5th SIPTA school
on imprecise probability

16-20 July 2012, Pescara (Italy)

# State sequence estimation in imprecise hidden Markov models

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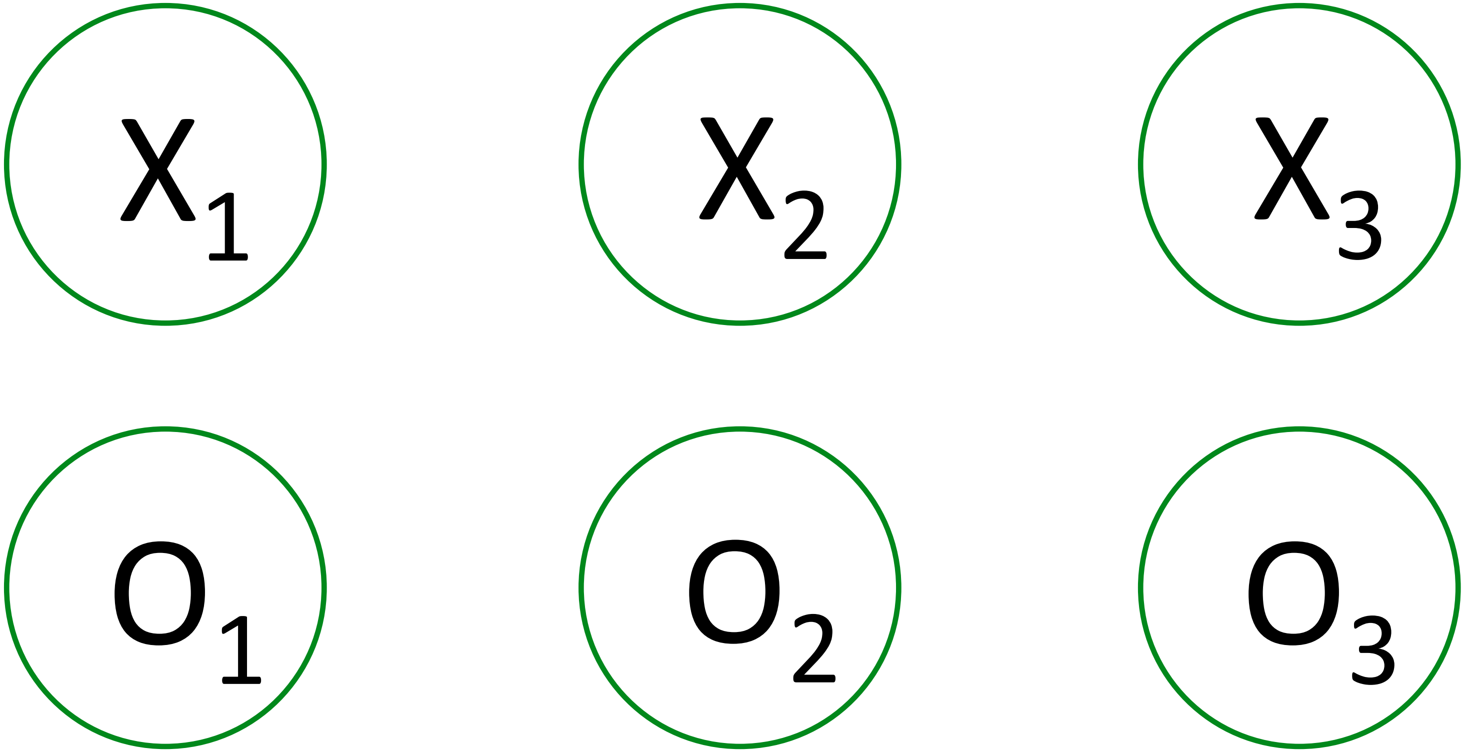
# State sequence estimation in imprecise hidden Markov models

