Extreme Lower Previsions and Minkowski Indecomposability

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Abstract Coherent lower previsions constitute a convex set that is closed and compact under the topology of point-wise convergence, and Maaß [2] has shown that any coherent lower prevision can be written as a 'countably additive convex combination' of the extreme points of this set. We show that when the possibility space has a finite number *n* of elements, these extreme points are either degenerate precise probabilities, or in a one-to-one correspondence with the (Minkowski) indecomposable compact convex subsets of $\mathbb{R}^{n-1}$.

Keywords: Extreme lower previsions, extreme credal sets, fully imprecise lower previsions, fully imprecise credal sets, Minkowski decomposition.

1 Introduction

In his Ph.D. dissertation, Maaß [2] proved a general, Choquet-like representation result for what he called inequality preserving functionals. When we apply his results to coherent lower previsions, which have an important part in the theory of imprecise probabilities, we find that the set of all coherent lower previsions defined on a subset of the linear space of all bounded real-valued maps (gambles) on a possibility space $\mathscr{X}$ constitute a convex set, that is furthermore closed and compact under the topology of point-wise convergence, and that any coherent lower prevision can be written as a 'countably additive convex combination' of the extreme points of this set.

It became apparent quite soon, however, that finding these extreme coherent lower previsions was a non-trivial task. Contributions to solving this problem were made by Quaeghebeur [5], who essentially concentrated on coherent lower previsions defined on finite domains. In this paper, we look at the extreme points of the set of all coherent lower previsions defined on the space of all real-valued maps on a finite set $\mathscr{X}$, containing *n* elements. We begin by defining (extreme) coherent lower previsions in Section 2. In Section 3, we recall that coherent lower previsions are in a one-to-one relationship with compact convex sets of probability mass functions, which allows us, in Sections 4 and 5, to establish a link between extreme coherent lower previsions on the one hand, and (Minkowski) indecomposable compact convex subsets of $\mathbb{R}^{n-1}$ on the other.

This link allows us to reduce the problem of finding all extreme coherent lower previsions to a problem that has received quite a bit of attention in the mathematical literature, and to use existing solutions for that problem. We give a short discussion of what can and could be learned from this connection in Section 6, and go on to discuss a number of avenues for further research and possible applications.

2 Coherent Lower Previsions

Consider a variable X taking values in some non-empty set $\mathscr{X}$, called *possibility space*. We will restrict ourselves to finite possibility spaces $\mathscr{X} = \{x_1, \dots, x_n\}$, with $n \in \mathbb{N}_{>1}$.[1,2] The theory of coherent lower previsions models a subject's beliefs regarding the uncertain value of X by means of lower and upper previsions of so-called gambles. A *gamble* is a real-valued map on $\mathscr{X}$ and we use $\mathscr{G}(\mathscr{X})$ to denote the set of all of them. A lower prevision $\underline{P}$ is a real-valued functional defined on this set $\mathscr{G}(\mathscr{X})$. $\underline{P}$ is said to be *coherent* if it satisfies the following three conditions: for all $f, g \in \mathscr{G}(\mathscr{X})$ and all real $\lambda > 0$

C1. $\underline{P}(f) \geq \min f$

C2. $\underline{P}(\lambda f) = \lambda \underline{P}(f)$ [non-negative homogeneity]

C3. $\underline{P}(f+g) \geq \underline{P}(f) + \underline{P}(g)$ [super-additivity]

The set of all coherent lower previsions on $\mathscr{G}(\mathscr{X})$ is denoted by $\underline{\mathbb{P}}(\mathscr{X})$. The conjugate of a lower prevision $\underline{P} \in \underline{\mathbb{P}}(\mathscr{X})$ is called an *upper prevision*. It is denoted by $\overline{P}$ and defined by $\overline{P}(f) := -\underline{P}(-f)$ for all gambles $f \in \mathscr{G}(\mathscr{X})$. Coherent lower and upper previsions can be given a behavioural interpretation in terms of buying and selling prices, turning the three conditions above into criteria for rational behaviour; see Ref. [9] for an in-depth study, and Ref. [4] for a recent survey.

2.1 Extreme Lower Previsions

Coherence is preserved under taking convex combinations [9, Section 2.6.4]. Consider two coherent lower previsions $\underline{P}_1$ and $\underline{P}_2$ in $\underline{\mathbb{P}}(\mathscr{X})$ and any $\lambda \in [0,1]$. Then the lower prevision $\underline{P} = \lambda \underline{P}_1 + (1-\lambda)\underline{P}_2$, defined by $\underline{P}(f) := \lambda \underline{P}_1(f) + (1-\lambda)\underline{P}_2(f)$ for all $f \in \mathscr{G}(\mathscr{X})$, will also be coherent. One can now wonder whether every coherent lower prevision can be written as such a convex combination of others: given a coherent lower prevision $\underline{P} \in \underline{\mathbb{P}}(\mathscr{X})$, is it possible to find coherent lower previsions $\underline{P}_1$ and $\underline{P}_2$ in $\underline{\mathbb{P}}(\mathscr{X})$ and $\lambda \in [0,1]$ such that $\underline{P} = \lambda \underline{P}_1 + (1-\lambda)\underline{P}_2$? If we exclude the trivial decompositions, where $\lambda = 0$, $\lambda = 1$ or $\underline{P}_1 = \underline{P}_2 = \underline{P}$, then the answer can be no. We will refer to those coherent lower previsions for which no non-trivial decomposition exists as *extreme lower previsions*. The goal of this paper is to characterise, and where possible to find, the set $\operatorname{ext}\underline{\mathbb{P}}(\mathscr{X})$ of all extreme lower previsions on $\mathscr{G}(\mathscr{X})$.

