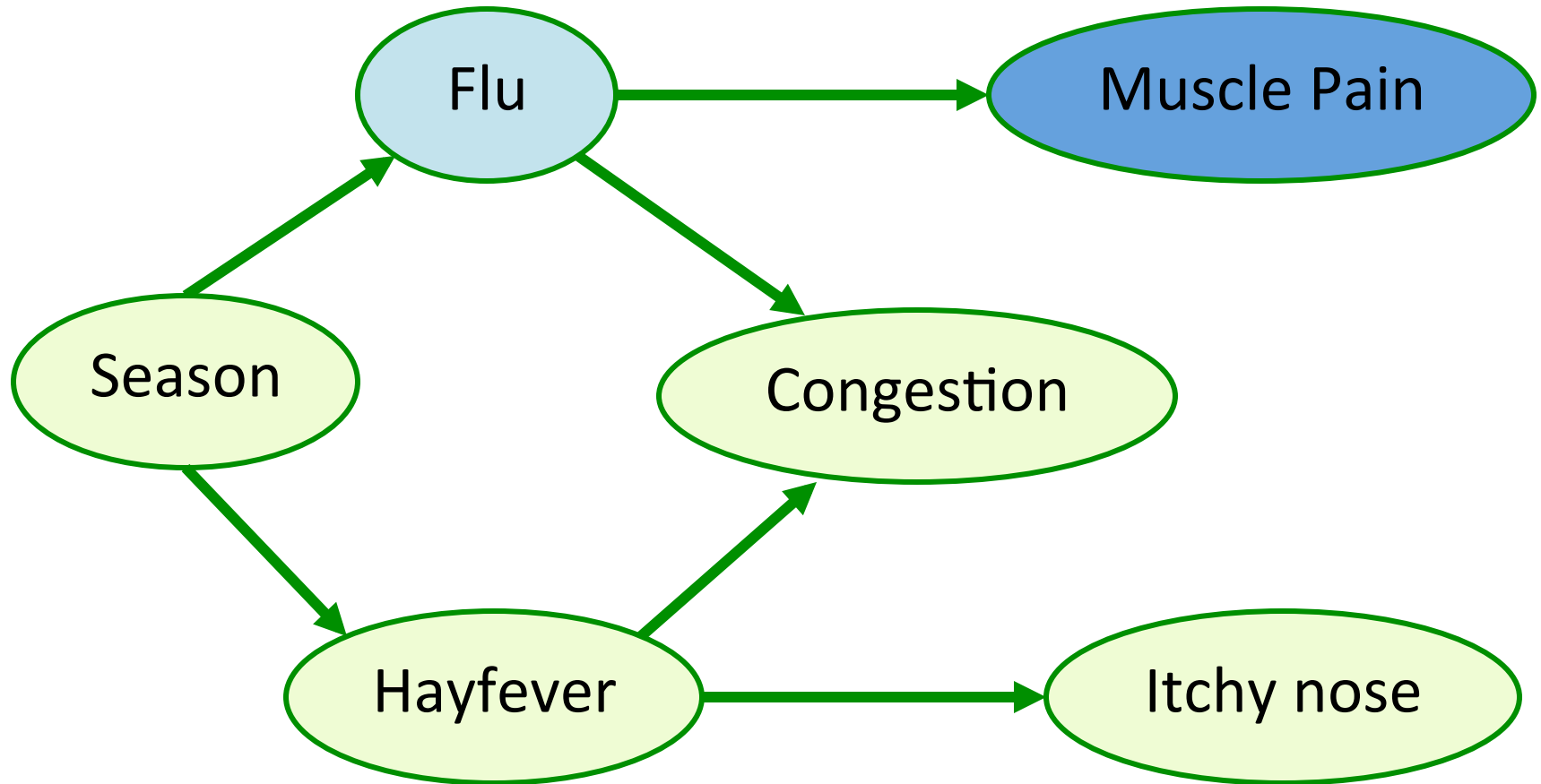
# Bayesian networks: in a perfect world...

What if we don't know them exactly?

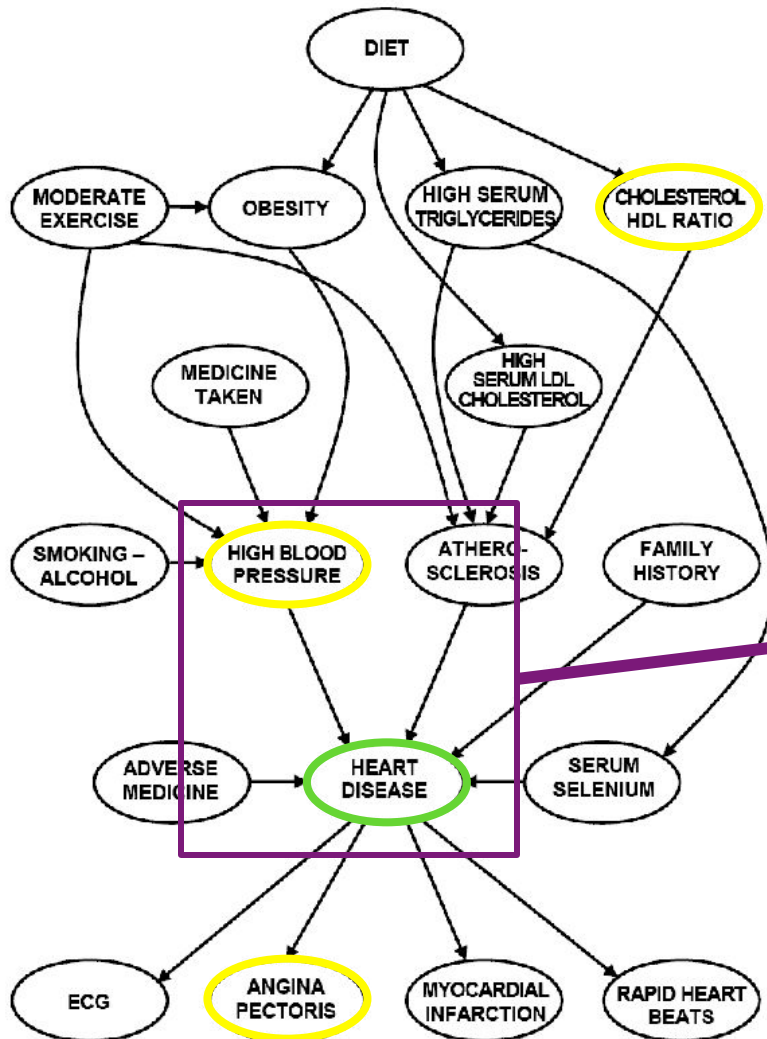
Local uncertainty models:  
mass functions  $p_{s|x_{P(s)}}$



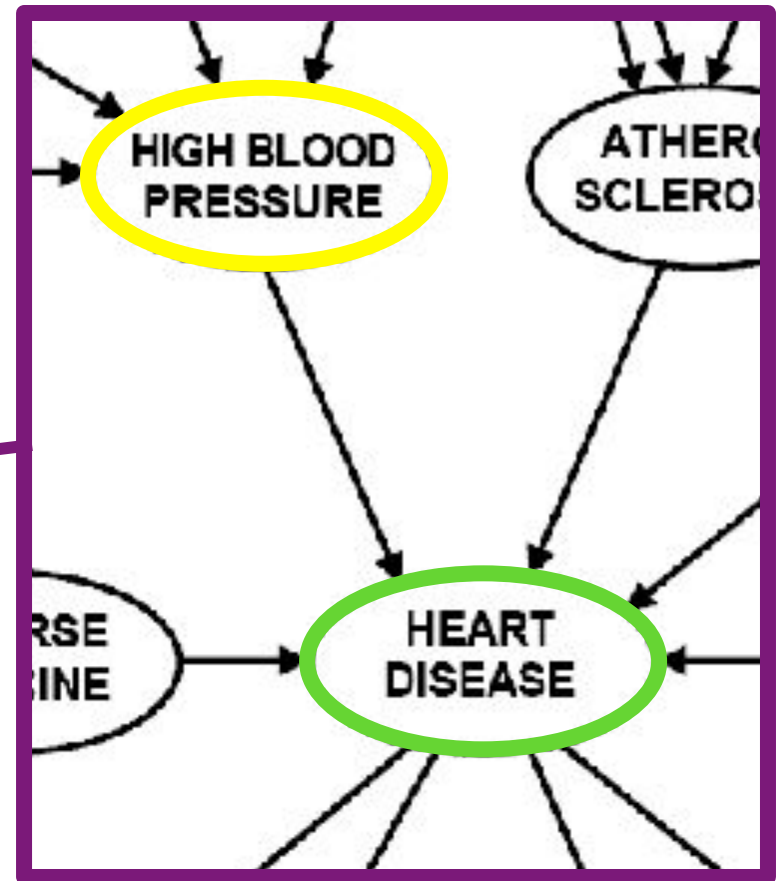
# Bayesian networks: in a perfect world...