# Precise hidden Markov model

A sequence of hidden state variables



A sequence of observable variables

# Precise hidden Markov model

A sequence of hidden state variables

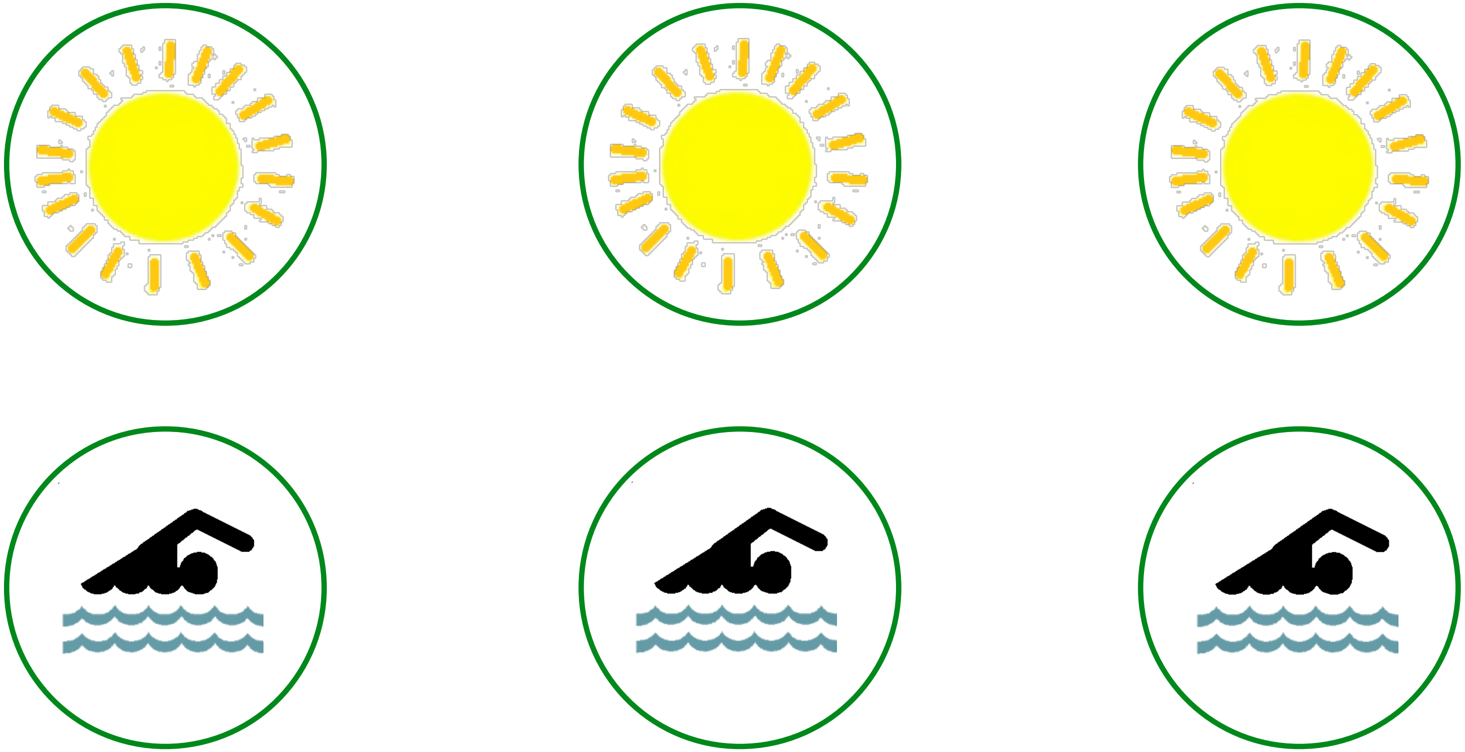
$X_i =$

$O_i =$

A sequence of observable variables

# Precise hidden Markov model

A sequence of hidden state variables



A sequence of observable variables

# Precise hidden Markov model

A sequence of hidden state variables