2.2 Special Kinds of Coherent Lower Previsions

In order to find these extreme lower previsions, it will be useful to split the set $\underline{\mathbb{P}}(\mathscr{X})$ into three disjoint subsets: linear previsions, lower previsions that are fully imprecise and lower previsions that are partially imprecise.

[1] $\mathbb{N}$ denotes the positive integers (excluding zero) and $\mathbb{R}$ the real numbers. Subsets are denoted by using predicates as subscripts; e.g., $\mathbb{N}_{\leq n} := \{i \in \mathbb{N} : i \leq n\} = \{1, \dots, n\}$ denotes the positive integers up to n and $\mathbb{R}_{>0} := \{r \in \mathbb{R} : r > 0\}$ the strictly positive real numbers.

[2] We do not consider $n = 1$ because this case is both trivial and of no practical use. Indeed, a variable that can only assume a single value has no uncertainty associated with it.

A coherent lower prevision $\underline{P} \in \underline{\mathbb{P}}(\mathscr{X})$ is called a *linear prevision* if it has the additional property that $\underline{P}(f+g) = \underline{P}(f) + \underline{P}(g)$ for all $f, g \in \mathscr{G}(\mathscr{X})$. It is then generically denoted by P and we use $\mathbb{P}(\mathscr{X})$ to denote the set of all of them. It can be shown that for every *mass function* p in the so-called $\mathscr{X}$-simplex

$$\Sigma_{\mathscr{X}} := \Big\{ p \in \mathbb{R}^{\mathscr{X}} : \sum_{i=1}^{n} p(x_i) = 1 \text{ and } p(x_i) \geq 0 \text{ for all } i \in \mathbb{N}_{\leq n} \Big\}, \tag{1}$$

the corresponding expectation operator P_p, defined by $P_p(f) := \sum_{i=1}^{n} f(x_i) p(x_i)$ for all $f \in \mathscr{G}(\mathscr{X})$, is a linear prevision in $\mathbb{P}(\mathscr{X})$. Conversely, every linear prevision $P \in \mathbb{P}(\mathscr{X})$ has a unique mass function $p \in \Sigma_{\mathscr{X}}$ for which $P = P_p$. It is defined by $p(x_i) := P(\mathbb{I}_{\{x_i\}})$, $i \in \mathbb{N}_{\leq n}$, where $\mathbb{I}_{\{x_i\}}$ denotes the *indicator* of $\{x_i\}$: for all $x \in \mathscr{X}$, $\mathbb{I}_{\{x_i\}}(x) = 1$ if $x = x_i$ and $\mathbb{I}_{\{x_i\}}(x) = 0$ otherwise.

Another special kind of coherent lower previsions are those that are *fully imprecise*. They are uniquely characterised by the property that $\underline{P}(\mathbb{I}_{\{x_i\}}) = 0$ for all $i \in \mathbb{N}_{\leq n}$. As we shall see further on, we can interpret $\underline{P}(\mathbb{I}_{\{x_i\}})$ as the lower probability of x_i, thereby making fully imprecise lower previsions those for which the lower probability of all elements in the possibility space is zero. We will use $\underline{\underline{\mathbb{P}}}(\mathscr{X})$ to denote the set of all such fully imprecise lower previsions. The reason why we call them fully imprecise is because they differ most from the precise, linear previsions. This distinction is already apparent from the following Proposition, but will become even clearer in Section 5.1, where we prove that every coherent lower prevision that is neither linear nor fully imprecise can be uniquely decomposed into a linear and a fully imprecise part.

Proposition 1. *$\mathbb{P}(\mathscr{X})$ and $\underline{\underline{\mathbb{P}}}(\mathscr{X})$ are disjoint subsets of $\underline{\mathbb{P}}(\mathscr{X})$: linear previsions are never fully imprecise.*

We refer to coherent lower previsions in $\underline{\mathbb{P}}(\mathscr{X})$ that are neither fully imprecise nor linear previsions as *partially imprecise*, and we denote by $\underset{\sim}{\mathbb{P}}(\mathscr{X})$ the set of all partially imprecise lower previsions. The next corollary is a direct consequence of Proposition 1.

Corollary 1. *$\mathbb{P}(\mathscr{X})$, $\underline{\underline{\mathbb{P}}}(\mathscr{X})$ and $\underset{\sim}{\mathbb{P}}(\mathscr{X})$ constitute a partition of $\underline{\mathbb{P}}(\mathscr{X})$.*

3 Credal Sets

Linear previsions are not the only coherent lower previsions that can be characterised by means of mass functions in $\Sigma_{\mathscr{X}}$. It is well known [9, Section 3.6] that every coherent lower prevision can be uniquely characterised by a so-called *credal set*, which is a closed (and therefore compact[3]) convex subset of $\Sigma_{\mathscr{X}}$. We denote a generic credal set by $\mathscr{M}$ and use $\underline{\mathbb{M}}(\mathscr{X})$ to denote the set of all of them. For any $\underline{P} \in \underline{\mathbb{P}}(\mathscr{X})$, its corresponding credal set $\mathscr{M}_{\underline{P}}$ is the set of all mass functions that define a dominating linear prevision:

$$\mathscr{M}_{\underline{P}} := \big\{ p \in \mathscr{M} : P_p(f) \geq \underline{P}(f) \text{ for all } f \in \mathscr{G}(\mathscr{X}) \big\}. \tag{2}$$

[3] Since we only consider finite possibility spaces $\mathscr{X}$, we can use the Euclidean topology instead of the weak*-topology that is usually adopted for credal sets.