# Bayesian networks: in a perfect world...



A real **medical** example



**Bayesian**

~~Credal~~ **Networks** under  
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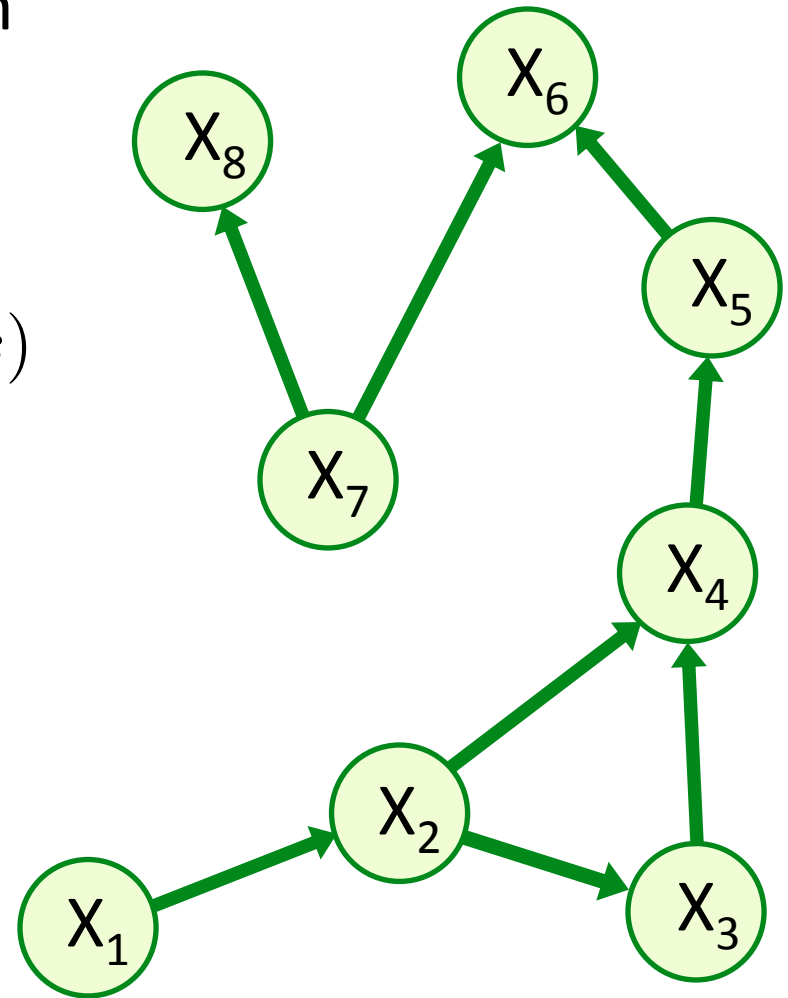
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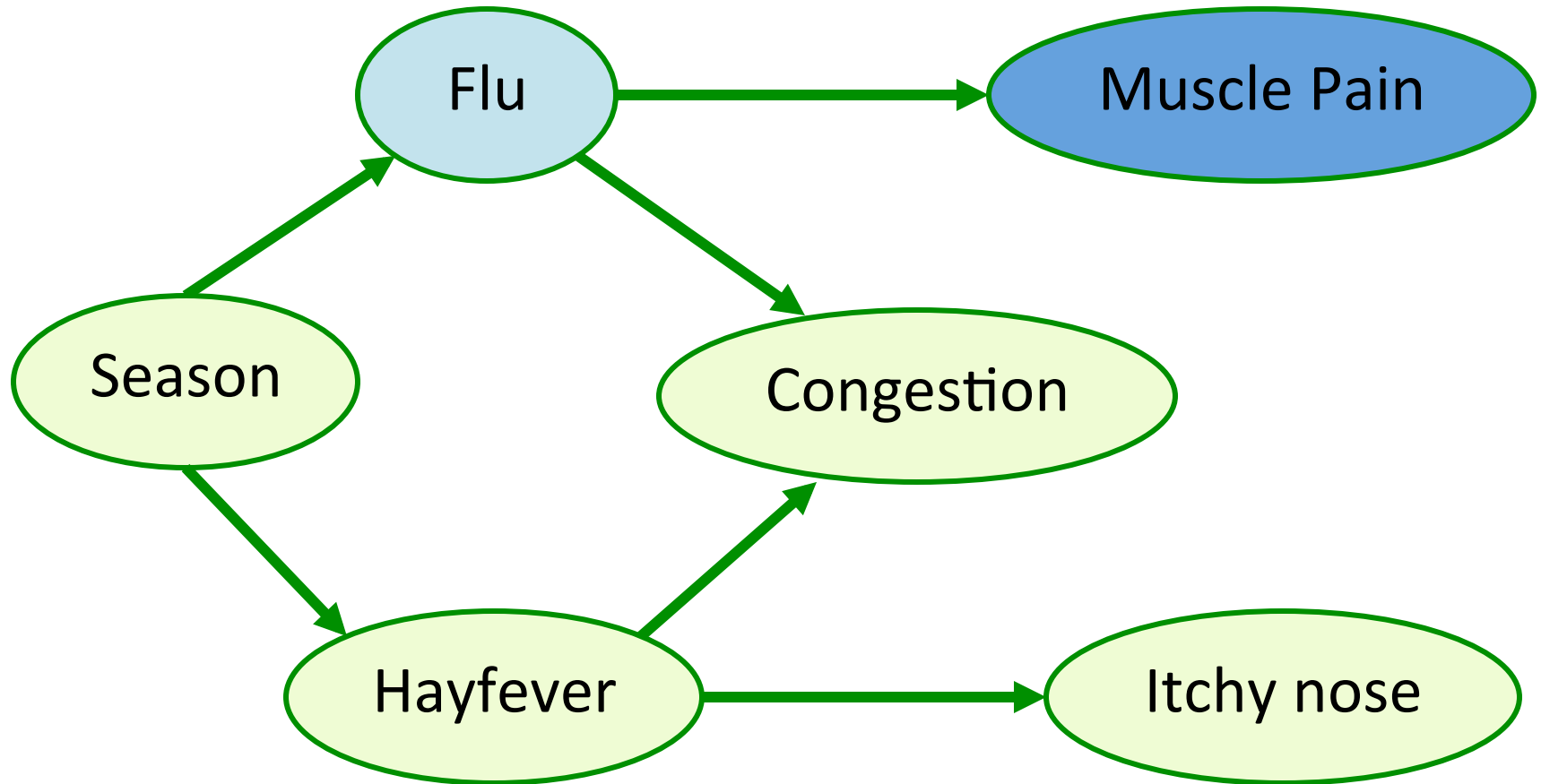
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# Credal networks: credal sets

- Variables  $X_s$  take values  $x_s$  in a finite non-empty set  $\mathcal{X}_s$
- Graphical structure: DAG  
 $\Rightarrow \forall s \in G: P(s), D(s), N(s)$
- Local uncertainty models:  
credal sets  $\mathcal{F}_{s|x_{P(s)}}$



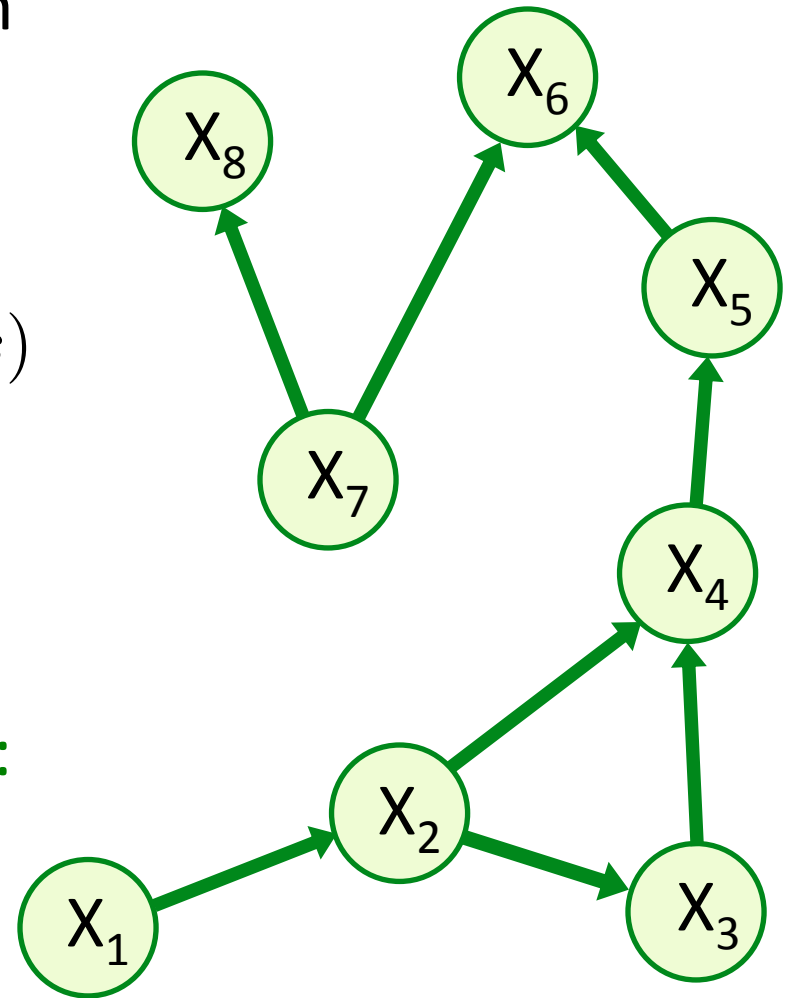
# Credal networks: credal sets





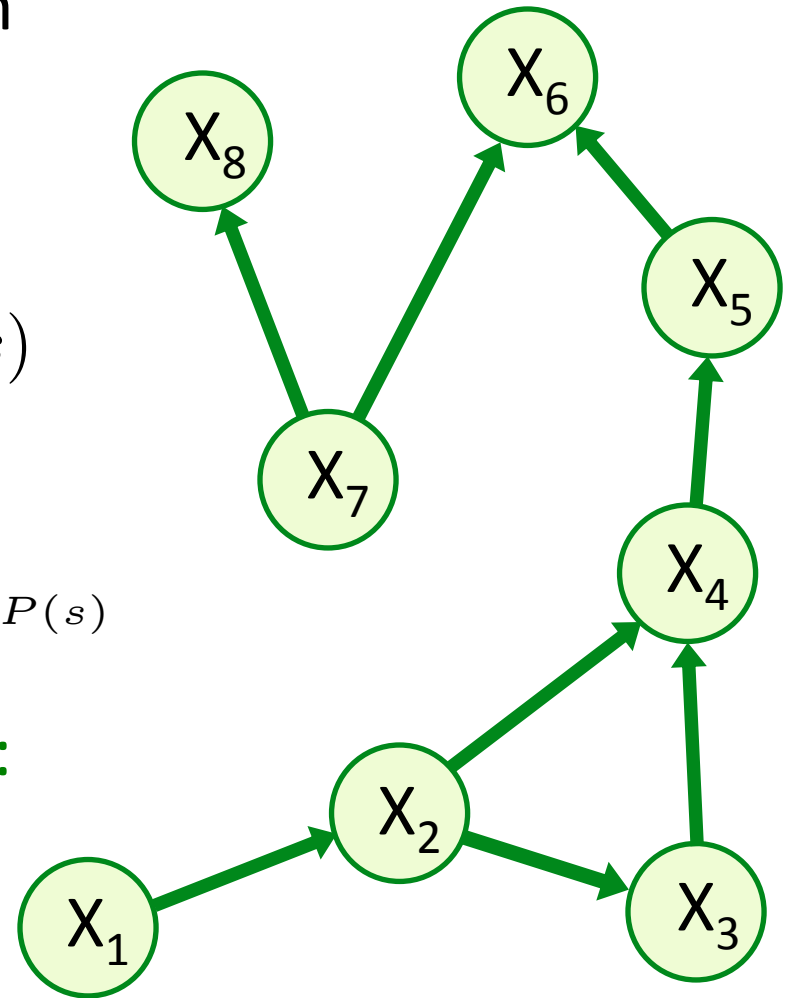
# Credal networks: independence?