$$X = \quad X_1 \quad X_2 \quad X_3$$

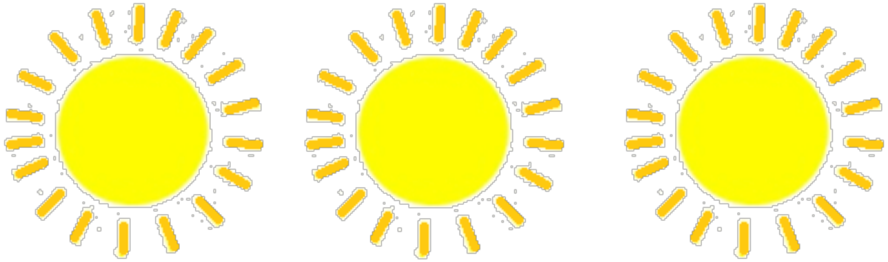
$$O = \quad O_1 \quad O_2 \quad O_3$$

A sequence of observable variables

# Precise hidden Markov model

A sequence of hidden state variables

X =

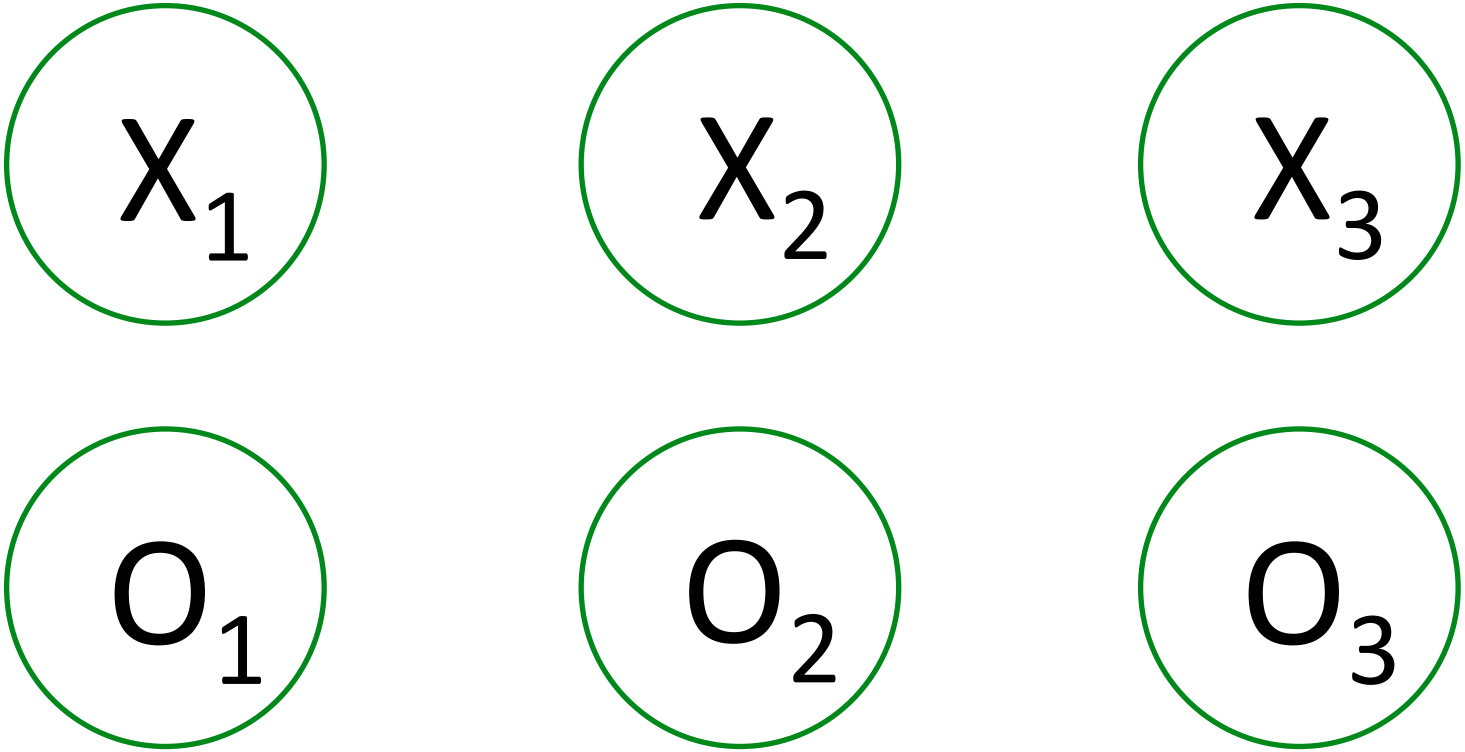


O =

A sequence of observable variables

# Precise hidden Markov model

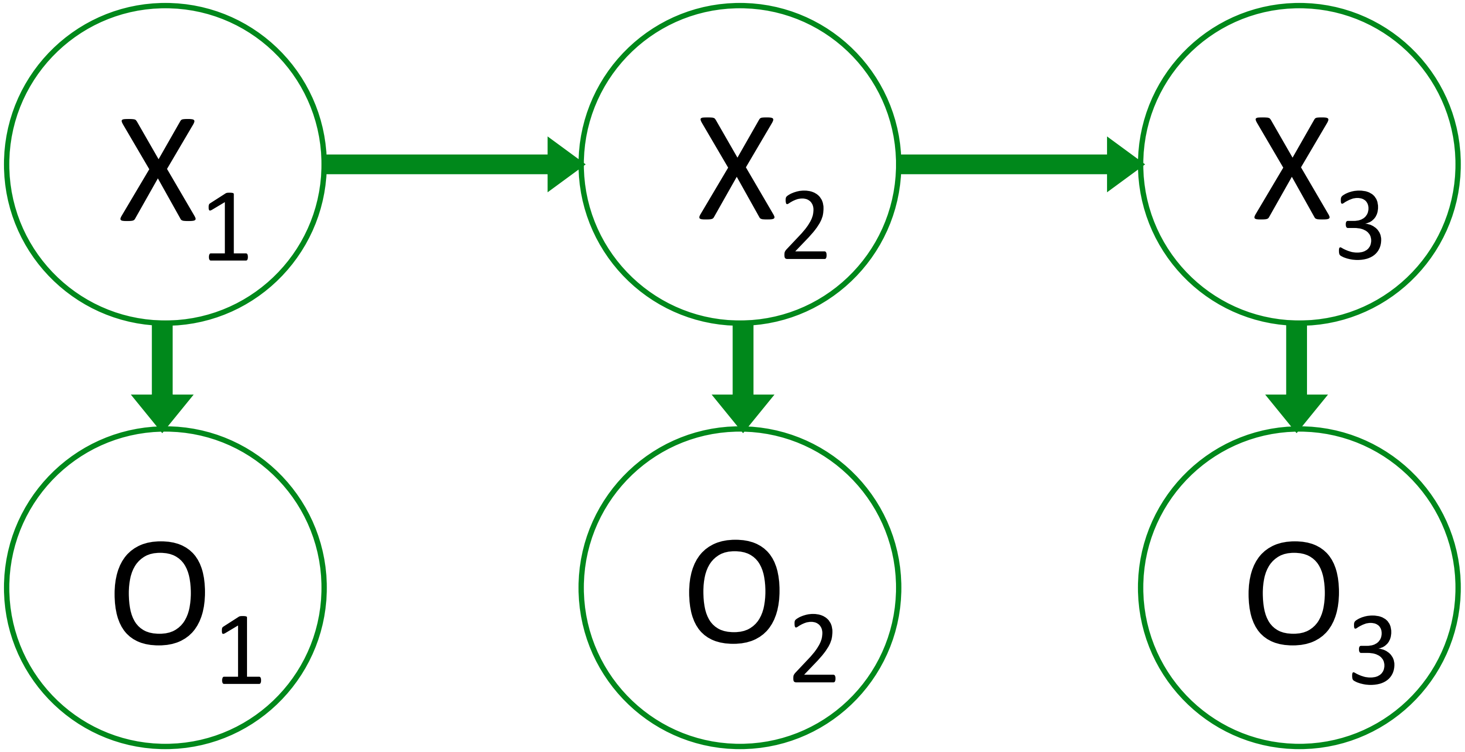
A sequence of hidden state variables



A sequence of observable variables