The original lower prevision $\underline{P}$ and its conjugate upper prevision $\overline{P}$ can be derived from the credal set $\mathscr{M}_{\underline{P}}$: for all $f \in \mathscr{G}(\mathscr{X})$

$$\underline{P}(f) = \min\{P_p(f) \colon p \in \mathscr{M}_{\underline{P}}\} \text{ and } \overline{P}(f) = \max\{P_p(f) \colon p \in \mathscr{M}_{\underline{P}}\}. \tag{3}$$

We can use this equation to justify our earlier statement in Section 2.2 that for all $i \in \mathbb{N}_{\leq n}$, we can interpret $\underline{P}(\mathbb{I}_{\{x_i\}})$ as the lower probability of x_i. Indeed, we find that

$$\underline{P}(\mathbb{I}_{\{x_i\}}) = \min\{P_p(\mathbb{I}_{\{x_i\}}) \colon p \in \mathscr{M}_{\underline{P}}\} = \min\{p(x_i) \colon p \in \mathscr{M}_{\underline{P}}\} \tag{4}$$

is the smallest probability of x_i corresponding with the mass functions in $\mathscr{M}_{\underline{P}}$.

Credal sets are therefore in a one-to-one correspondence with coherent lower previsions, allowing us to think of a coherent lower prevision as a closed and convex set of mass functions instead of as an operator on gambles. This geometric approach will be useful in our search for extreme lower previsions, since it will enable us to establish links with results already proved in fields other than coherent lower prevision theory.

3.1 Extreme Credal Sets

Similarly to what we have done in Section 2.1 for coherent lower previsions, we can also take convex combinations of credal sets. Consider two credal sets $\mathscr{M}_1$ and $\mathscr{M}_2$ in $\underline{\mathbb{M}}(\mathscr{X})$ and any $\lambda \in [0,1]$. Then the set $\mathscr{M} := \lambda\mathscr{M}_1 + (1-\lambda)\mathscr{M}_2$, given by

$$\mathscr{M} := \{\lambda p_1 + (1-\lambda)p_2 \colon p_1 \in \mathscr{M}_1 \text{ and } p_2 \in \mathscr{M}_2\}, \tag{5}$$

will again be a credal set in $\underline{\mathbb{M}}(\mathscr{X})$. Due to the equivalence between credal sets and coherent lower previsions, the following proposition should not cause any surprise.

Proposition 2. *Consider coherent lower previsions $\underline{P}$, $\underline{P}_1$ and $\underline{P}_2$ in $\underline{\mathbb{P}}(\mathscr{X})$ and their corresponding credal sets $\mathscr{M}_{\underline{P}}$, $\mathscr{M}_{\underline{P}_1}$ and $\mathscr{M}_{\underline{P}_2}$ in $\underline{\mathbb{M}}(\mathscr{X})$. Then for all $\lambda \in [0,1]$:*

$$\underline{P} = \lambda\underline{P}_1 + (1-\lambda)\underline{P}_2 \Leftrightarrow \mathscr{M}_{\underline{P}} = \lambda\mathscr{M}_{\underline{P}_1} + (1-\lambda)\mathscr{M}_{\underline{P}_2}. \tag{6}$$

We now define an *extreme credal set* as a credal set $\mathscr{M} \in \underline{\mathbb{M}}(\mathscr{X})$ that cannot be written as a convex combination of two other credal sets $\mathscr{M}_1$ and $\mathscr{M}_2$ other than in a trivial way, trivial meaning that $\lambda = 0$, $\lambda = 1$ or $\mathscr{M}_1 = \mathscr{M}_2 = \mathscr{M}$. We will denote the set of all such extreme credal sets as $\operatorname{ext}\underline{\mathbb{M}}(\mathscr{X})$. The following immediate corollary of Proposition 2 shows that they are in a one-to-one correspondence with extreme lower previsions.

Corollary 2. *A coherent lower prevision is extreme iff its credal set is. For all $\underline{P} \in \underline{\mathbb{P}}(\mathscr{X})$:*

$$\underline{P} \in \operatorname{ext}\underline{\mathbb{P}}(\mathscr{X}) \Leftrightarrow \mathscr{M}_{\underline{P}} \in \operatorname{ext}\underline{\mathbb{M}}(\mathscr{X}). \tag{7}$$

3.2 Special Kinds of Credal Sets

Because of the one-to-one correspondence between coherent lower previsions and credal sets, the special subsets of $\underline{\mathbb{P}}(\mathscr{X})$ that were introduced in Section 2.2 immediately lead to corresponding subsets of $\underline{\mathbb{M}}(\mathscr{X})$. The set

$$\mathbb{M}(\mathscr{X}) := \{\mathscr{M}_P \colon P \in \mathbb{P}(\mathscr{X})\} = \{\{p\} \colon p \in \Sigma_{\mathscr{X}}\} \tag{8}$$

of credal sets that correspond to linear previsions in $\mathbb{P}(\mathscr{X})$ is the easiest one.

Another subset of $\underline{\mathbb{M}}(\mathscr{X})$, which will become very important further on, contains those credal sets that correspond to fully imprecise coherent lower previsions:

$$\begin{aligned}\underline{\underline{\mathbb{M}}}(\mathscr{X}) &:= \{\mathscr{M}_{\underline{P}} \colon \underline{P} \in \underline{\underline{\mathbb{P}}}(\mathscr{X})\} && (9)\\ &= \{\mathscr{M} \in \underline{\mathbb{M}}(\mathscr{X}) \colon \min\{p(x_i) \colon p \in \mathscr{M}\} = 0 \text{ for all } i \in \mathbb{N}_{\leq n}\}, && (10)\end{aligned}$$

where the second equality is a consequence of Eq. (4) and the definition of fully imprecise lower previsions. It should also clarify our statement in Section 2.2 that for fully imprecise lower previsions the lower probability of all elements of the possibility space is zero. We refer to elements of $\underline{\underline{\mathbb{M}}}(\mathscr{X})$ as *fully imprecise credal sets*.