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- Independence assumptions:



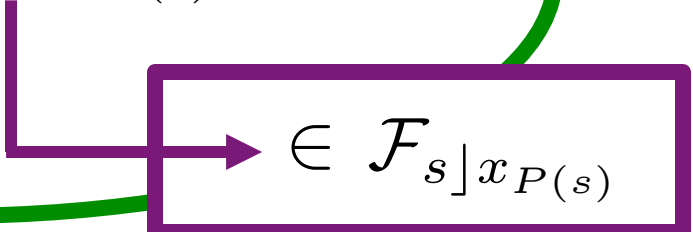
# Credal networks: complete independence

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- Graphical structure: DAG  
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- Local uncertainty models:  
credal sets  $\mathcal{F}_{s|x_{P(s)}} \ni \mathcal{P}_{s|x_{P(s)}}$
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# Credal networks: complete independence

$$p(x_G) = \prod_{s \in G} p_{s | x_{P(s)}}(x_s)$$

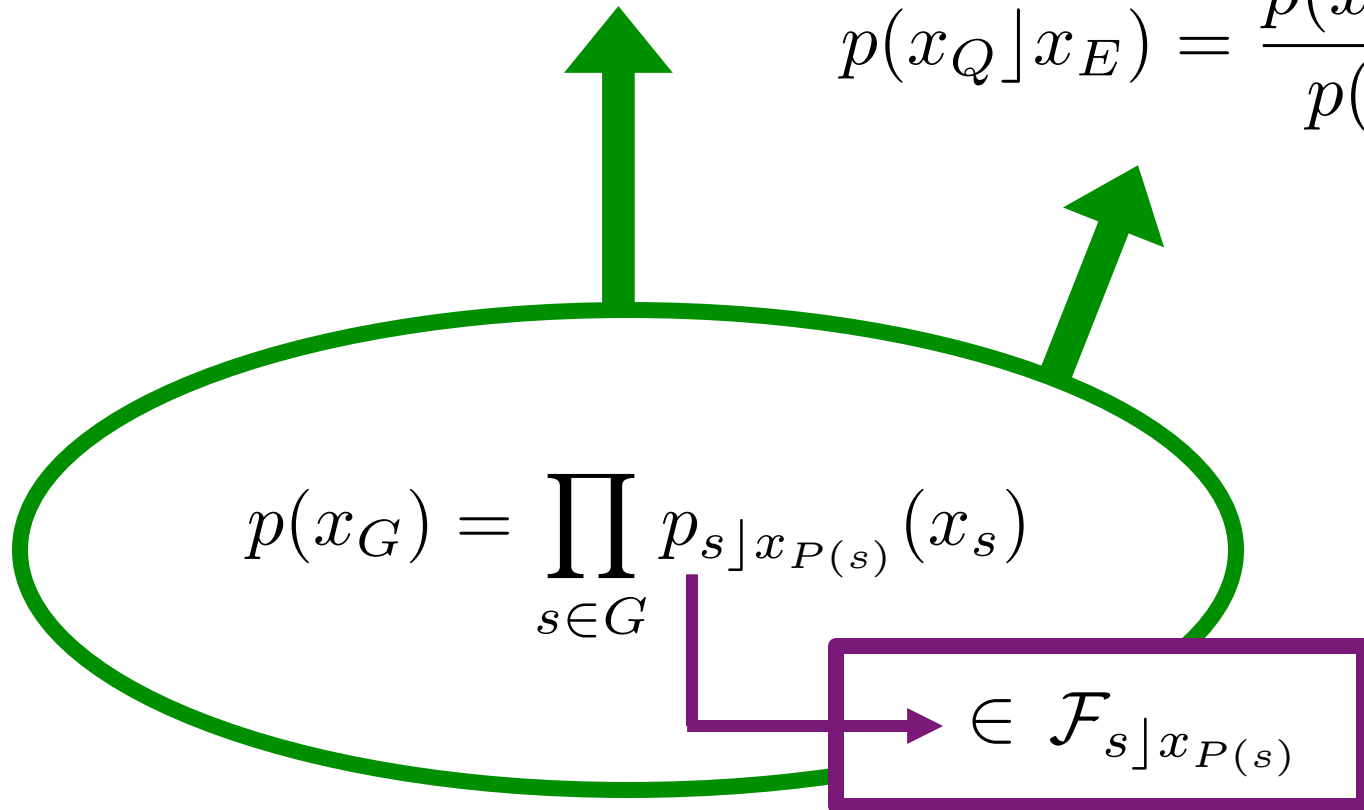

$$\in \mathcal{F}_{s | x_{P(s)}}$$

# Credal networks: inference

$$p(x_S) = \sum_{x_{G \setminus S}} p(x_S, x_{G \setminus S})$$

?

$$p(x_Q | x_E) = \frac{p(x_{Q \cup E})}{p(x_E)}$$



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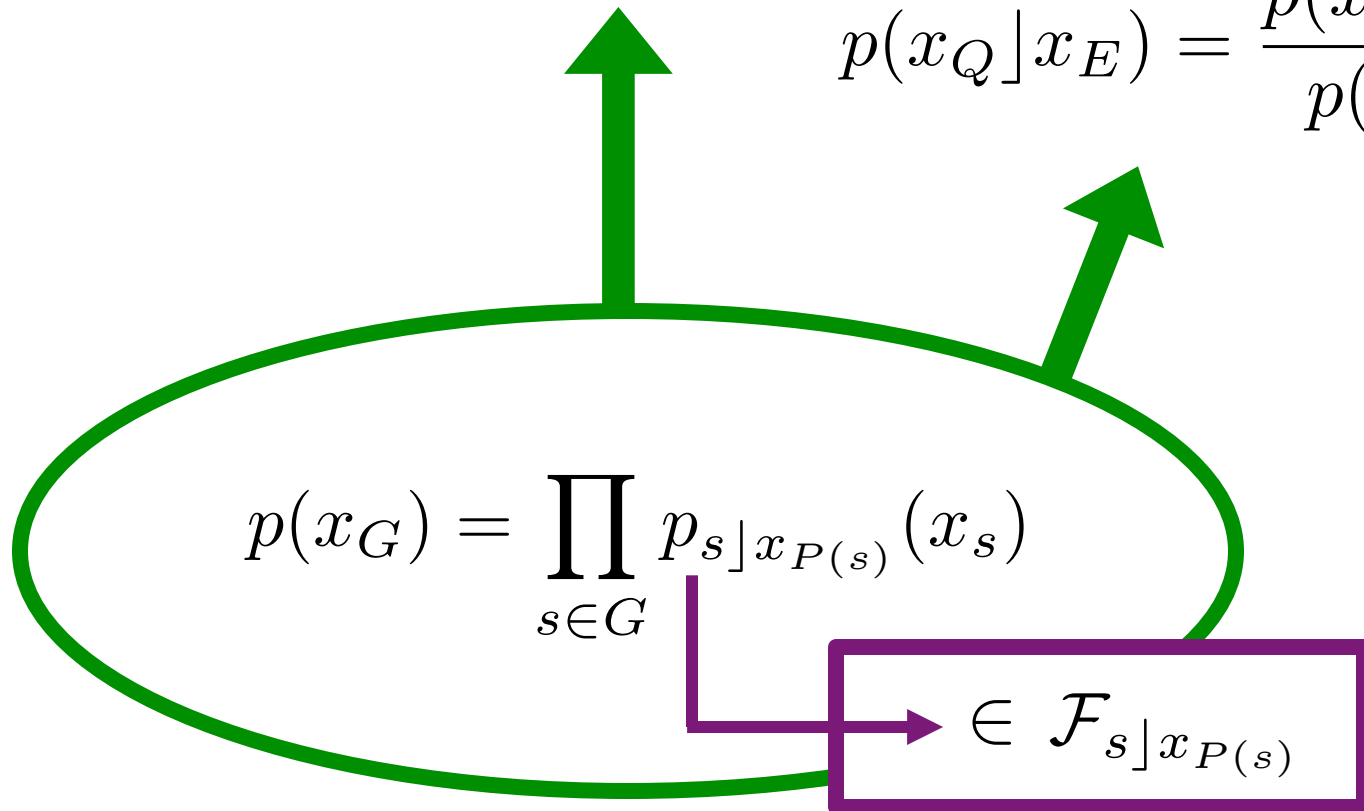
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# Credal networks: inference

$$p(x_S) = \sum_{x_{G \setminus S}} p(x_S, x_{G \setminus S})$$

Lower and upper bounds!

$$p(x_Q | x_E) = \frac{p(x_{Q \cup E})}{p(x_E)}$$



# Credal networks: inference

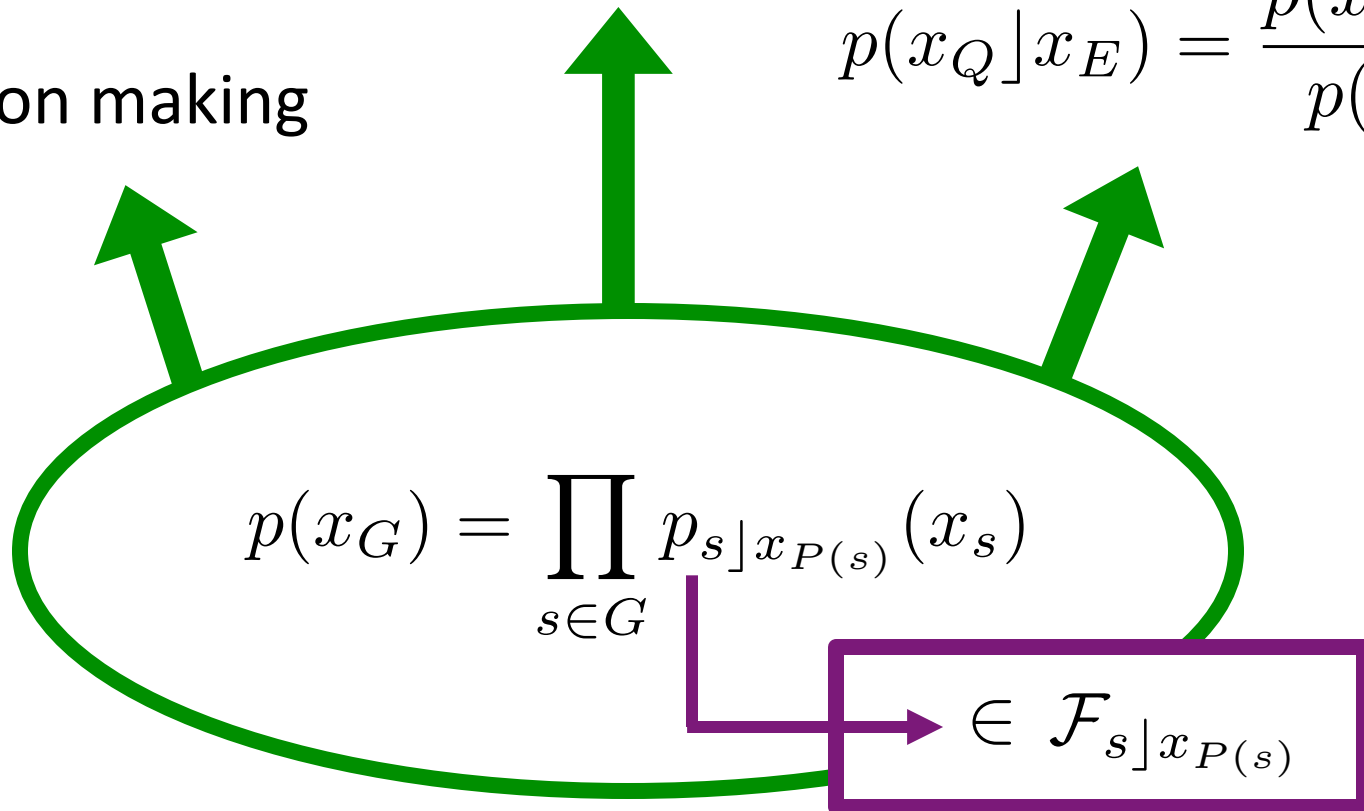
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Lower and upper bounds!

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Decision making

?