# Precise hidden Markov model

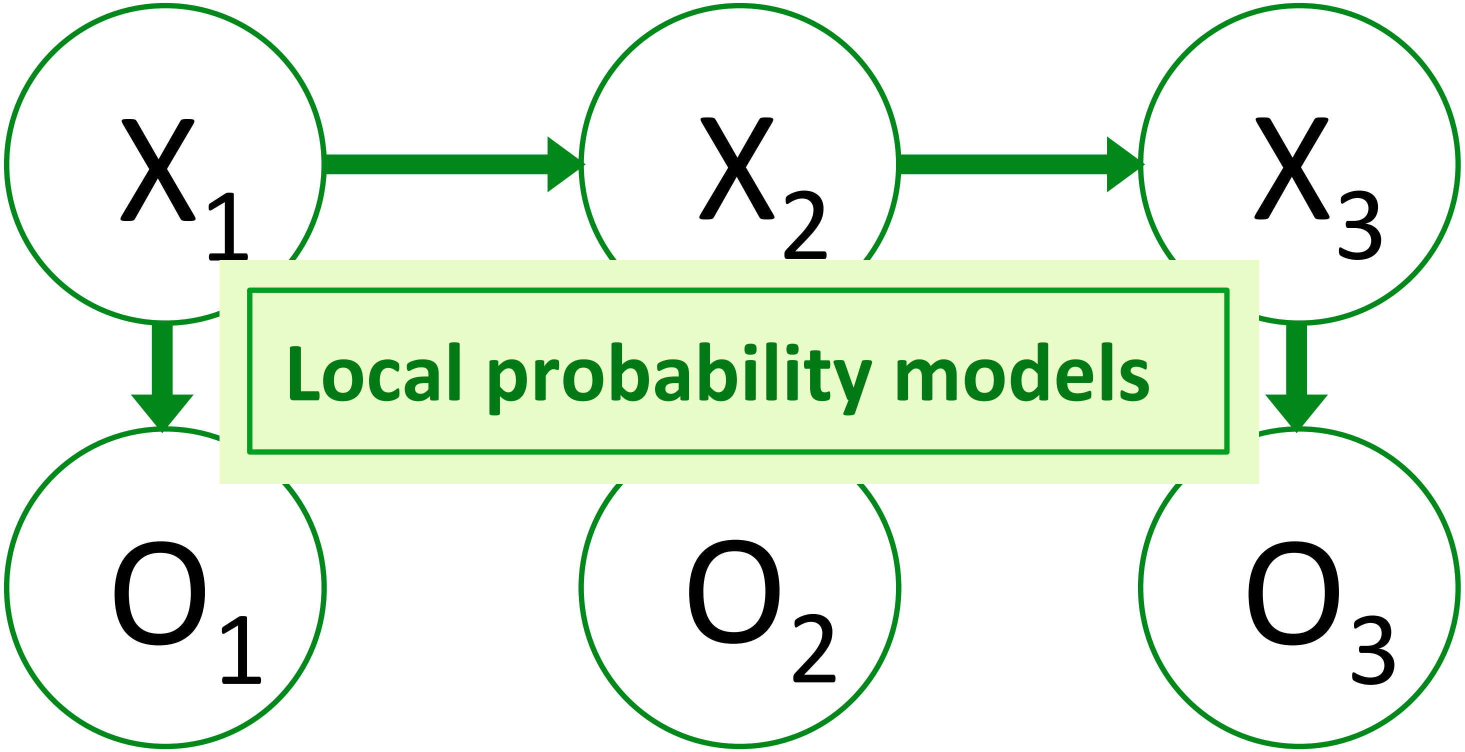
A sequence of hidden state variables



A sequence of observable variables

# Precise hidden Markov model

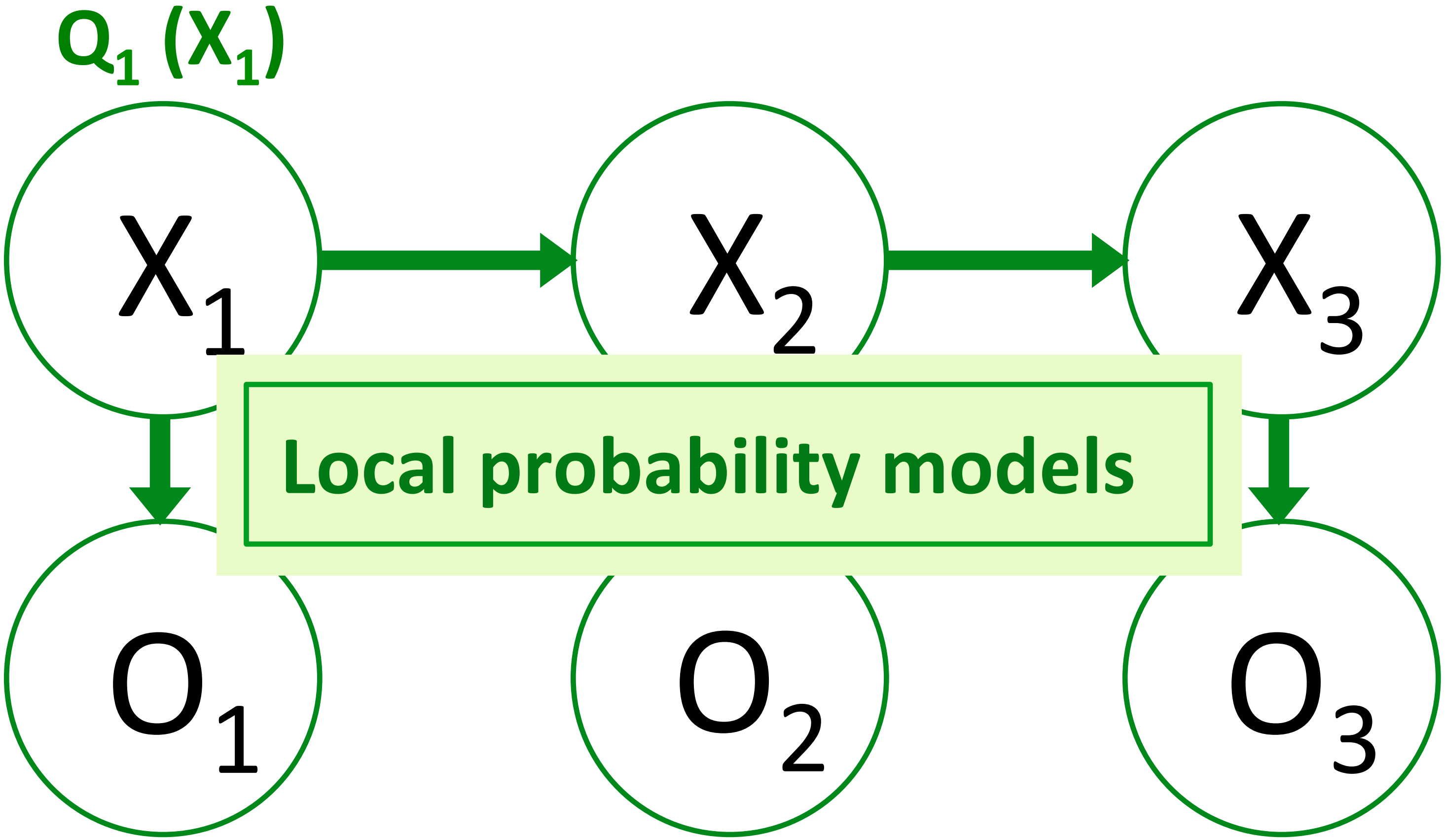
A sequence of hidden state variables



**Local probability models**

A sequence of observable variables

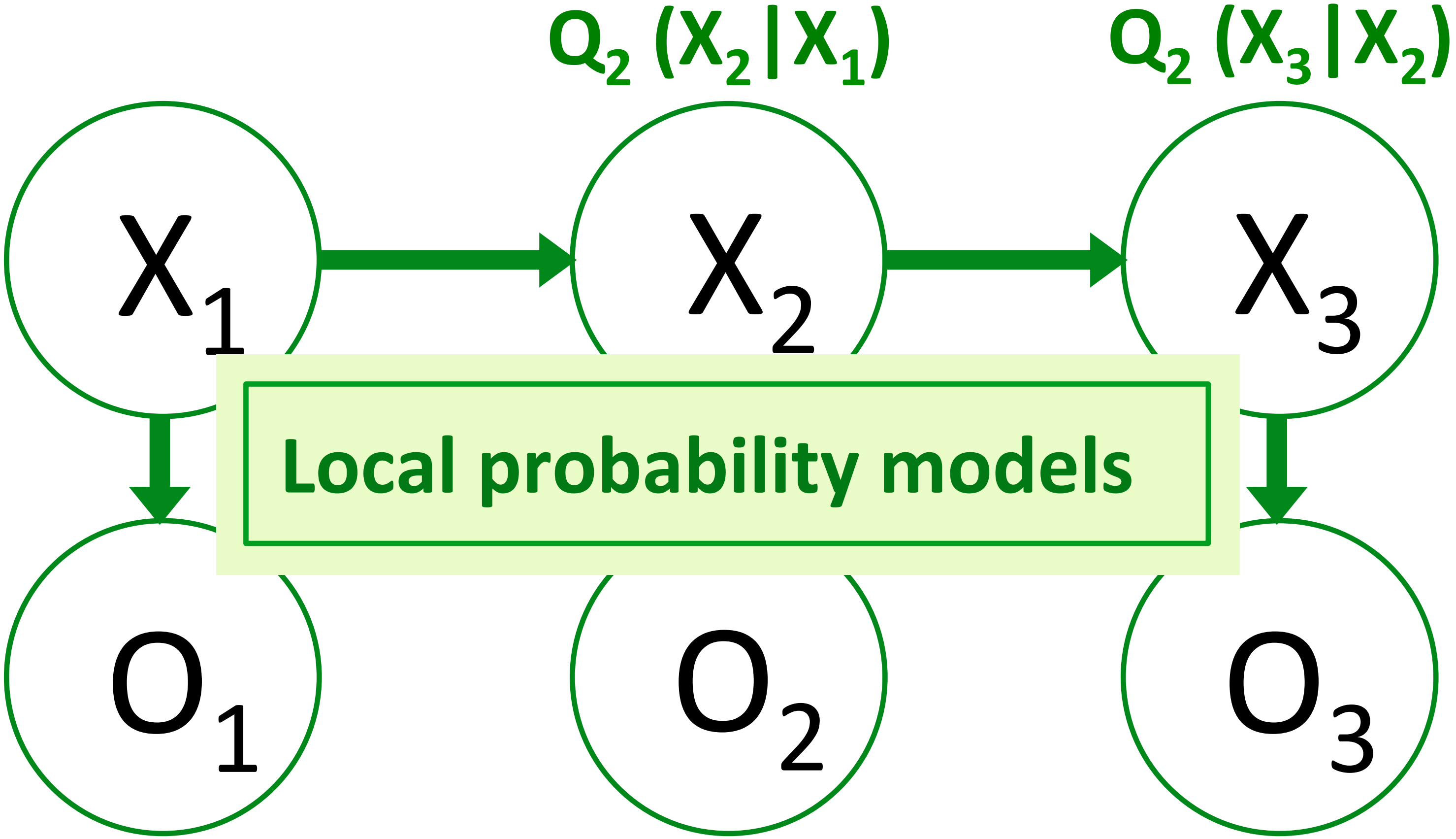
# Precise hidden Markov model



**Local probability models**

**Marginal model for the first hidden