The final subset of $\underline{\mathbb{M}}(\mathscr{X})$ that we need to consider contains the *partially imprecise credal sets*, corresponding to partially imprecise lower previsions in $\underset{\sim}{\mathbb{P}}(\mathscr{X})$:

$$\underset{\sim}{\mathbb{M}}(\mathscr{X}) := \{\mathscr{M}_{\underline{P}} \colon \underline{P} \in \underset{\sim}{\mathbb{P}}(\mathscr{X})\} = \underline{\mathbb{M}}(\mathscr{X}) \setminus \{\mathbb{M}(\mathscr{X}) \cup \underline{\underline{\mathbb{M}}}(\mathscr{X})\}. \tag{11}$$

Finally, the following result is a direct consequence of Corollary 1.

Corollary 3. *$\mathbb{M}(\mathscr{X})$, $\underline{\underline{\mathbb{M}}}(\mathscr{X})$ and $\underset{\sim}{\mathbb{M}}(\mathscr{X})$ constitute a partition of $\underline{\mathbb{M}}(\mathscr{X})$.*

3.3 Projected Credal Sets

Mass functions on the possibility space $\mathscr{X} = \{x_1, \ldots, x_n\}$ are uniquely characterised by the probability of the first $n-1$ elements because the final probability follows from the requirement that $\sum_{i=1}^{n} p(x_i) = 1$. This leads us to identify a mass function p on $\mathscr{X}$ with a point v_p in $\mathbb{R}^{n-1}$, defined by $(v_p)_i := p(x_i)$ for all $i \in \mathbb{N}_{<n}$. Similarly, a credal set $\mathscr{M}$ can be identified with a subset of $\mathbb{R}^{n-1}$ by letting

$$K_{\mathscr{M}} := \{v_p \colon p \in \mathscr{M}\}. \tag{12}$$

We call $K_{\mathscr{M}}$ the *projected credal set* of $\mathscr{M}$. We will use $K_{\underline{P}}$ as a shorthand notation for $K_{\mathscr{M}_{\underline{P}}}$ and call it the projected credal set of $\underline{P}$. For all $\mathscr{M} \in \underline{\mathbb{M}}(\mathscr{X})$, $K_{\mathscr{M}}$ is a closed and convex subset of the so-called *projected $\mathscr{X}$-simplex*

$$\mathbf{K}_{\mathscr{X}} = \left\{ v \in \mathbb{R}^{n-1} \colon \sum_{i=1}^{n-1} v_i \leq 1 \text{ and } v_i \geq 0 \text{ for all } i \in \mathbb{N}_{<n} \right\}, \tag{13}$$

which is a compact, closed and convex subset of $\mathbb{R}^{n-1}$. The set of all closed (and therefore compact) convex subsets of $\mathbf{K}_{\mathscr{X}}$ is denoted by $\underline{\mathbb{K}}(\mathscr{X})$. To show that both representations are indeed equivalent, let us define for every point $v \in \mathbf{K}_{\mathscr{X}}$ a corresponding mass function p_v on $\mathscr{X}$, defined by $p_v(x_i) := v_i$ for all $i \in \mathbb{N}_{<n}$ and $p_v(x_n) := 1 - \sum_{i=1}^{n-1} v_i$. It should be clear that $v_{p_v} = v$ and $p_{v_p} = p$, whence the equivalence. Similarly, we can define for all $K \in \mathbf{K}_{\mathscr{X}}$ a corresponding credal set

$$\mathscr{M}_K := \{p_v \colon v \in K\}. \tag{14}$$

Again, we have that $K_{\mathscr{M}_K} = K$ and $\mathscr{M}_{K_{\mathscr{M}}} = \mathscr{M}$. Finally, the following intuitive result shows that projecting credal sets on $\mathbf{K}_{\mathscr{X}}$ preserves convex combinations.

Proposition 3. *Consider credal sets $\mathscr{M}$, $\mathscr{M}_1$ and $\mathscr{M}_2$ in $\underline{\mathbb{M}}(\mathscr{X})$ and their corresponding projected credal sets $K_{\mathscr{M}}$, $K_{\mathscr{M}_1}$ and $K_{\mathscr{M}_2}$ in $\underline{\mathbb{K}}(\mathscr{X})$. Then for all $\lambda \in [0,1]$:*

$$\mathscr{M} = \lambda \mathscr{M}_1 + (1-\lambda)\mathscr{M}_2 \Leftrightarrow K_{\mathscr{M}} = \lambda K_{\mathscr{M}_1} + (1-\lambda) K_{\mathscr{M}_2}. \tag{15}$$

3.4 Special Kinds of Projected Credal Sets

Due the equivalence between credal sets and their projected versions, we can use the partition of $\underline{\mathbb{M}}(\mathscr{X})$ in Corollary 3 to construct a similar partition of $\underline{\mathbb{K}}(\mathscr{X})$. The first set in that partition corresponds to the credal sets of linear previsions and is equal to

$$\mathbb{K}(\mathscr{X}) := \{K_{\mathscr{M}} \colon \mathscr{M} \in \mathbb{M}(\mathscr{X})\} = \{K \in \underline{\mathbb{K}}(\mathscr{X}) \colon K = \{v\}, \text{with } v \in \mathbf{K}_{\mathscr{X}}\}. \tag{16}$$

The second set consists of the projections of the credal sets in $\underline{\underline{\mathbb{M}}}(\mathscr{X})$:

$$\underline{\underline{\mathbb{K}}}(\mathscr{X}) := \{K_{\mathscr{M}} \colon \mathscr{M} \in \underline{\underline{\mathbb{M}}}(\mathscr{X})\} \tag{17}$$

$$= \left\{K \in \underline{\mathbb{K}}(\mathscr{X}) \colon \min_{v \in K} v_i = 0 \text{ for all } i \in \mathbb{N}_{<n} \text{ and } \max_{v \in K} \sum_{i=1}^{n-1} v_i = 1\right\}. \tag{18}$$

