Convergence of Continuous-Time Imprecise Markov Chains

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Lower transition (rate) operators

Let $\mathscr{X}$ be some finite set. A map $\underline{Q}\colon \mathbb{R}^{\mathscr{X}} \to \mathbb{R}^{\mathscr{X}}$ is a **lower transition rate operator** if

- **LR1.** $\underline{Q}(\mu) = 0$ for all constant $\mu \in \mathbb{R}^{\mathscr{X}}$
- **LR2.** $\underline{Q}(f+g) \geq \underline{Q}(f) + \underline{Q}(g)$ for all $f,g \in \mathbb{R}^{\mathscr{X}}$
- **LR3.** $\underline{Q}(\lambda f) = \lambda \underline{Q}(f)$ for all $f \in \mathbb{R}^{\mathscr{X}}$ and $\lambda \geq 0$
- **LR4.** $\underline{Q}(\mathbb{I}_y)(x) \geq 0$ for all $x,y \in \mathscr{X}$ such that $x \neq y$

A map $\underline{T}\colon \mathbb{R}^{\mathscr{X}} \to \mathbb{R}^{\mathscr{X}}$ is a **lower transition operator** if

- **L1.** $\underline{T}(f) \geq \min f$ for all $f \in \mathbb{R}^{\mathscr{X}}$
- **L2.** $\underline{T}(f+g) \geq \underline{T}(f) + \underline{T}(g)$ for all $f,g \in \mathbb{R}^{\mathscr{X}}$
- **L3.** $\underline{T}(\lambda f) = \lambda \underline{T}(f)$ for all $f \in \mathbb{R}^{\mathscr{X}}$ and $\lambda \geq 0$

These two notions are connected: **every lower transition rate operator $\underline{Q}$ has corresponding lower transition operators $\underline{T}_t$, $t \geq 0$**, as defined for all $f \in \mathbb{R}^{\mathscr{X}}$ by

$$\underline{T}_0 f = f \text{ and } (\forall t \geq 0)\ \frac{d}{dt}\underline{T}_t f = \underline{Q}(\underline{T}_t f)$$

The conjugate **upper transition (rate) operators** $\overline{T}_t$, $t \geq 0$, and $\overline{Q}$ are defined by $\overline{T}_t(f) := -\underline{T}_t(-f)$ and $\overline{Q}(f) := -\underline{Q}(-f)$ for all $f \in \mathbb{R}^{\mathscr{X}}$.

Continuous-Time Imprecise MCs

Definition A **continuous-time imprecise Markov chain** is a continuous-time Markov chain with state space $\mathscr{X}$ of which the transition rate matrix Q_t is an unspecified function of time that takes values in some closed, convex, bounded set of transition rate matrices $\mathscr{Q}$ that has separately specified rows, meaning that

$$Q \in \mathscr{Q} \Leftrightarrow (\forall x \in \mathscr{X})\ Q(x,\cdot) \in \mathscr{Q}_x$$

where, for all $x \in \mathscr{X}$, $\mathscr{Q}_x$ is a set of row vectors.

Bounds on expectations For all $f \in \mathbb{R}^{\mathscr{X}}$ and $x \in \mathscr{X}$, the expected value $E_t(f|X_0 = x)$ of f at time t, conditional on $X_0 = x$, **takes values in a closed interval with lower bound $\underline{T}_t(f)(x)$ and upper bound $\overline{T}_t(f)(x)$ (Škulj 2015)**, where $\underline{T}_t$ is the lower transition operator that corresponds to the lower transition rate operator $\underline{Q}$, with $\underline{Q}(h)$ defined for all $h \in \mathbb{R}^{\mathscr{X}}$ by

$$\underline{Q}(h)(x) := \min_{Q \in \mathscr{Q}} \sum_{y \in \mathscr{X}} Q(x,y)h(y) \text{ for all } x \in \mathscr{X}.$$