variable**

# Precise hidden Markov model



$Q_2\ (X_2|X_1)$ $Q_2\ (X_3|X_2)$

$X_1$ $X_2$ $X_3$

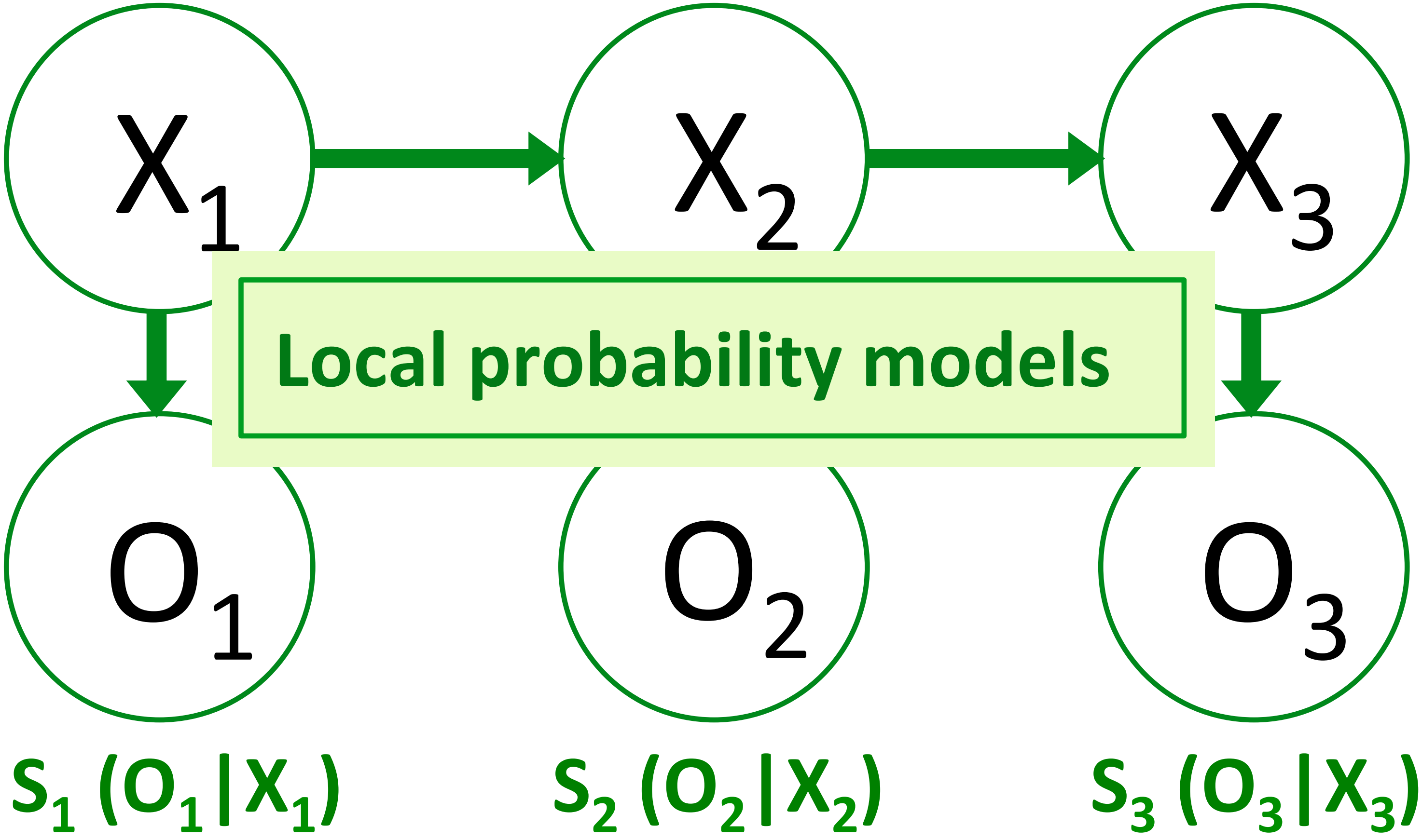
$O_1$ $O_2$ $O_3$

**Local probability models**

**Transition models for the next hidden variables**

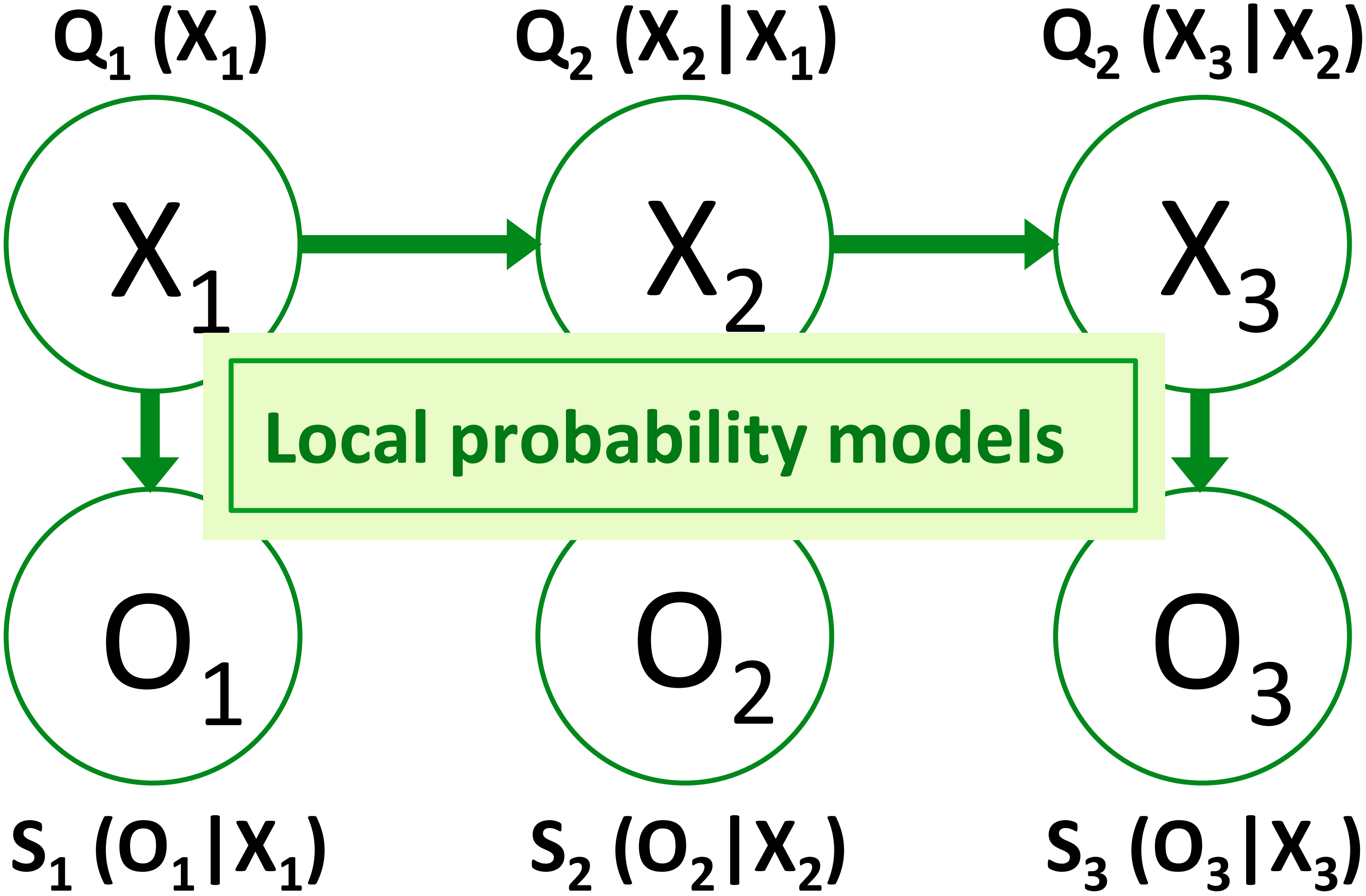
# Precise hidden Markov model

**Output models for the observable output variables**



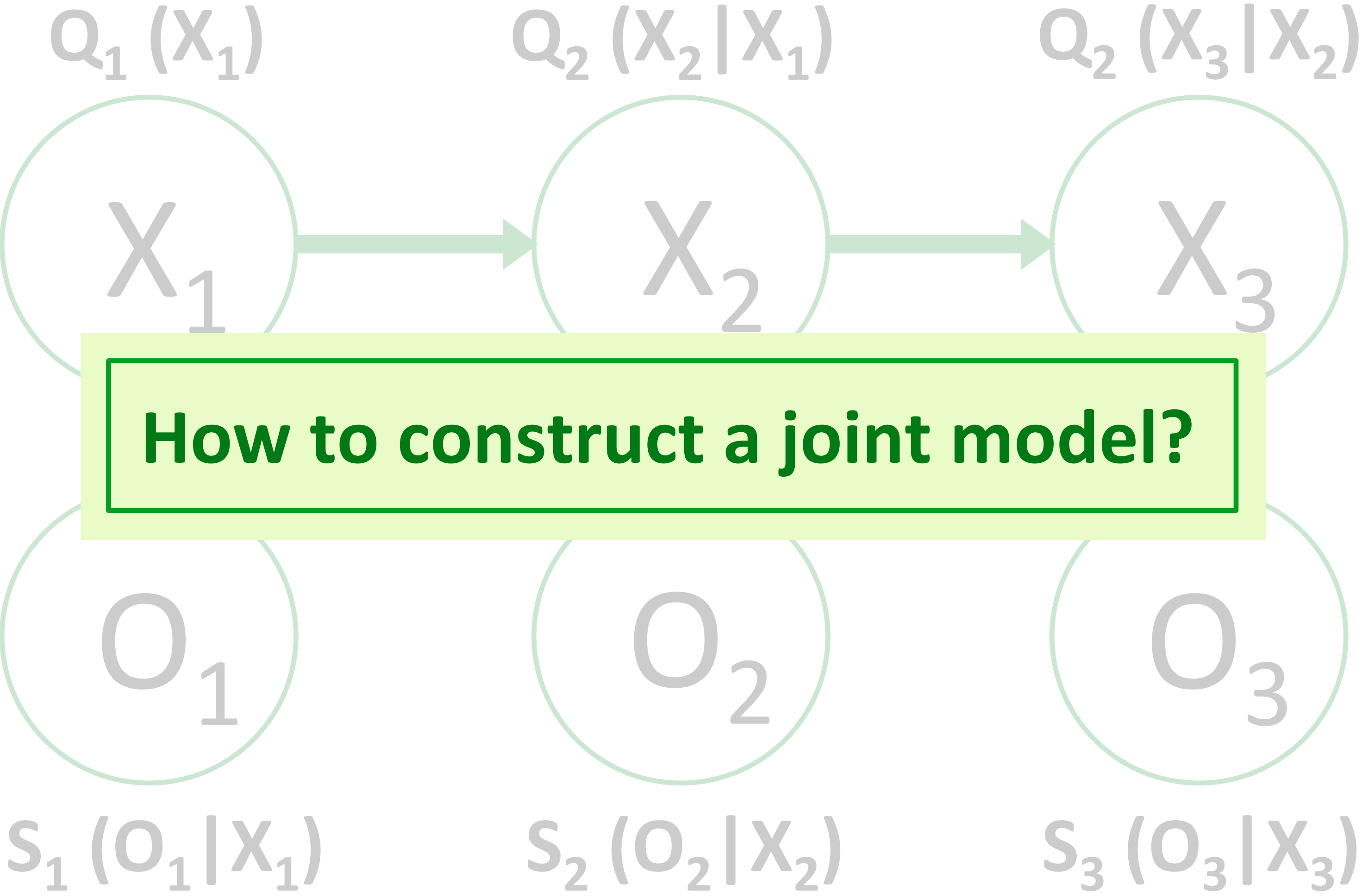