The final set contains the projected credal sets of partially imprecise lower previsions:

$$\underset{\sim}{\mathbb{K}}(\mathscr{X}) := \{K_{\mathscr{M}} \colon \mathscr{M} \in \underset{\sim}{\mathbb{M}}(\mathscr{X})\} = \underline{\mathbb{K}}(\mathscr{X}) \setminus \{\mathbb{K}(\mathscr{X}) \cup \underline{\underline{\mathbb{K}}}(\mathscr{X})\}. \tag{19}$$

4 Minkowski Decomposition

Given two compact convex subsets A_1 and A_2 of $\mathbb{R}^{n-1}$, their *Minkowski sum* or *vector sum* is given by $A_1 + A_2 := \{a_1 + a_2 \colon a_1 \in A_1 \text{ and } a_2 \in A_2\}$. They are called *homothetic* if $A_1 = v + \lambda A_2 := \{v + \lambda a_2 \colon a_2 \in A_2\}$ for some $\lambda > 0$ and $v \in \mathbb{R}^{n-1}$. If $A = A_1 + A_2$, with A, A_1 and A_2 compact convex subsets of $\mathbb{R}^{n-1}$, then A_1 and A_2 are called *summands* of A. We say that A is written as a Minkowski sum in a non-trivial way, if neither of its summands is homothetic to A or a singleton. If such a non-trivial decomposition exists, we say that A is *Minkowski decomposable*. Otherwise, A is called *Minkowski indecomposable*. Sections 6.2 and 6.3 point to relevant literature, where, incidentally, the prefix "Minkowski" is not always used. We add it in the present paper to avoid confusion with the decomposition of credal sets and lower previsions.

4.1 Connecting Both Theories

One of the main contributions of this paper will be to show how the extensive literature on Minkowski decomposition of convex sets can be related to the search for extreme lower previsions in imprecise probability theory. The results in this section take the first step towards doing so, and will turn out to be crucial for our main theorem further on.

We start by associating with any compact set $A \subseteq \mathbb{R}^{n-1}$ a point $m(A) \in \mathbb{R}^{n-1}$, defined by $m_i(A) := \min\{v_i \colon v \in A\}$ for all $i \in \mathbb{N}_{<n}$ and a real number $\mu(A)$, given by

$$\mu(A) := \max\left\{\sum_{i=1}^{n-1} v_i \colon v \in A\right\} - \sum_{i=1}^{n-1} m_i(A). \tag{20}$$

Both $m(A)$ and $\mu(A)$ are well-defined due to the compactness of A. If A is not a singleton, then it is easy to see that $\mu(A) > 0$ and we can define

$$\underline{\underline{A}} := \frac{1}{\mu(A)}\big(A - m(A)\big) = \left\{\frac{1}{\mu(A)}(v - m(A)) \colon v \in A\right\}. \tag{21}$$

Proposition 4. *For any compact convex subset A of $\mathbb{R}^{n-1}$ that is not a singleton, the corresponding set $\underline{\underline{A}}$ is an element of $\underline{\underline{\mathbb{K}}}(\mathscr{X})$.*

Proposition 5. *A compact convex subset A of $\mathbb{R}^{n-1}$ that is not a singleton is Minkowski decomposable iff the corresponding set $\underline{\underline{A}}$ is Minkowski decomposable.*

The following result shows how the transformation that we have just introduced can be usefully exploited to reformulate the property of Minkowski decomposability.

Theorem 1. *A compact convex subset A of $\mathbb{R}^{n-1}$ that is not a singleton is Minkowski decomposable iff its corresponding set $\underline{\underline{A}}$ can be written as a non-trivial convex combination $\lambda K_1 + (1-\lambda)K_2$, with K_1 and K_2 both elements of $\underline{\underline{\mathbb{K}}}(\mathscr{X})$, $K_1 \neq K_2$ and $0 < \lambda < 1$.*

5 Characterising Extreme Lower Previsions

We now have all the tools needed to characterise the set $\operatorname{ext}\underline{\mathbb{P}}(\mathscr{X})$ of all extreme lower previsions on $\mathscr{G}(\mathscr{X})$, or equivalently, the set $\operatorname{ext}\underline{\mathbb{M}}(\mathscr{X})$ of all extreme credal sets. We will show that partially imprecise lower previsions are never extreme as they can be split up in a linear and a fully imprecise part. The only extreme linear previsions are the degenerate ones, and the extreme fully imprecise models will turn out to be closely related to the Minkowski indecomposable convex compact sets of Section 4.

5.1 Partially Imprecise Lower Previsions

We claimed earlier on in Section 2.2 that every partially imprecise lower prevision can be uniquely decomposed in a linear and a fully imprecise part. To see why this is true, first consider the following proposition, which is the counterpart of that statement in the language of credal sets. The desired property is then a direct consequence of this result.

Proposition 6. *Any partially imprecise credal set $\mathscr{M} \in \underset{\sim}{\mathbb{M}}(\mathscr{X})$ can be uniquely written as a convex combination $\lambda\mathscr{M}_1 + (1-\lambda)\mathscr{M}_2$ of a credal set $\mathscr{M}_1 \in \mathbb{M}(\mathscr{X})$ that contains only a single mass function $p_1 \in \Sigma_{\mathscr{X}}$ and a fully imprecise credal set $\mathscr{M}_2 \in \underline{\underline{\mathbb{M}}}(\mathscr{X})$. Moreover, $0 < \lambda := \sum_{i=1}^{n} \min\{p(x_i) : p \in \mathscr{M}\} < 1$, the mass function p_1 that characterises $\mathscr{M}_1$ is given by $p_1(x_i) = \frac{1}{\lambda}\min\{p(x_i) : p \in \mathscr{M}\}$ for all $i \in \mathbb{N}_{\leq n}$, and*

$$\mathscr{M}_2 = \left\{ \frac{1}{1-\lambda} p - \frac{\lambda}{1-\lambda} p_1 : p \in \mathscr{M} \right\}. \tag{22}$$