# Credal networks: inference

$$p(x_S) = \sum_{x_{G \setminus S}} p(x_S, x_{G \setminus S})$$

Lower and upper bounds!

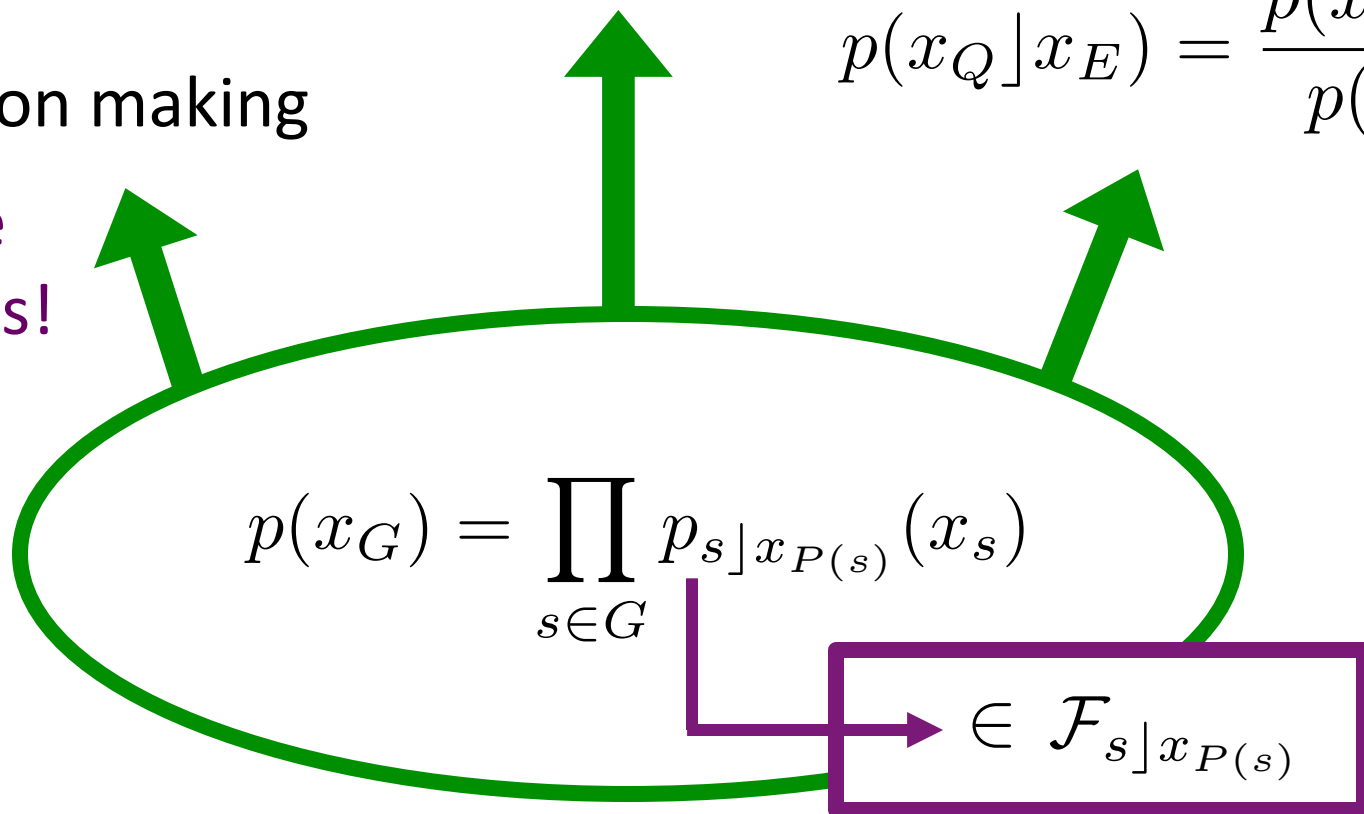
$$p(x_Q | x_E) = \frac{p(x_{Q \cup E})}{p(x_E)}$$

Decision making

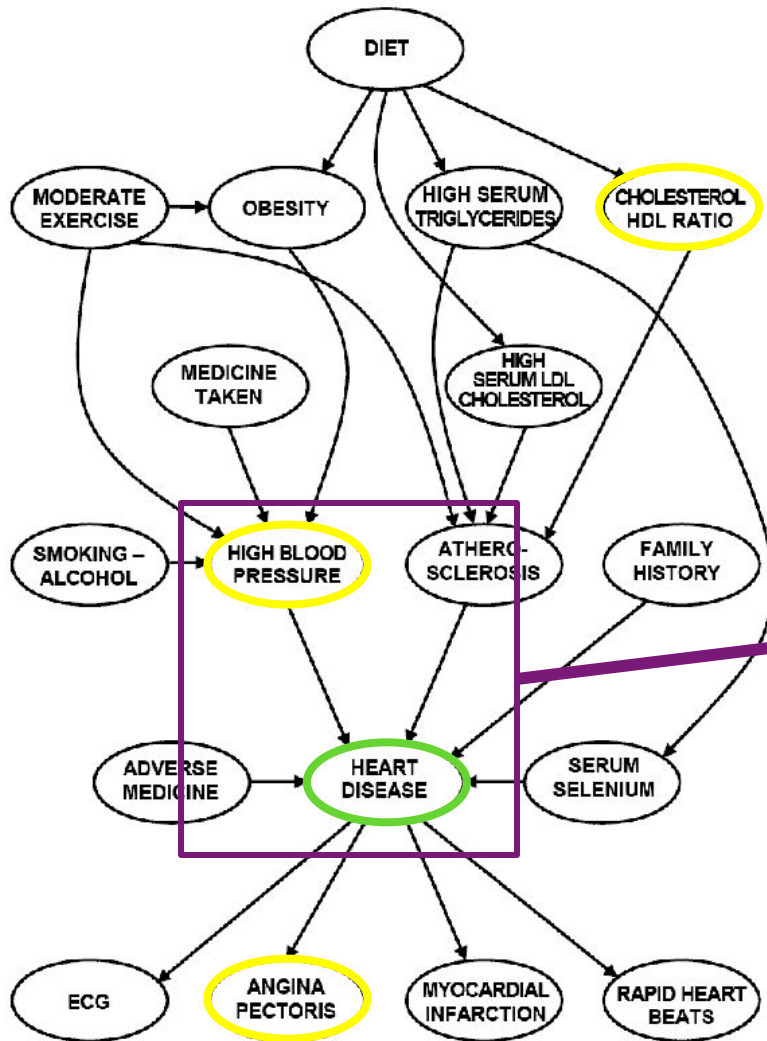
Multiple decisions!

$$p(x_G) = \prod_{s \in G} p_{s | x_{P(s)}}(x_s)$$

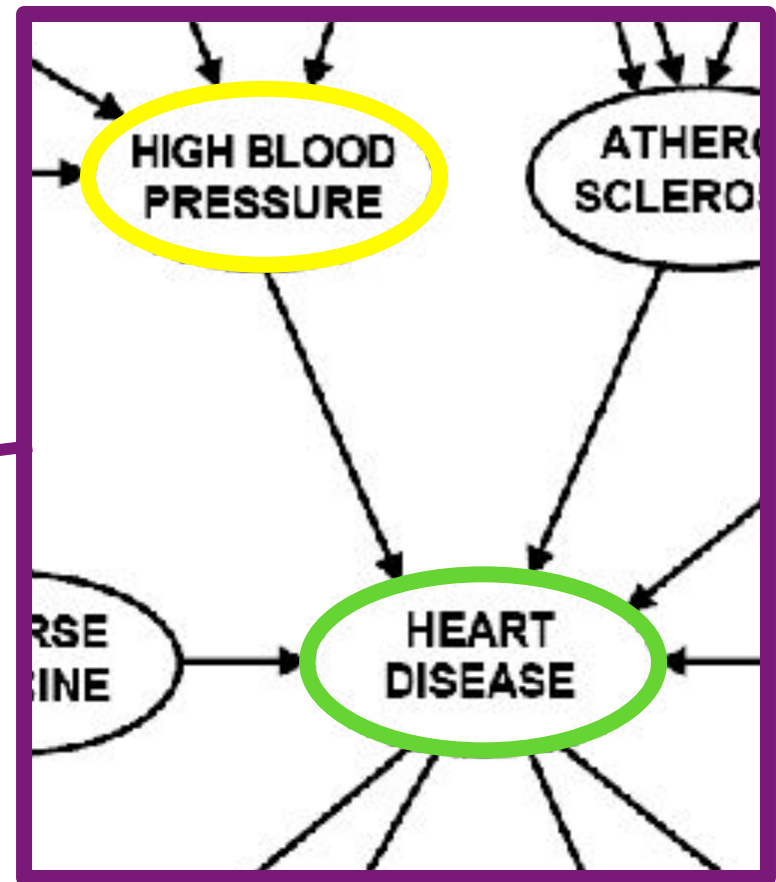
$$\in \mathcal{F}_{s | x_{P(s)}}$$



# Credal networks: inference



A real **medical** example





# Credal networks: in a perfect world...

Are you sure they  
are completely  
independent?

Maybe they are almost  
independent?

What does  
'almost'  
mean?



Independence assumptions:

$$\forall s \in G: I(N(s), s | P(s))$$

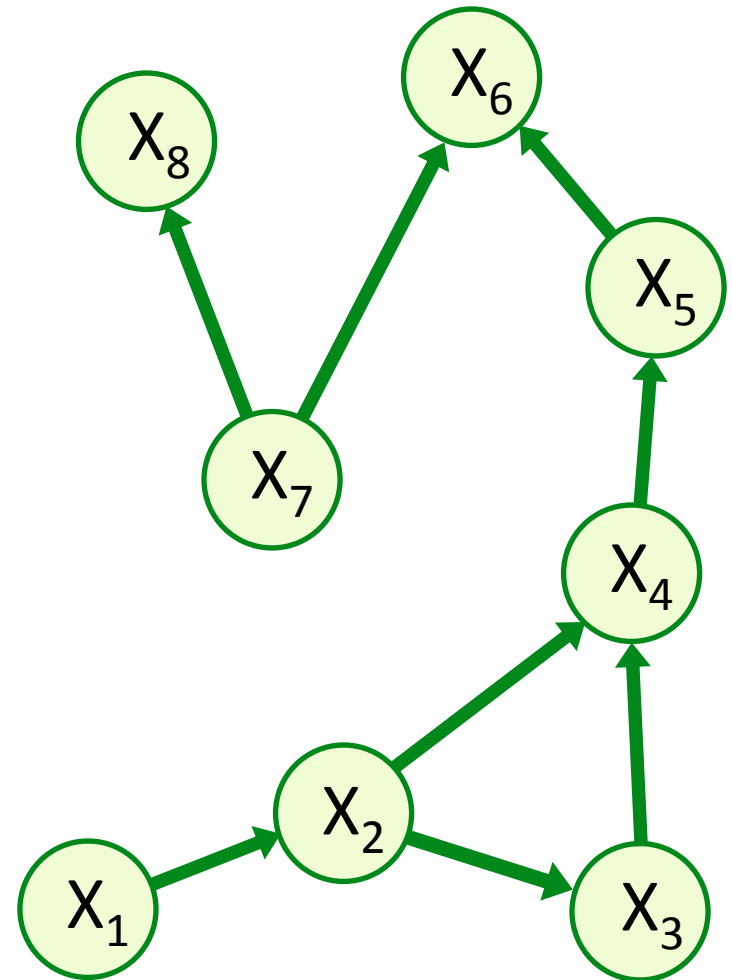
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# Credal networks: epistemic irrelevance

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- Graphical structure: DAG  
 $\Rightarrow \forall s \in G: P(s), D(s), N(s)$
- Local uncertainty models:  
credal sets  $\mathcal{F}_{s|x_{P(s)}}$
- Epistemic irrelevance:  
 $\forall s \in G: IR(N(s), s | P(s))$



# Credal networks: epistemic irrelevance

$$\mathcal{F}_s \upharpoonright x_{P(s)}, x_{N(s)} = \mathcal{F}_s \upharpoonright x_{P(s)}$$

Epistemic irrelevance:

$$\forall s \in G: IR(N(s), s | P(s))$$

# Credal networks: epistemic irrelevance

$$\mathcal{F}_s \downarrow x_{P(s)}, x_{N(s)} = \mathcal{F}_s \downarrow x_{P(s)}$$

$$p(x_s \downarrow x_{P(s)}, x_{N(s)}) \not\equiv p(x_s \downarrow x_{P(s)})$$

Almost independence!