**Local probability models**

# Precise hidden Markov model



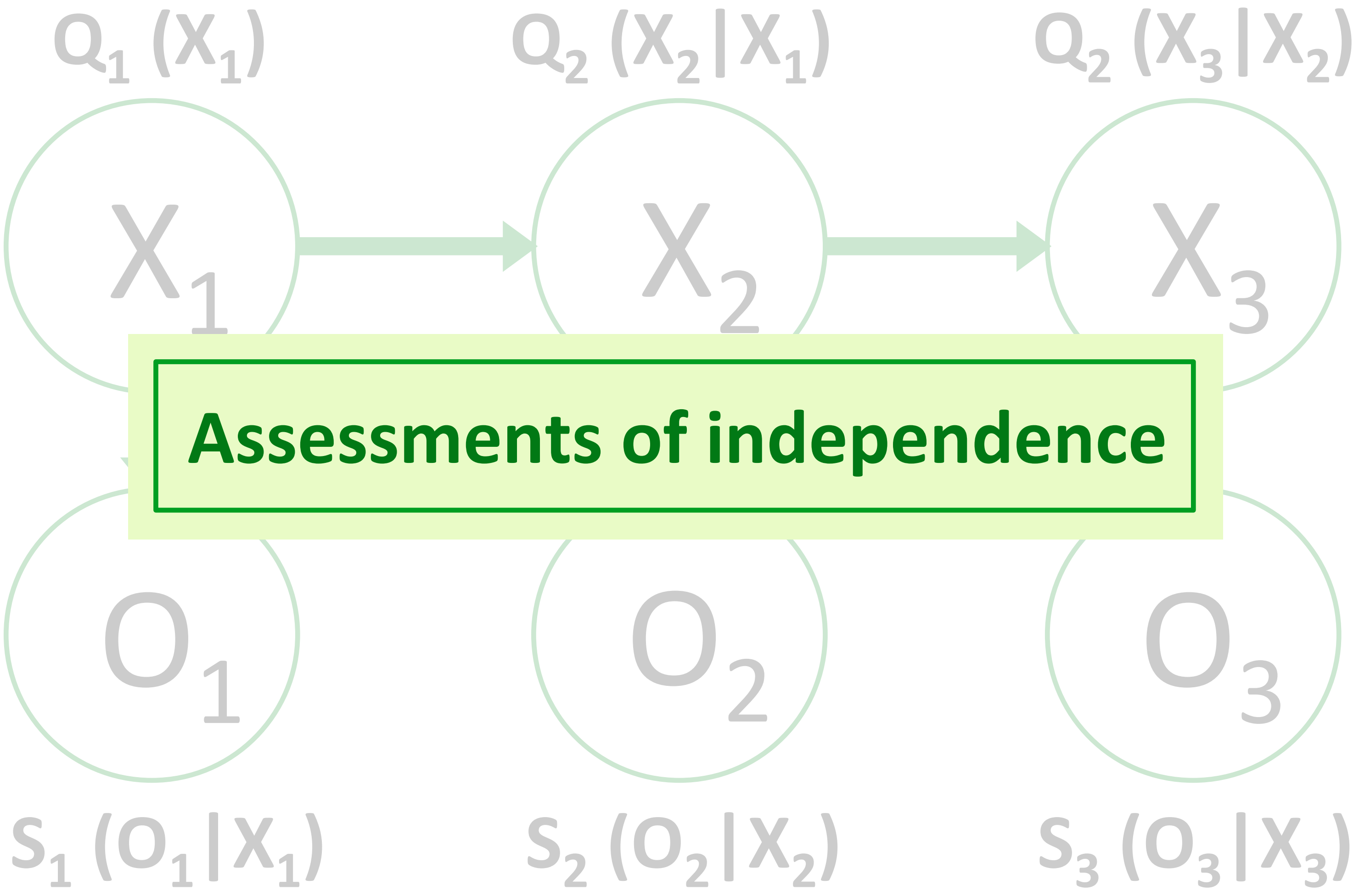
**Local probability models**

# Precise hidden Markov model



**How to construct a joint model?**

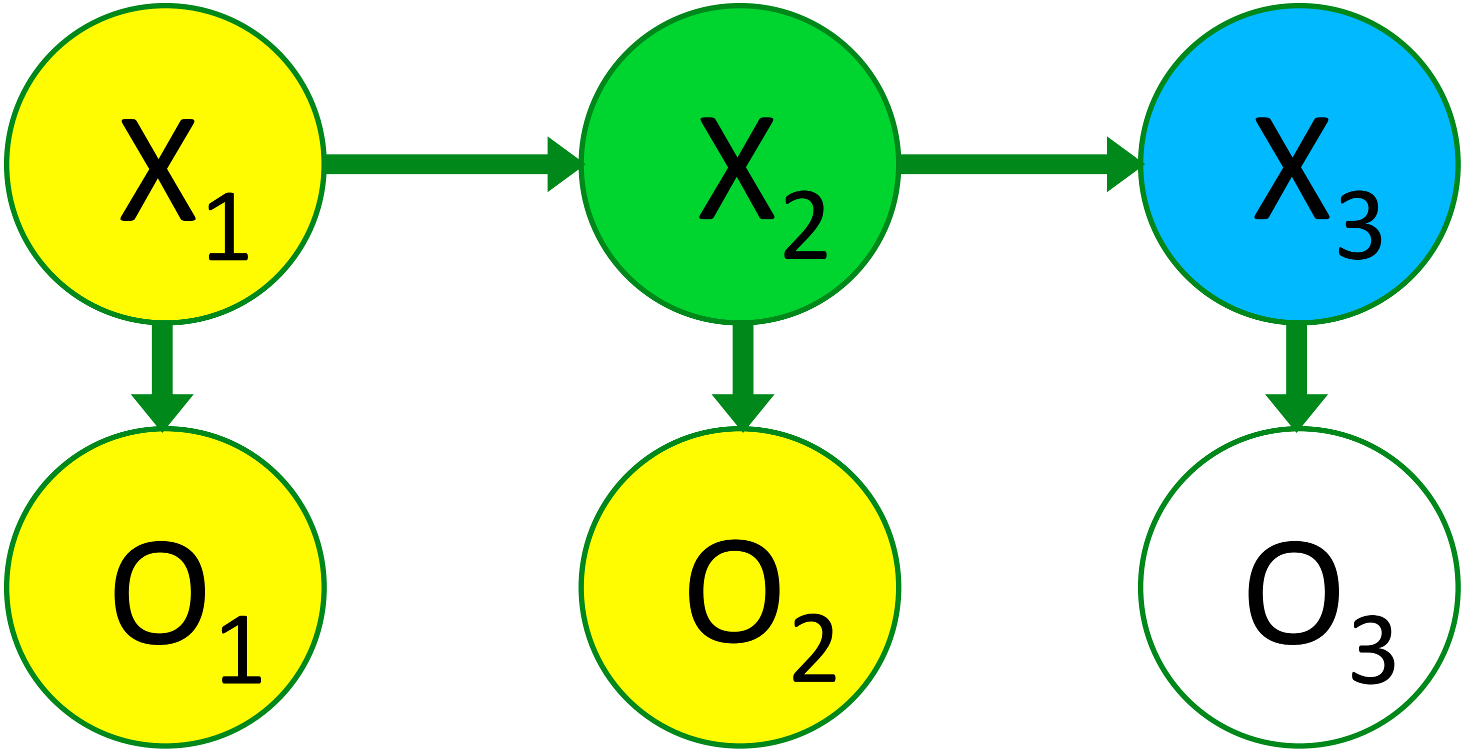
# Precise hidden Markov model



**Assessments of independence**

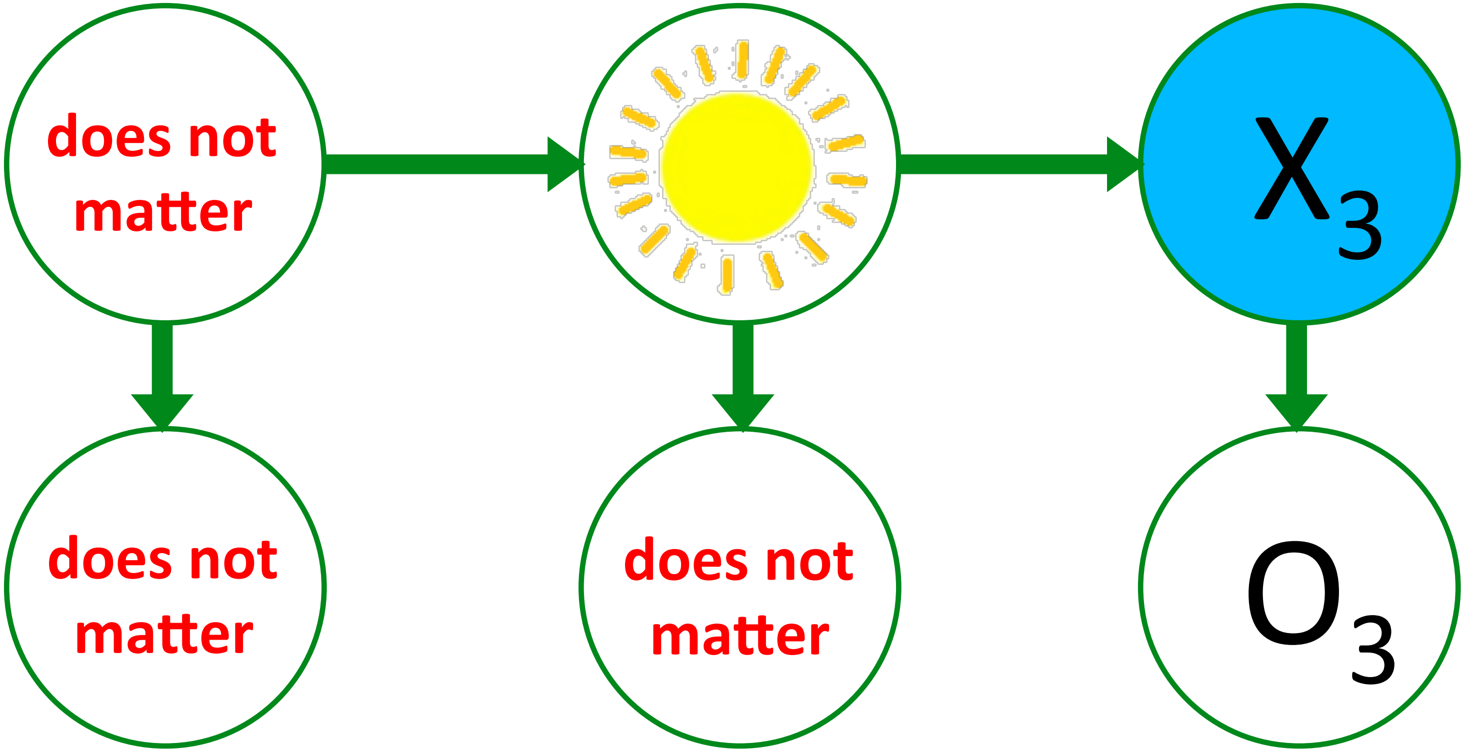
# Precise hidden Markov model

Conditional on its **mother variable**, any **variable** is **independent** of its **non-parent non-descendants**