Lower and Upper Reachability

For any $x \in \mathscr{X}$ and $y \in \mathscr{X}$, we say that x is **upper reachable** from y and write $y \xrightarrow{\text{up}} x$ if there is some sequence $y = x_0, \dots, x_n = x$ such that, for all $k \in \{1, \dots, n\}$:

$$x_k \neq x_{k-1} \text{ and } \overline{Q}(\mathbb{I}_{x_k})(x_{k-1}) > 0.$$

For any $x \in \mathscr{X}$ and $A \subseteq \mathscr{X}$, we say that A is **lower reachable** from x and write $x \xrightarrow{\text{low}} A$ if $x \in A_n$, where A_k, $k \in \mathbb{N}_0$, is the nested sequence defined by $A_0 := A$ and

$$A_{k+1} := A_k \cup \{y \in \mathscr{X} \setminus A_k \colon \underline{Q}(\mathbb{I}_{A_k})(y) > 0\} \text{ for all } k \in \mathbb{N},$$

and where n is the first index such that $A_n = A_{n+1}$.

$\underline{E}_\infty$?

$\underline{E}_t$

Want to know what happens at infinity? Then you've come to the right place...

$\underline{E}_0$

The following conditions are necessary and sufficient:

$\underline{Q}$ is Perron-Frobenius-like

There is a unique lower expectation operator $\underline{E}_\infty$ on $\mathbb{R}^{\mathscr{X}}$ such that, for all $x \in \mathscr{X}$:

$$\lim_{t \to +\infty} \underline{T}_t f(x) = \underline{E}_\infty f \text{ for all } f \in \mathbb{R}^{\mathscr{X}},$$

or, equivalently, such that for any initial lower expectation operator $\underline{E}_0$, $\underline{E}_t := \underline{E}_0(\underline{E}_t(\cdot|X_0)) = \underline{E}_0 \underline{T}_t$ converges to the stationary distribution $\underline{E}_\infty$.

$\Updownarrow$ **NEW** $\Updownarrow$

$\exists t > 0 \quad \Leftrightarrow \quad \forall t > 0$

$\underline{T}_t$ is Perron-Frobenius-like

There is a unique lower expectation operator $\underline{E}_\infty$ on $\mathbb{R}^{\mathscr{X}}$ such that, for all $x \in \mathscr{X}$:

$$\lim_{k \to +\infty} \underline{T}_t^k f(x) = \underline{E}_\infty f \text{ for all } f \in \mathbb{R}^{\mathscr{X}},$$

or, equivalently, such that for any initial lower expectation operator $\underline{E}_0$, $\underline{E}_k := \underline{E}_0 \underline{T}_t^k$ converges to the stationary distribution $\underline{E}_\infty$.

$\Updownarrow$ **Hermans (2012)**

$\underline{T}_t$ is regularly absorbing

$$\mathscr{R} := \{x \in \mathscr{X} \colon (\exists n \in \mathbb{N})\ \min \overline{T}_t^n \mathbb{I}_x > 0\} \neq \emptyset$$

$$(\forall x \in \mathscr{X} \setminus \mathscr{R})(\exists n \in \mathbb{N})\ \underline{T}_t^n \mathbb{I}_{\mathscr{R}}(x) > 0$$

$\Updownarrow$ **NEW**

$\underline{T}_t$ is 1-step absorbing

$$\mathscr{R} := \{x \in \mathscr{X} \colon \min \overline{T}_t \mathbb{I}_x > 0\} \neq \emptyset$$

$$(\forall x \in \mathscr{X} \setminus \mathscr{R})\ \underline{T}_t \mathbb{I}_{\mathscr{R}}(x) > 0$$

$\Updownarrow$ **NEW**

$\underline{Q}$ is regularly absorbing

$$\mathscr{R} := \{x \in \mathscr{X} \colon (\forall y \in \mathscr{X})\ y \xrightarrow{\text{up}} x\} \neq \emptyset$$

$$(\forall x \in \mathscr{X} \setminus \mathscr{R})\ x \xrightarrow{\text{low}} \mathscr{R}$$