# Credal networks: epistemic irrelevance

$$\begin{array}{ccc} \mathcal{F}_s \upharpoonright x_{P(s)}, x_{N(s)} & = & \mathcal{F}_s \upharpoonright x_{P(s)} \\ \cup & & \cup \\ p(x_s \upharpoonright x_{P(s)}, x_{N(s)}) & \not= & p(x_s \upharpoonright x_{P(s)}) \end{array}$$

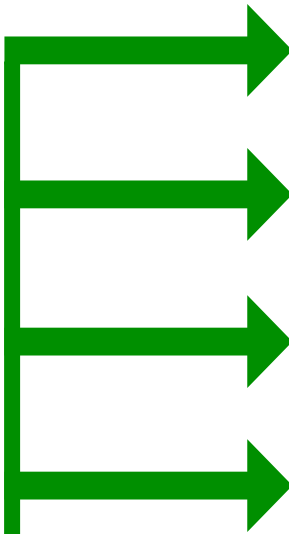
Almost independence!

What is  
probability?

What is  
independence?

What is  
uncertainty?

# Credal networks: epistemic irrelevance


$$\begin{aligned}\mathcal{M}_s \mid x_{P(s)}, x_{N(s)} &= \mathcal{M}_s \mid x_{P(s)} \\ \mathcal{F}_s \mid x_{P(s)}, x_{N(s)} &= \mathcal{F}_s \mid x_{P(s)} \\ \mathcal{D}_s \mid x_{P(s)}, x_{N(s)} &= \mathcal{D}_s \mid x_{P(s)} \\ \underline{P}_s \mid x_{P(s)}, x_{N(s)} &= \underline{P}_s \mid x_{P(s)}\end{aligned}$$

What is  
probability?

What is  
independence?

What is  
uncertainty?

# Credal networks: epistemic irrelevance

$$\mathcal{F}_s \downarrow x_{P(s)}, x_{N(s)} = \mathcal{F}_s \downarrow x_{P(s)}$$

$$p(x_s \downarrow x_{P(s)}, x_{N(s)}) \not\equiv p(x_s \downarrow x_{P(s)})$$

Almost independence!



# Credal networks: the global model?

$$\mathcal{F}_s \mid x_{P(s)}, x_{N(s)} = \mathcal{F}_s \mid x_{P(s)}$$

$$\cup$$
$$\cup$$

$$p(x_s \mid x_{P(s)}, x_{N(s)}) \not\equiv p(x_s \mid x_{P(s)})$$

**?**  $p(x_G) = \prod_{s \in G} p_{s \mid x_{P(s)}}(x_s)$

$\in \mathcal{F}_s \mid x_{P(s)}$

# The irrelevant natural extension

$$\begin{array}{ccc} \mathcal{F}_{s \mid x_{P(s)}, x_{N(s)}} & = & \mathcal{F}_{s \mid x_{P(s)}} \\ \cup & & \cup \\ p(x_s \mid x_{P(s)}, x_{N(s)}) & \not= & p(x_s \mid x_{P(s)}) \end{array}$$

the set of all global probability mass functions that are compatible with our assessments

# The irrelevant natural extension: inference?

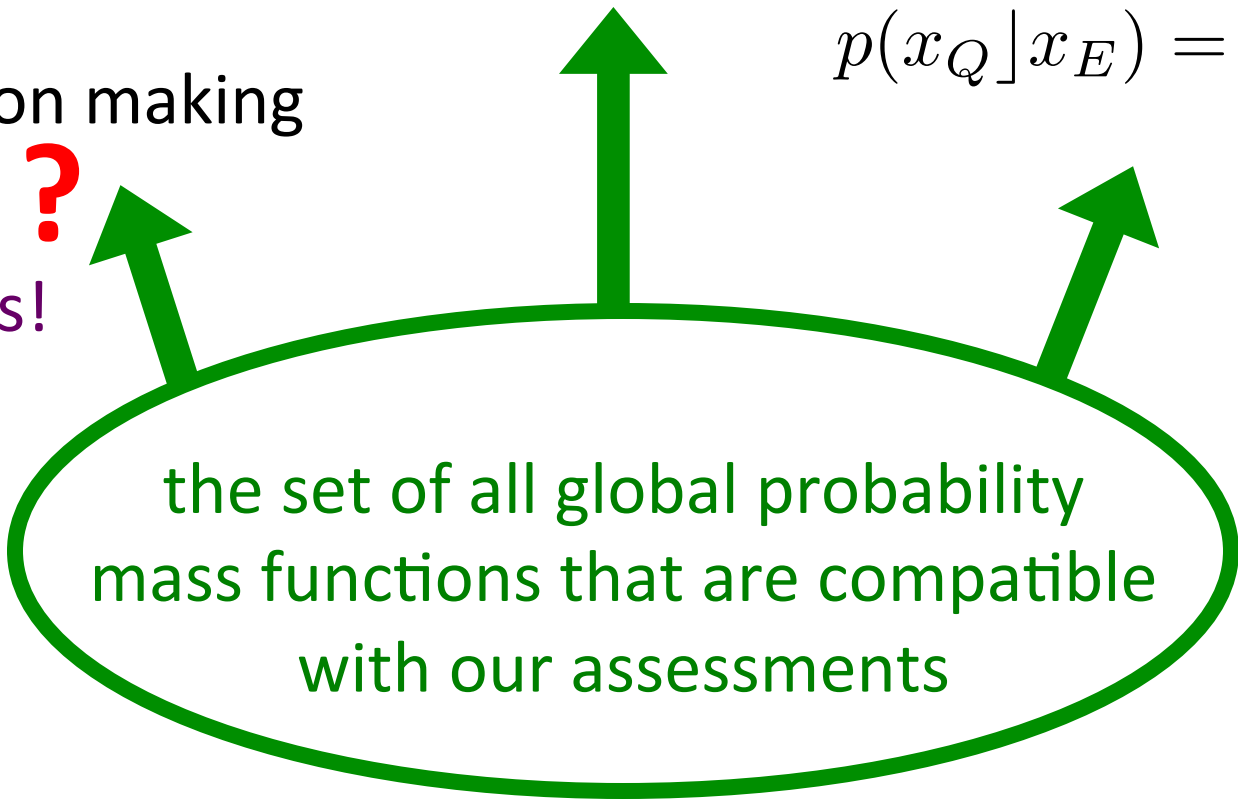
$$? p(x_S) = \sum_{x_{G \setminus S}} p(x_S, x_{G \setminus S})$$

Lower and upper bounds!

$$p(x_Q | x_E) = \frac{p(x_{Q \cup E})}{p(x_E)} ?$$

Decision making

Multiple ?  
decisions!

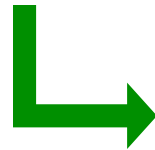


the set of all global probability mass functions that are compatible with our assessments

# The irrelevant natural extension

The **global model** can be described in terms of  
**linear constraints**

(Cozman, 2000)

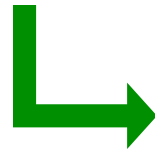


Inference can be performed  
using linear programming  
techniques

# The irrelevant natural extension

The **global model** can be described in terms of **linear constraints**

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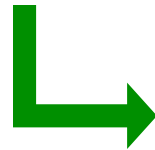
Inference can be performed using linear programming techniques

- Without a **positivity assumption!**
- Equally simple representations in terms of **three other frameworks!**

# The irrelevant natural extension

The **global model** can be described in terms of **linear constraints**

(Cozman, 2000)



Inference can be performed using linear programming techniques

- Without a **positivity assumption!**
- Equally simple representations in terms of **three other frameworks!**

**# constraints is exponential in the size of the network!**

**Credal Networks under  
Epistemic Irrelevance:  
Theory and Algorithms**

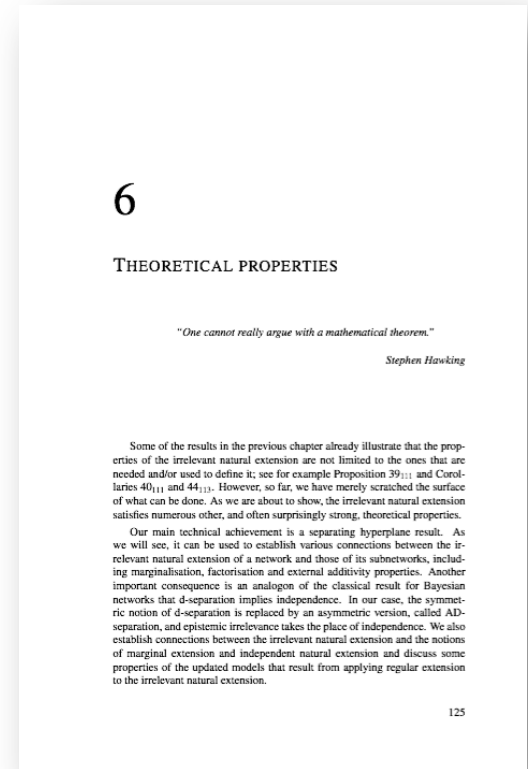
Jasper De Bock

13 May 2015, Ghent, Belgium

# The irrelevant natural extension

## Theoretical properties

- Connections with marginal and independent natural extension
- Marginalisation properties
- AD-separation implies epistemic irrelevance
- ...





# The irrelevant natural extension

## Inference algorithms

- For recursively decomposable networks, inference is very efficient!

For trees: (Cooman et al., 2010)

- Non-decomposable networks can also be dealt with (on a case by case basis)
- Complex types of inference are possible!
- ...