Corollary 4. *Any partially imprecise lower prevision $\underline{P} \in \underset{\sim}{\mathbb{P}}(\mathscr{X})$ can be uniquely written as a convex combination $\lambda P_1 + (1-\lambda)\underline{P}_2$ of a linear prevision $P_1 \in \mathbb{P}(\mathscr{X})$ and a fully imprecise lower prevision $\underline{P}_2 \in \underline{\underline{\mathbb{P}}}(\mathscr{X})$. Moreover, $0 < \lambda := \sum_{i=1}^{n} \underline{P}(\mathbb{I}_{\{x_i\}}) < 1$ and*

$$P_1(f) = \frac{1}{\lambda}\sum_{i=1}^{n} f(x_i)\underline{P}(\mathbb{I}_{\{x_i\}}) \text{ and } \underline{P}_2(f) = \frac{1}{1-\lambda}\underline{P}(f) - \frac{\lambda}{1-\lambda}P_1(f) \text{ for all } f \in \mathscr{G}(\mathscr{X}). \tag{23}$$

The fact that partially imprecise models can be decomposed in this way has some immediate important consequences for extreme credal sets and lower previsions.

Corollary 5. *Extreme credal sets and lower previsions are never partially imprecise:*

$$\mathscr{M} \in \underset{\sim}{\mathbb{M}}(\mathscr{X}) \Rightarrow \mathscr{M} \notin \operatorname{ext}\underline{\mathbb{M}}(\mathscr{X}) \text{ and } \underline{P} \in \underset{\sim}{\mathbb{P}}(\mathscr{X}) \Rightarrow \underline{P} \notin \operatorname{ext}\underline{\mathbb{P}}(\mathscr{X}). \tag{24}$$

In our search for extreme lower previsions, we therefore only need to look at the subsets of the linear previsions and of the fully imprecise lower previsions.

5.2 Linear Previsions

A special class of linear previsions are those that correspond to degenerate mass functions. For every $i \in \mathbb{N}_{\leq n}$, the corresponding *degenerate mass function* $p_i^\circ \in \Sigma_{\mathscr{X}}$ has all its probability mass in x_i and is therefore defined by $p_i^\circ := \mathbb{I}_{\{x_i\}}$. They satisfy the following important property.

Proposition 7. *A credal set $\mathscr{M} \in \mathbb{M}(\mathscr{X})$ containing only a single mass function is extreme iff that single mass function is degenerate. Furthermore, any other mass function can be written as a convex combination of those degenerate ones.*

The linear previsions that correspond to such a degenerate mass function are called *degenerate linear previsions*. For every $i \in \mathbb{N}_{\leq n}$, we have a corresponding degenerate linear prevision P_i°, defined for all $f \in \mathscr{G}(\mathscr{X})$ by $P_i^\circ(f) := f(x_i)$. As a direct consequence of Proposition 7, we find that these degenerate linear previsions are the only linear previsions that are extreme.

Corollary 6. *A linear prevision $P \in \underline{\mathbb{P}}(\mathscr{X})$ is extreme iff it is degenerate. Furthermore, any other linear prevision can be written as a convex combination of degenerate ones.*

For coherent lower previsions that are defined on a finite domain $\mathscr{K} \subset \mathscr{G}(\mathscr{X})$, a result that combines Corollary 5 and 6 was already mentioned in Ref. [5, Proposition 1].

5.3 Fully Imprecise Lower Previsions

So far, we have shown that partially imprecise models are never extreme and that the extreme linear models are those that are degenerate. The only models that are thus left to investigate are those that are fully imprecise. We start with a property of decompositions of fully imprecise credal sets.

Proposition 8. *If a fully imprecise credal set $\mathscr{M} \in \underline{\underline{\mathbb{M}}}(\mathscr{X})$ can be written as a non-trivial convex combination $\lambda\mathscr{M}_1 + (1-\lambda)\mathscr{M}_2$, with $\mathscr{M}_1, \mathscr{M}_2 \in \underline{\mathbb{M}}(\mathscr{X})$, $\mathscr{M}_1 \neq \mathscr{M}_2$ and $0 < \lambda < 1$, then $\mathscr{M}_1$ and $\mathscr{M}_2$ are both fully imprecise and therefore elements of $\underline{\underline{\mathbb{M}}}(\mathscr{X})$.*

In the language of coherent lower previsions, this turns into the following corollary.

Corollary 7. *If a fully imprecise coherent lower prevision $\underline{P} \in \underline{\underline{\mathbb{P}}}(\mathscr{X})$ can be written as a non-trivial convex combination $\lambda\underline{P}_1 + (1-\lambda)\underline{P}_2$, with $\underline{P}_1, \underline{P}_2 \in \underline{\mathbb{P}}(\mathscr{X})$, $\underline{P}_1 \neq \underline{P}_2$ and $0 < \lambda < 1$, then $\underline{P}_1$ and $\underline{P}_2$ are both fully imprecise and therefore elements of $\underline{\underline{\mathbb{P}}}(\mathscr{X})$.*

Combined with Proposition 3 and Theorem 1, Proposition 8 leads to a crucial result.

Theorem 2. *A fully imprecise credal set $\mathscr{M} \in \underline{\underline{\mathbb{M}}}(\mathscr{X})$ can be written as a non-trivial convex combination $\lambda\mathscr{M}_1 + (1-\lambda)\mathscr{M}_2$, with $\mathscr{M}_1, \mathscr{M}_2 \in \underline{\mathbb{M}}(\mathscr{X})$, $\mathscr{M}_1 \neq \mathscr{M}_2$ and $0 < \lambda < 1$ iff its projected credal set $K_{\mathscr{M}}$ is Minkowski decomposable.*

When stated in terms of coherent lower previsions, this result looks as follows.