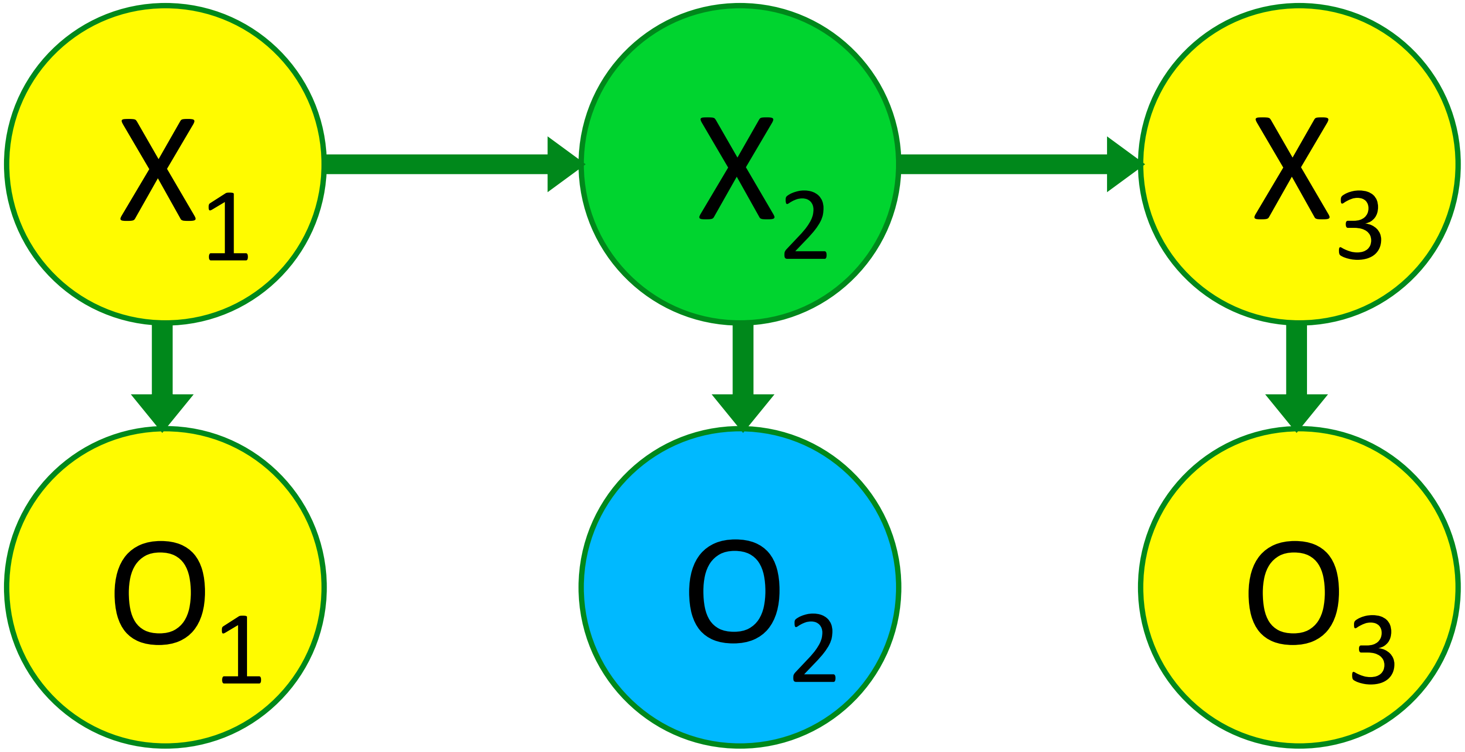
# Precise hidden Markov model

Conditional on its **mother variable**, any **variable** is **independent** of its **non-parent non-descendants**



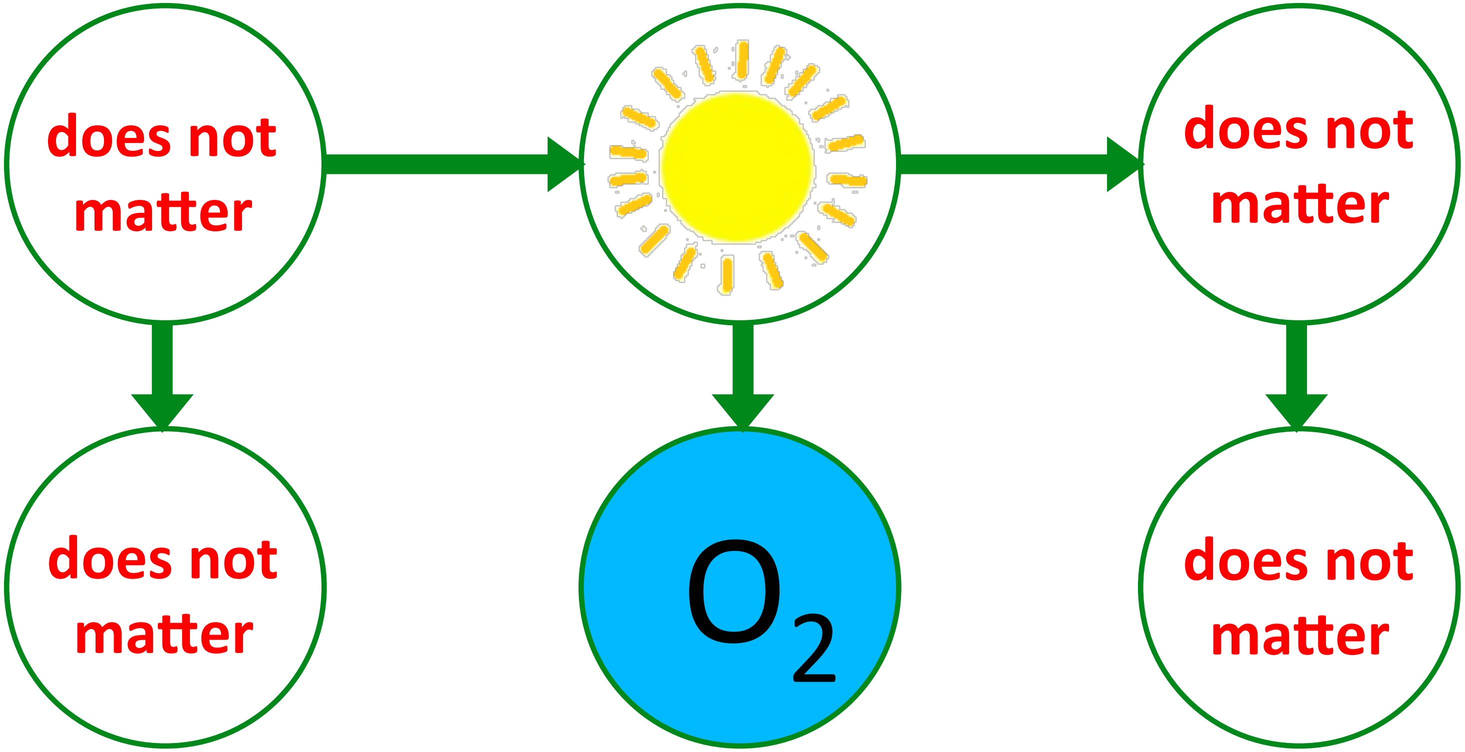
# Precise hidden Markov model

Conditional on its **mother variable**, any **variable** is **independent** of its **non-parent non-descendants**