For the second part of the proof, we start by considering the following collection of gambles on  $\mathcal{X}_K$ :

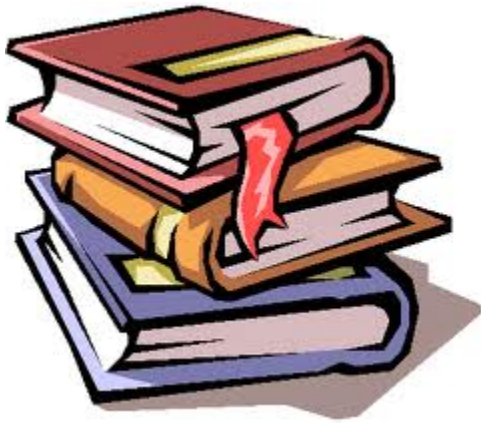
$$\mathcal{A}_{K \rfloor x_{P(K)}}^* := \left\{ \mathbb{I}_{\{z_{PN(s) \cap K_1}\}} f_{s, z_{PN(s)}} : s \in K, z_{PN(s)} \in \mathcal{X}_{PN(s)}, \right. \\ \left. z_{P(s) \setminus P_K(s)} = x_{P(s) \setminus P_K(s)}, P(s) \cap K \subseteq K_1 \subseteq K, \right. \\ \left. f_{s, z_{PN(s)}} \neq 0 \right\},$$

which is a finite subset of  $\mathcal{D}_{K \rfloor x_{P(K)}}^{\text{irr}} := \text{posi}(\mathcal{A}_{K \rfloor x_{P(K)}}^{\text{irr}})$ . To see why, first notice that because  $PN_K(s) = PN(s) \cap K$  due to Lemma 79(iii)<sub>184</sub>,  $\mathbb{I}_{\{z_{PN(s) \cap K_1}\}}$  is clearly the (finite) sum of all indicators  $\mathbb{I}_{\{y_{PN_K(s)}\}}$  such that  $y_{PN_K(s)} \in \mathcal{X}_{PN_K(s)}$  and  $y_{PN(s) \cap K_1} = z_{PN(s) \cap K_1}$ . By definition of the posi operator, we are now left to show that for any  $y_{PN_K(s)} \in \mathcal{X}_{PN_K(s)}$  such that  $y_{PN(s) \cap K_1} = z_{PN(s) \cap K_1}$ , we have  $\mathbb{I}_{\{y_{PN_K(s)}\}} f_{s, z_{PN(s)}} \in \mathcal{A}_{K \rfloor x_{P(K)}}^{\text{irr}}$ . By construction of  $\mathcal{A}_{K \rfloor x_{P(K)}}^*$ , we know that  $z_{P(s) \setminus P_K(s)} = x_{P(s) \setminus P_K(s)}$ , and it therefore suffices to show that  $y_{P_K(s)} = z_{P_K(s)}$ . To see why this last equality holds, first notice that  $P_K(s) = P(s) \cap K$  due to Lemma 76<sub>181</sub>. Also,  $P(s) \cap K \subseteq PN(s) \cap K_1$  because  $P(s) \cap K \subseteq K_1$  by construction of  $\mathcal{A}_{K \rfloor x_{P(K)}}^*$  and  $P(s) \cap K \subseteq PN(s)$  by definition of  $PN(s)$ . Therefore, we find that  $P_K(s) \subseteq PN(s) \cap K_1$ , implying that  $y_{P_K(s)} = z_{P_K(s)}$  is a direct consequence of  $y_{PN(s) \cap K_1} = z_{PN(s) \cap K_1}$ .



**Cosa si può fare  
con esso?**

# An application: correcting OCR errors

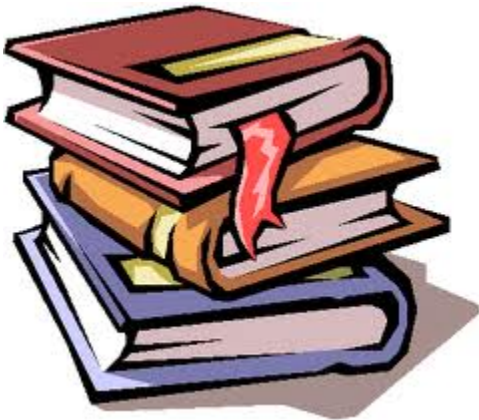


Scan and apply  
Optical Character  
Recognition  
software



# An application: correcting OCR errors

DOCTORAAT



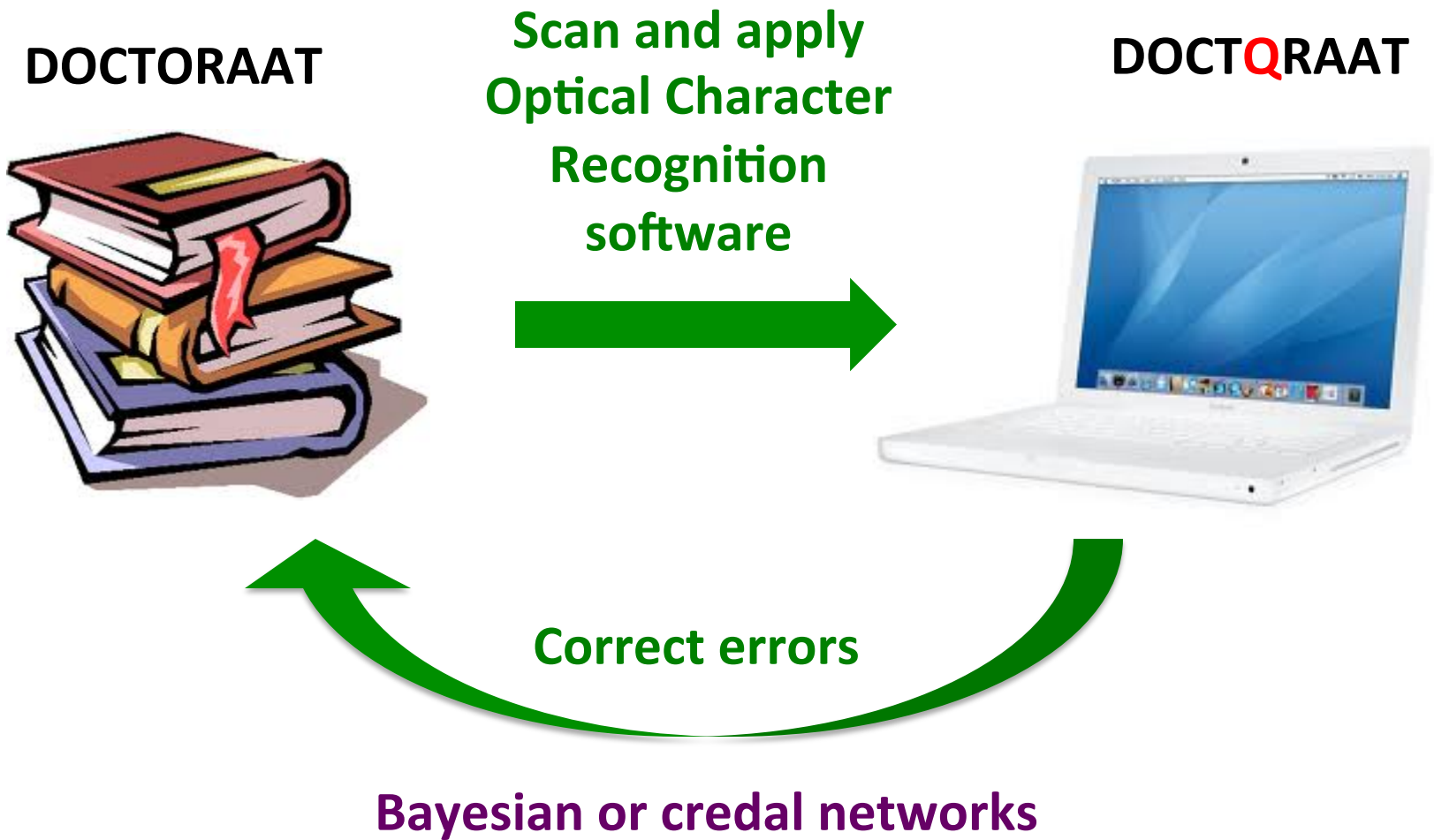
Scan and apply  
Optical Character  
Recognition  
software



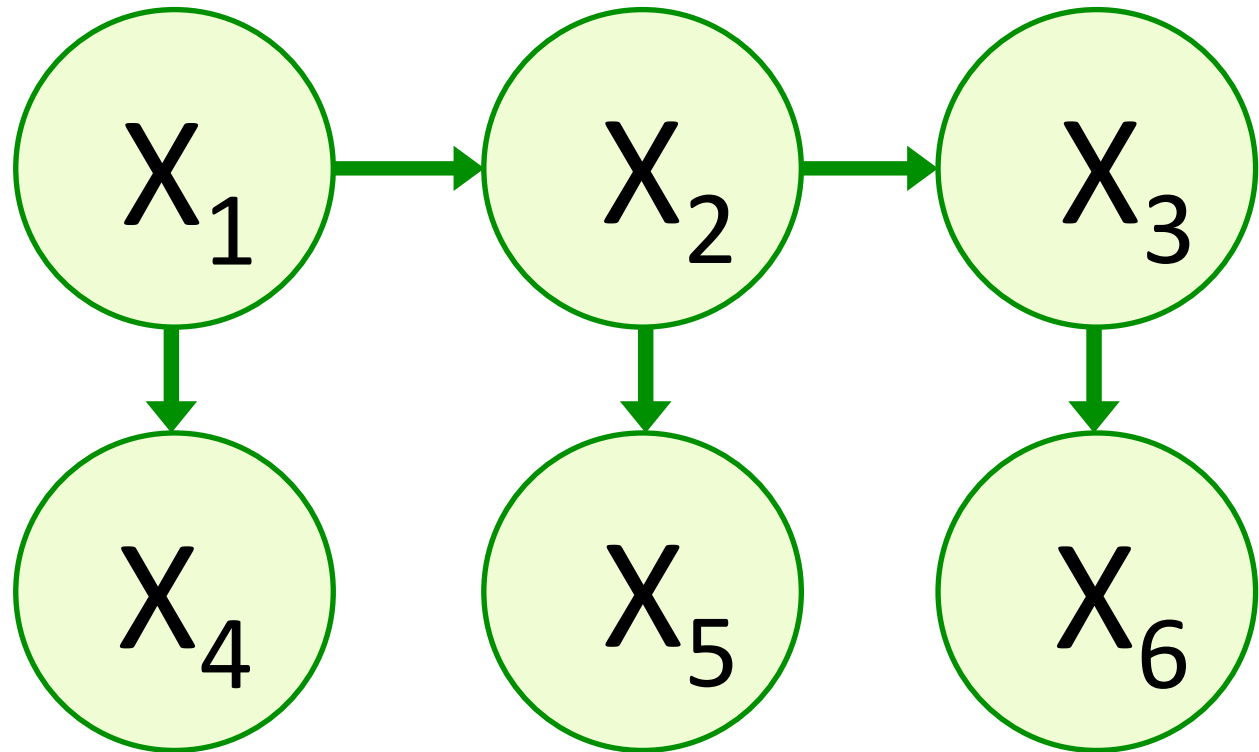
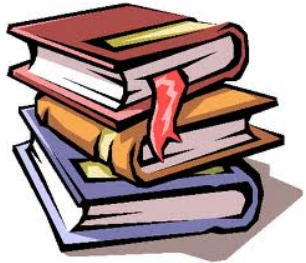
DOCTQRAAT