Corollary 8. *A fully imprecise coherent lower prevision $\underline{P} \in \underline{\underline{\mathbb{P}}}(\mathscr{X})$ can be written as a non-trivial convex combination $\lambda\underline{P}_1 + (1-\lambda)\underline{P}_2$, with $\underline{P}_1, \underline{P}_2 \in \underline{\mathbb{P}}(\mathscr{X})$, $\underline{P}_1 \neq \underline{P}_2$ and $0 < \lambda < 1$ iff its projected credal set $K_{\underline{P}}$ is Minkowski decomposable.*

The importance of these two results is that they provide us with an easy characterisation of the extreme models that are fully imprecise.

Corollary 9. *A fully imprecise credal set $\mathscr{M} \in \underline{\underline{\mathbb{M}}}(\mathscr{X})$ is extreme iff its projected credal set $K_{\mathscr{M}}$ is Minkowski indecomposable. Equivalently, a fully imprecise lower prevision $\underline{P} \in \underline{\underline{\mathbb{P}}}(\mathscr{X})$ is extreme iff its projected credal set $K_{\underline{P}}$ is Minkowski indecomposable.*

These alternative characterisations of fully imprecise extreme credal sets and lower previsions will allow us to import known results from the literature on Minkowski decomposability, using them to find the sets $\operatorname{ext}\underline{\mathbb{M}}(\mathscr{X})$ and $\operatorname{ext}\underline{\mathbb{P}}(\mathscr{X})$, containing all extreme credal sets and lower previsions respectively.

To conclude this section, we want to mention a very special fully imprecise credal set. It contains every single mass function in $\Sigma_{\mathscr{X}}$ and will be denoted as $\mathscr{M}_V := \Sigma_{\mathscr{X}}$. It is used to model complete ignorance and is called the *vacuous* credal set. The corresponding (fully imprecise) lower prevision $\underline{P}_V$ is referred to as the *vacuous* lower prevision and is given, for all $f \in \mathscr{G}(\mathscr{X})$, by $\underline{P}_V(f) = \min f$.

Proposition 9. *The vacuous credal set is extreme: $\mathscr{M}_V \in \operatorname{ext}\underline{\mathbb{M}}(\mathscr{X})$.*

Corollary 10. *The vacuous lower prevision is extreme: $\underline{P}_V \in \operatorname{ext}\underline{\mathbb{P}}(\mathscr{X})$.*

6 Finding All Extreme Lower Previsions

The size of $\operatorname{ext}\underline{\mathbb{M}}(\mathscr{X})$ and $\operatorname{ext}\underline{\mathbb{P}}(\mathscr{X})$ and the complexity of their elements, turns out to depend heavily on the number of elements in the possibility space $\mathscr{X} = \{x_1, \dots, x_n\}$. We consider three distinct cases: $n = 2$, $n = 3$ and $n > 3$. We focus on constructing $\operatorname{ext}\underline{\mathbb{M}}(\mathscr{X})$, since $\operatorname{ext}\underline{\mathbb{P}}(\mathscr{X})$ can be derived from it by applying Corollary 2.

6.1 Possibility Spaces with Two States

For $n = 2$, constructing $\operatorname{ext}\underline{\mathbb{M}}(\mathscr{X})$ is almost trivial. Nevertheless, it serves as a good didactic exercise to get to know the basic tools in this paper.

It follows from the results in Section 5 that in our search for the extreme credal sets, we do not need to consider the partially imprecise ones. It suffices to look at the precise and the fully imprecise credal sets. We know from Proposition 7 that of all the precise credal sets (those consisting of only a single mass function) the only extreme ones are those that correspond to a degenerate mass function. In the current binary case, with $\mathscr{X} = \{x_1, x_2\}$, this yields the extreme credal sets $\mathscr{M}_1^\circ := \{p_1^\circ\}$ and $\mathscr{M}_2^\circ := \{p_2^\circ\}$. All other extreme credal sets will be fully imprecise. We know from Proposition 9 that $\mathscr{M}_V$ is one of those fully imprecise extreme credal sets, but finding the other ones would normally require the use of Corollary 9. However, in this simple binary case, $\mathscr{M}_V$ is the only fully imprecise credal set (we leave the simple proof of this statement as an exercise for the reader) and we can therefore conclude that for binary possibility spaces:

$$\operatorname{ext}\underline{\mathbb{M}}(\mathscr{X}) = \{\mathscr{M}_1^\circ, \mathscr{M}_2^\circ, \mathscr{M}_V\}. \tag{25}$$

By applying Corollary 2, we obtain the corresponding result for lower previsions:

$$\operatorname{ext}\underline{\mathbb{P}}(\mathscr{X}) = \{P_1^\circ, P_2^\circ, \underline{P}_V\}. \tag{26}$$

6.2 Possibility Spaces with Three States

For $n = 3$, finding $\operatorname{ext}\underline{\mathbb{M}}(\mathscr{X})$ becomes a bit more involved. As always, the partially imprecise credal sets are never extreme and the only precise extreme credal sets are the degenerate ones. Finding the fully imprecise credal sets that are extreme is however more difficult then it was in the binary case. Here, the vacuous credal set $\mathscr{M}_V$ will not be the only fully imprecise extreme credal set. In order to find the others, we rely on Corollary 9, using it to import the following result by Silverman into our framework.

Theorem 3 ([8, Theorem 4]). *A compact convex subset of $\mathbb{R}^2$ is Minkowski indecomposable if and only if it is a triangle or a line segment.*

This theorem is highly non-trivial since it holds for general compact convex subsets of $\mathbb{R}^2$ and not only for convex polygons. It allows us to derive the next result, which concludes our search for the extreme credal sets of ternary possibility spaces.