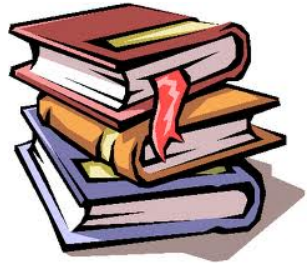
# An application: correcting OCR errors



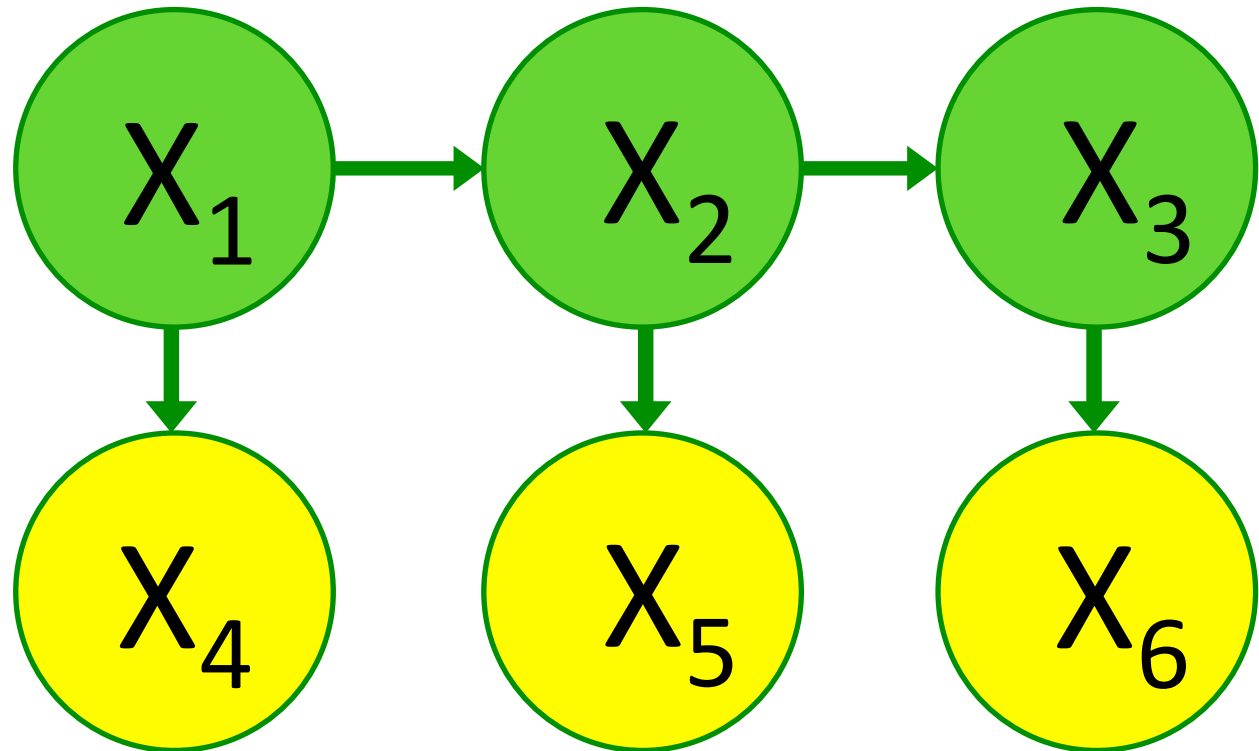
# An application: correcting OCR errors



# An application: correcting OCR errors



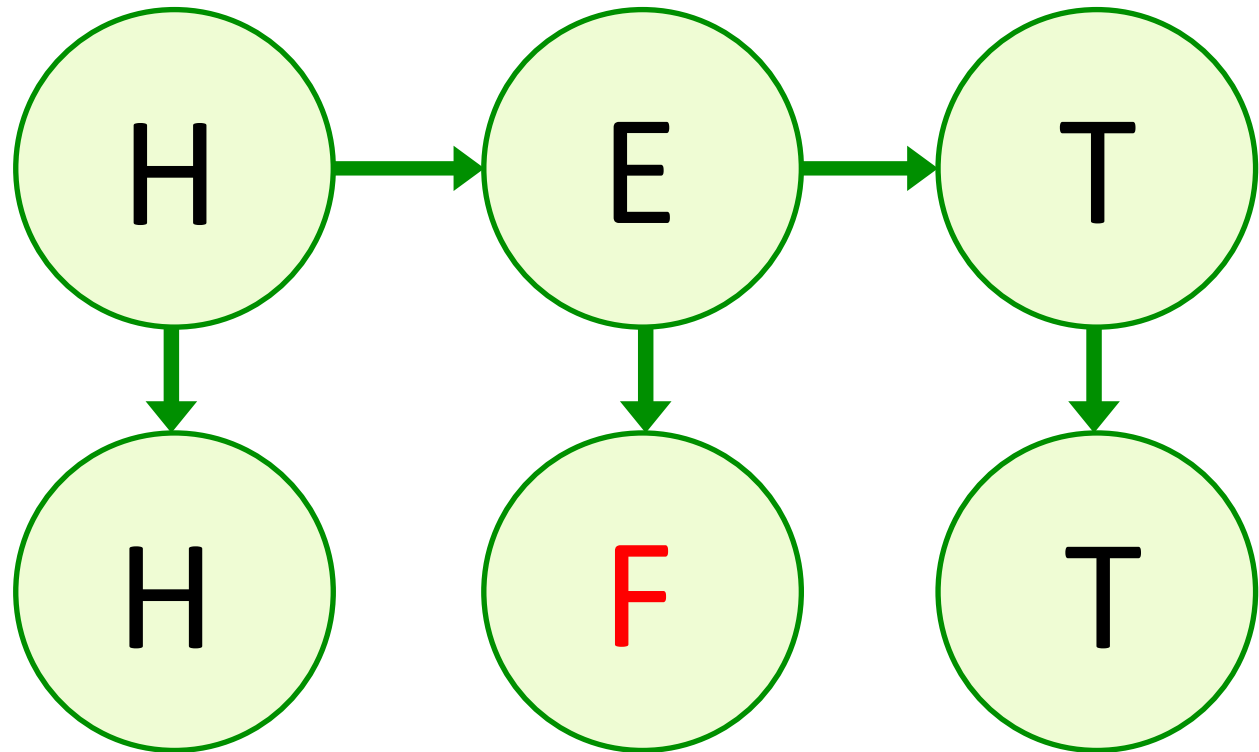
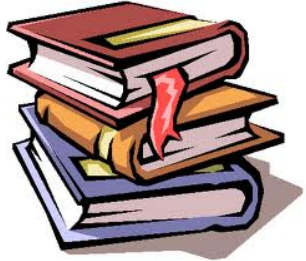
Query nodes



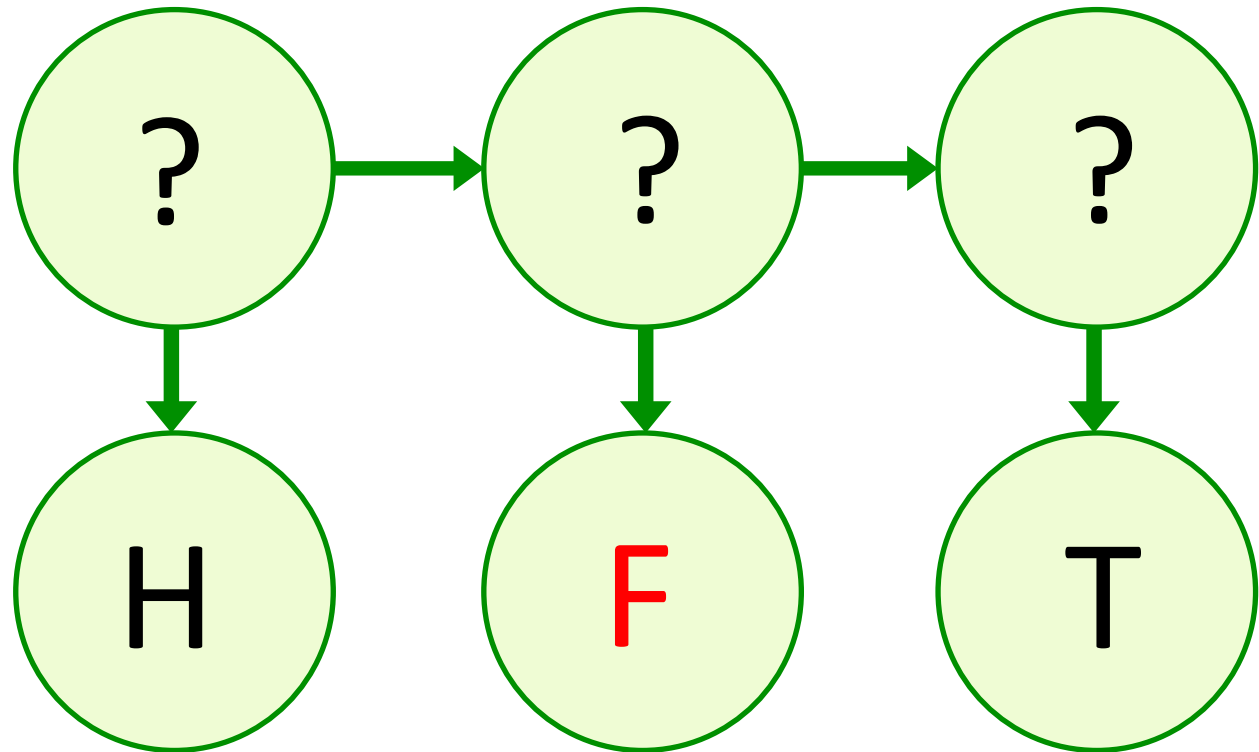
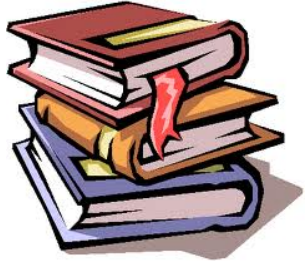
Evidence nodes