Corollary 11. *For possibility spaces $\mathscr{X} = \{x_1, x_2, x_3\}$ containing only three elements, a fully imprecise credal set $\mathscr{M} \in \underline{\underline{\mathbb{M}}}(\mathscr{X})$ is extreme if and only if it is the convex closure of three probability mass functions: we can find $p_1, p_2, p_3 \in \Sigma_{\mathscr{X}}$ such that*

$$\mathscr{M} = \left\{ \sum_{i=1}^{3} \lambda_i p_i \colon (\lambda_1, \lambda_2, \lambda_3) \in \Sigma_{\mathscr{X}} \right\}. \tag{27}$$

Figure 1 should provide this result with some intuition. It presents an example of a fully imprecise credal set with four vertices and its decomposition into two extreme ones with three vertices. We depict the credal sets using the well-known simplex representation [9, Section 4.2.3].

In order to obtain the extreme lower previsions of a ternary possibility space, all we need to do now is apply Corollary 2. We find that apart from the three degenerate linear previsions P_1°, P_2° and P_3°, all other extreme lower previsions are characterised by the following translation of Corollary 11.

Corollary 12. *For possibility spaces $\mathscr{X} = \{x_1, x_2, x_3\}$ containing only three elements, a fully imprecise lower prevision $\underline{P} \in \underline{\underline{\mathbb{M}}}(\mathscr{X})$ is extreme if and only if it is the lower envelope of three linear previsions: one can find $P_1, P_2, P_3 \in \mathbb{P}(\mathscr{X})$ such that*

$$\underline{P}(f) = \min_{i \in \mathbb{N}_{\leq 3}} P_i(f) \text{ for all } f \in \mathscr{G}(\mathscr{X}). \tag{28}$$

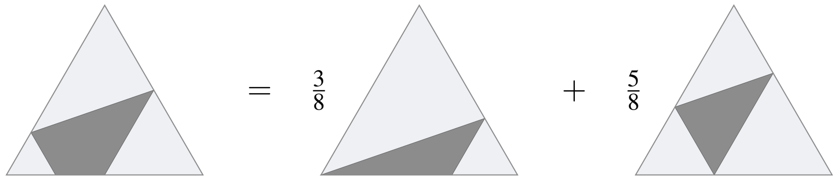

Figure 1. Decomposition of a fully imprecise credal set into two extreme ones

6.3 General Possibility Spaces

Due to the page limit constraint, we are not able to discuss the case $n > 3$ in full detail. In contrast to the cases $n = 2$ and $n = 3$, we will not construct the set of all extreme credal sets. It should however be clear that all extreme credal sets will again be fully imprecise, except for the degenerate precise ones. We restrict ourselves to stating some relevant results from the theory of Minkowski decomposability. Their implications for extreme credal sets (and thus also extreme lower previsions) are fairly intuitive, but we defer any more formal result to future work.

We know from Corollary 9 that fully imprecise extreme credal sets correspond to Minkowski indecomposable compact and convex subsets of $\mathbb{R}^{n-1}$. For $n = 3$, we were dealing with Minkowski indecomposability in the plane, which is completely determined by Theorem 3. In higher dimensions, Minkowski indecomposability is not yet fully resolved in the literature.

Most known results deal only with polytopes. Grünbaum [1, Chapter 15] provides a good summary, explaining (amongst other interesting results) why every simplicial polytope is indecomposable and every simple polytope, with the exception of a simplex, is decomposable. Meyer [3, Theorem 3] provides two rather complicated algebraic conditions, which are both necessary and sufficient for a polytope to be indecomposable.

For non-polytopes, the most important reference seems to be Ref. [7], in which Sallee shows that a wide class of compact convex sets is decomposable, the only condition being that they have on their boundary a sufficiently smooth neighbourhood. However, unlike in the case of $\mathbb{R}^2$, in higher dimensions Minkowski indecomposable compact convex sets need not be polytopes.

7 Conclusions

We have shown that when $\mathscr{X}$ has a finite number n of elements, then the extreme coherent lower previsions on $\mathscr{G}(\mathscr{X})$ are either degenerate linear previsions or fully imprecise and in a one-to-one correspondence with (Minkowski) indecomposable compact convex subsets of $\mathbb{R}^{n-1}$. Using this connection, we have constructed the set of all extreme lower previsions for the cases $n = 2$ and $n = 3$ and suggested what these sets might look like for $n > 3$. For the case $n = 3$, we have found that a fully imprecise coherent lower prevision is extreme if and only if it is the lower envelope of three linear previsions.

A first and rather obvious avenue of future research would be to use the results mentioned in Section 6.3 to try and construct $\operatorname{ext}\underline{\mathbb{M}}(\mathscr{X})$ and $\operatorname{ext}\underline{\mathbb{P}}(\mathscr{X})$ if $n > 3$, or to at least get a better idea of what kind of elements they contain. Consider for example the case $n = 4$. Can one find non-degenerate extreme lower previsions that are not the lower envelope of four linear ones? And are fully imprecise lower previsions that are the lower envelope of four linear previsions always extreme? We intend to answer these questions in an extended journal version of this paper.

It would also be interesting to compare our results with those in Ref. [5], which concentrated on coherent lower previsions defined on finite domains, and Ref. [6], which investigated the even more particular case of extreme lower probabilities. We conjecture that our results subsume (at least some of) those obtained in Refs. [5] and [6], but a detailed study is beyond the scope of this conference paper. Ref. [6] also looked at the extreme points of sets formed by all lower probabilities that satisfy certain properties, such as k-monotonicity and permutation invariance. We suspect that our results can be adapted to conduct a similar study for extreme coherent lower previsions as well.

Finally, we would like to see to what extent extreme lower previsions can be used to tackle practical problems. One idea would be to adapt the existing algorithms for Minkowski decomposition to decompose coherent lower previsions into convex combinations of extreme ones. Such decompositions can then be used to approximate coherent lower previsions in such a way as to satisfy certain properties or to develop a generalisation of the so-called random set product from the theory of belief functions.

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