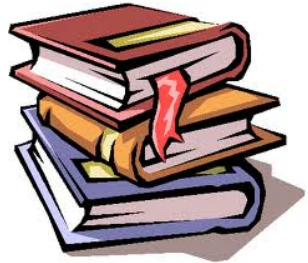
# An application: correcting OCR errors



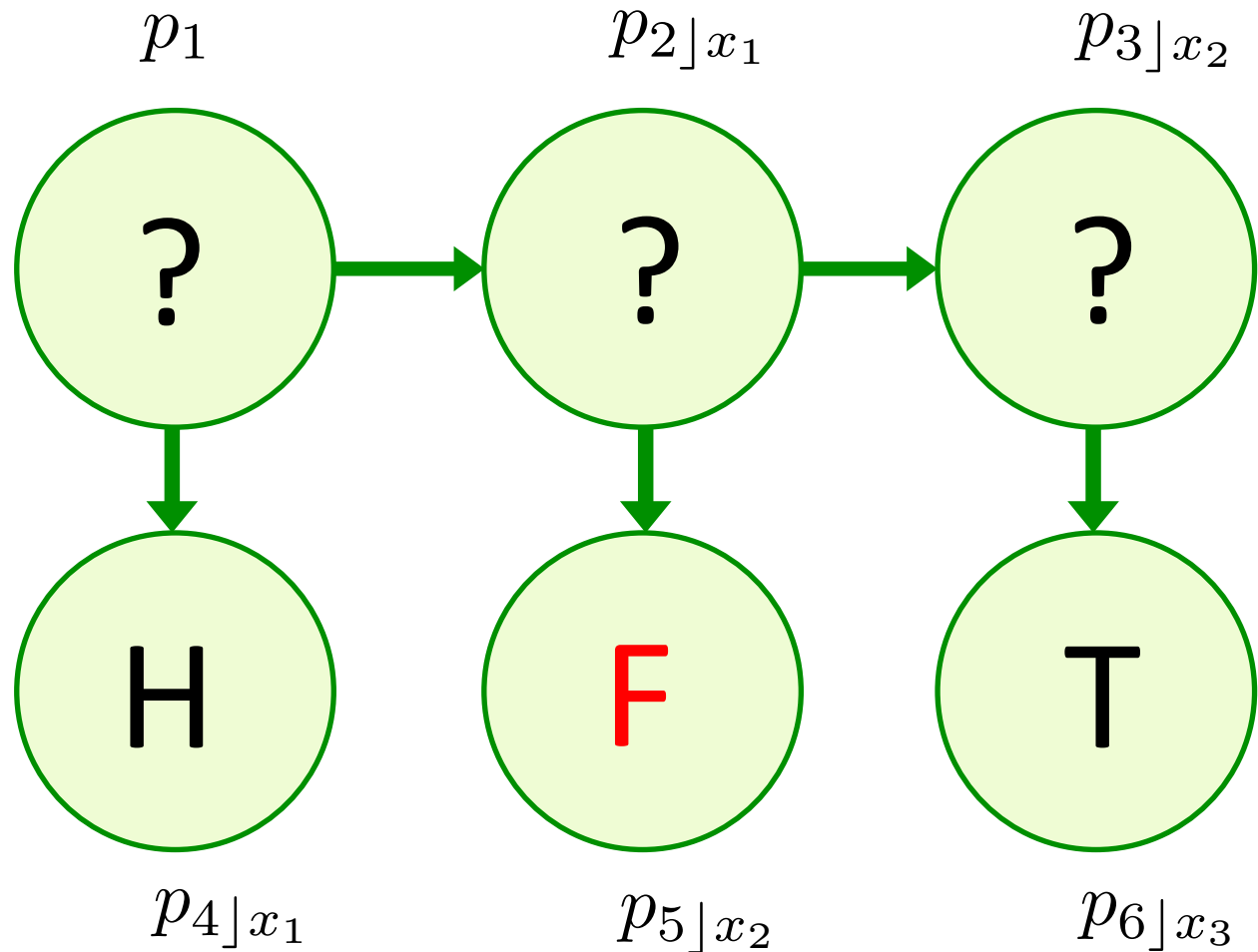
# An application: correcting OCR errors



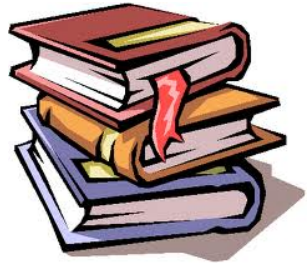
# An application: correcting OCR errors



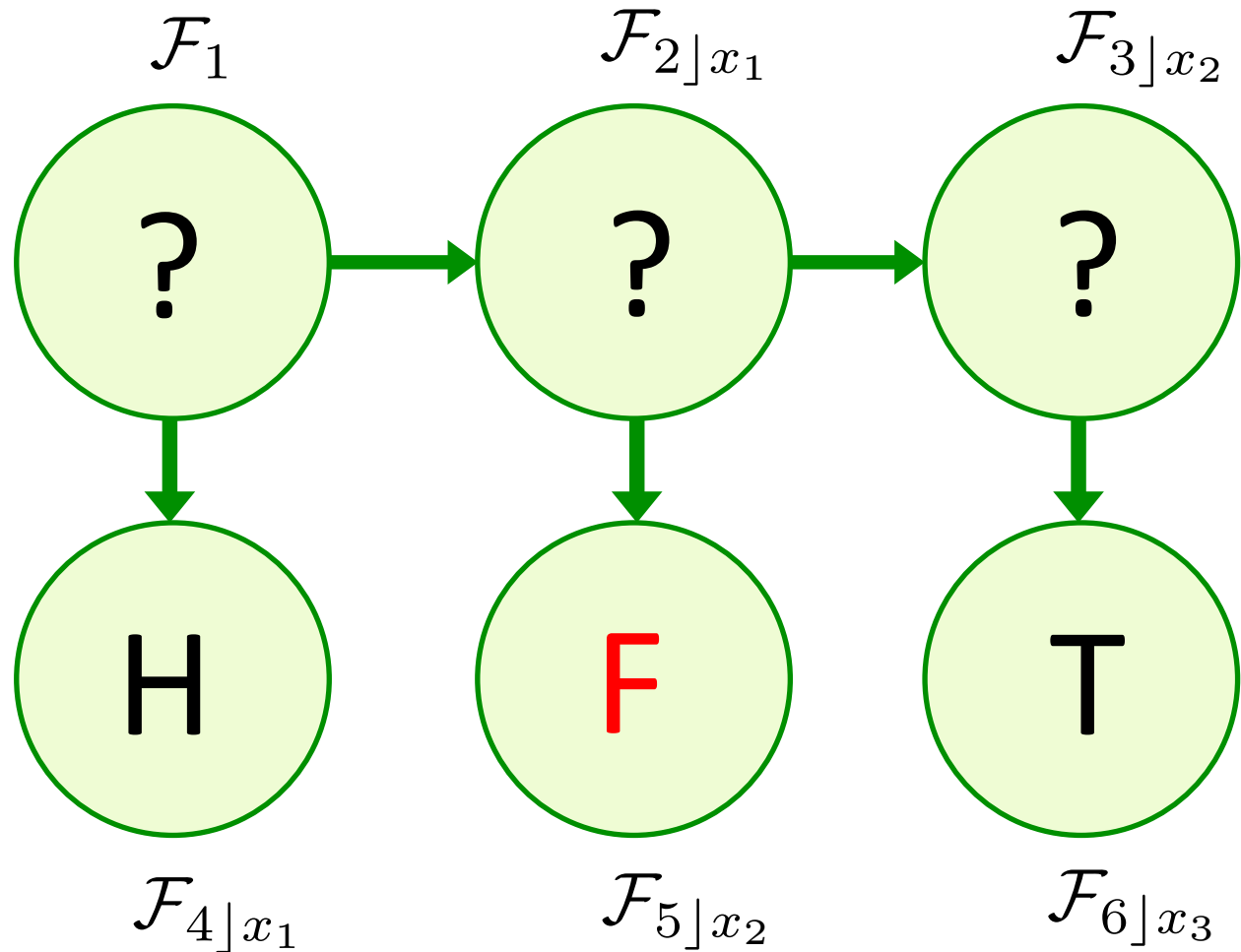
Bayesian  
network



# An application: correcting OCR errors



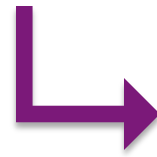
Credal  
network



# An application: correcting OCR errors

## La Divina Commedia

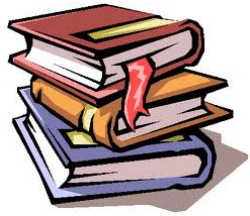
Data is scarce (or expensive)



Obtaining accurate probabilities is unrealistic



# An application: correcting OCR errors



original

**VITA**

correctly read

digital

**VITA**



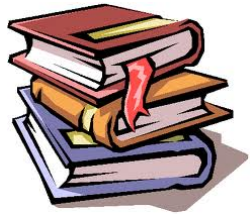
Solution Bayesian network

**VITA**

Solution(s) credal network

**VITA**

# An application: correcting OCR errors



original

**CON**

incorrectly read

digital

**CCN**



Solution Bayesian network

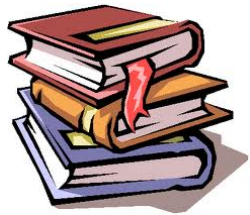
**CON**

Solution(s) credal network

**CON**



# An application: correcting OCR errors



original

**CHE**

incorrectly read

digital

**CNE**



Solution Bayesian network

**ONE**

Solution(s) credal network

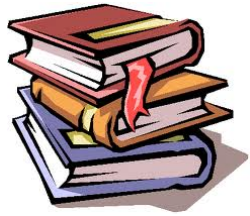
**CBE** **CHE**

**CNE** **CZE**

**ONE**



# An application: correcting OCR errors



original

**EH**

correctly read

digital

**EH**



Solution Bayesian network

**EN**

Solution(s) credal network

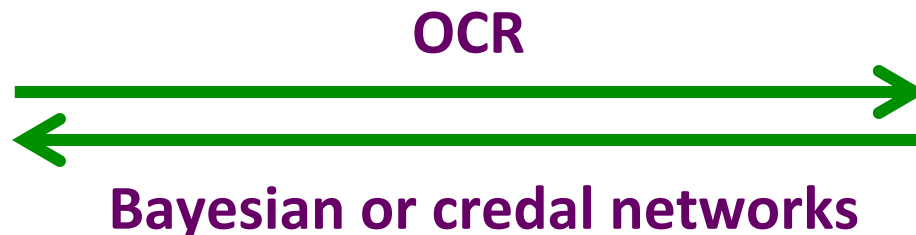
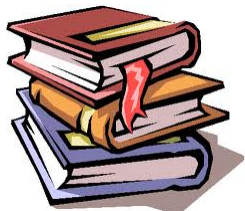
**CH**

**EH**

**EN**

# An application: correcting OCR errors

	TOTAL	OCR correct	OCR wrong
TOTAL	200 (100%)	137 (68.5%)	63 (31.5%)
<b>Credal network</b>			
Includes correct answer	172 (86%)	137	35
Only wrong answers	28 (14%)	0	28
<b>Bayesian network</b>			
Correct answer	157 (78.5%)	132	25
Wrong answer	43 (21.5%)	5	38



# An application: correcting OCR errors

Words for which the credal network suggests multiple answers

	TOTAL	OCR correct	OCR wrong
TOTAL	45 (100%)	8 (17.8%)	37 (82.2%)
<b>Credal network</b>			
Includes correct answer	38 (84.4%)	8	30
Only wrong answers	7 (15.6%)	0	7
<b>Bayesian network</b>			
Correct answer	23 (51.1%)	3	20
Wrong answer	22 (48.9%)	5	17



**Cosa si può fare  
con esso?**



**Ci sono  
domande?**