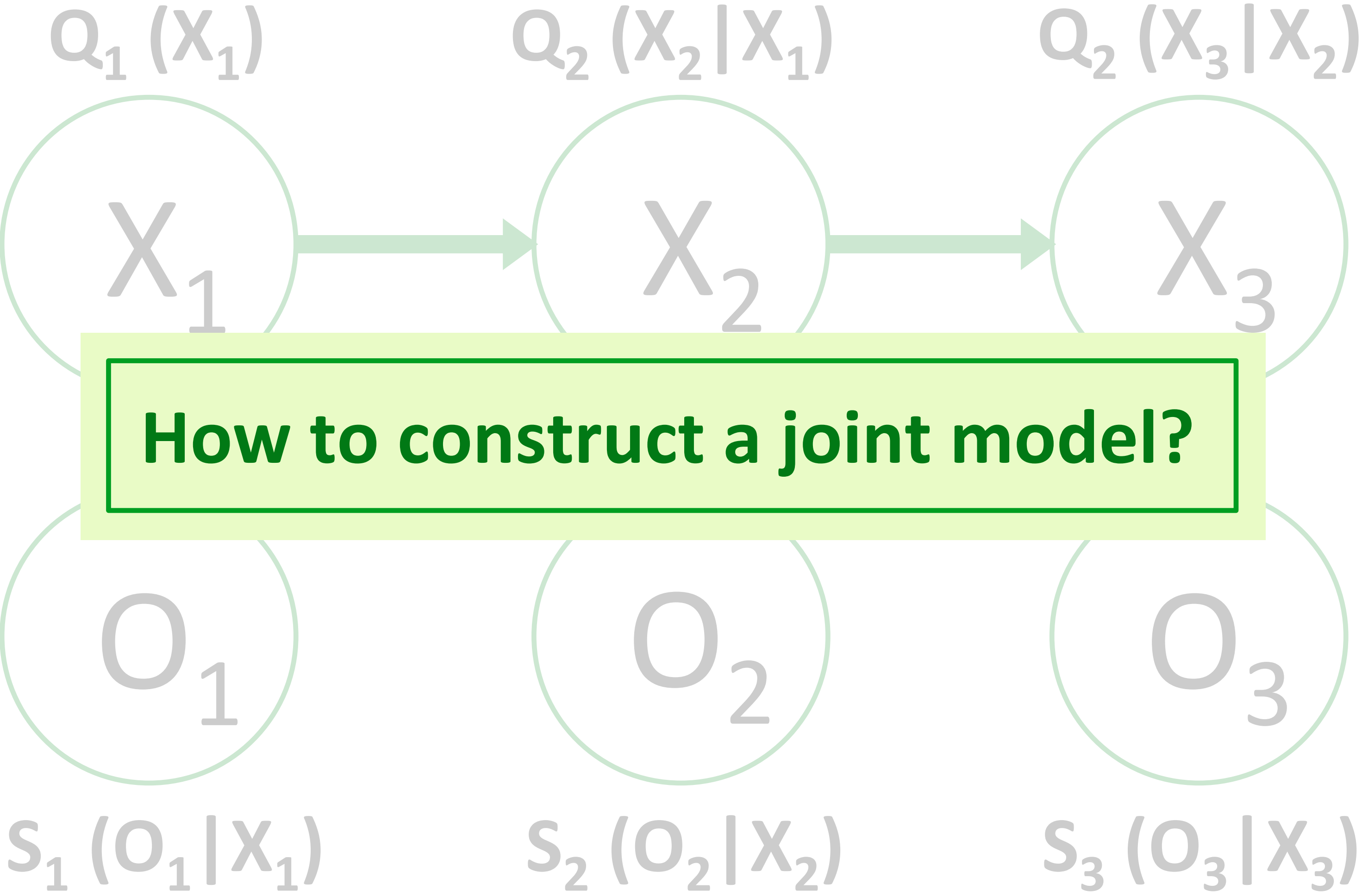


# Precise hidden Markov model

Conditional on its **mother variable**, any **variable** is **independent** of its **non-parent non-descendants**



# Precise hidden Markov model



**How to construct a joint model?**

# Precise hidden Markov model

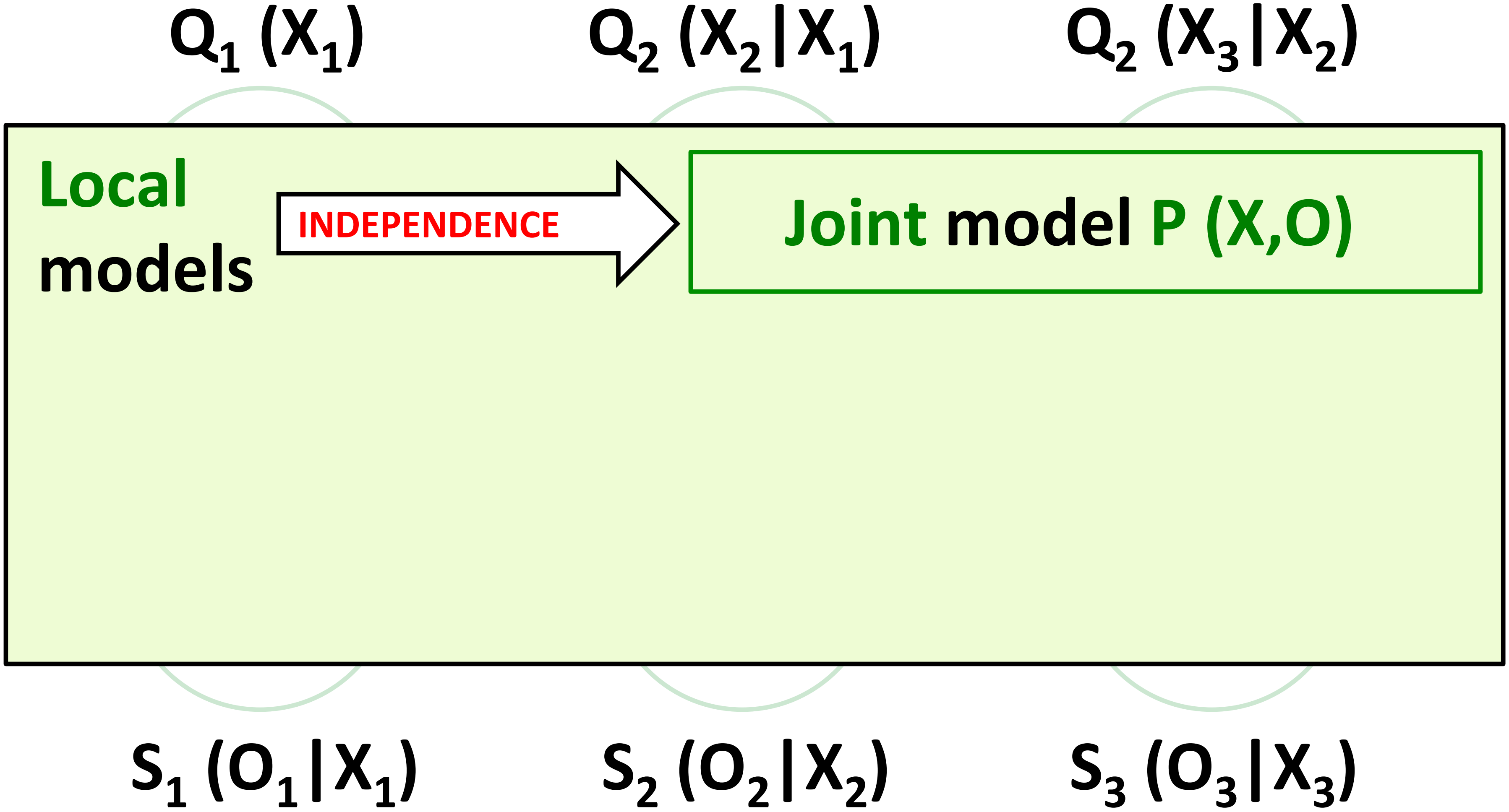
$Q_1\ (X_1)$ $Q_2\ (X_2|X_1)$ $Q_2\ (X_3|X_2)$

**Local**

**models**

$S_1\ (O_1|X_1)$ $S_2\ (O_2|X_2)$ $S_3\ (O_3|X_3)$

# Precise hidden Markov model



$Q_1\ (X_1)$ $Q_2\ (X_2|X_1)$ $Q_2\ (X_3|X_2)$

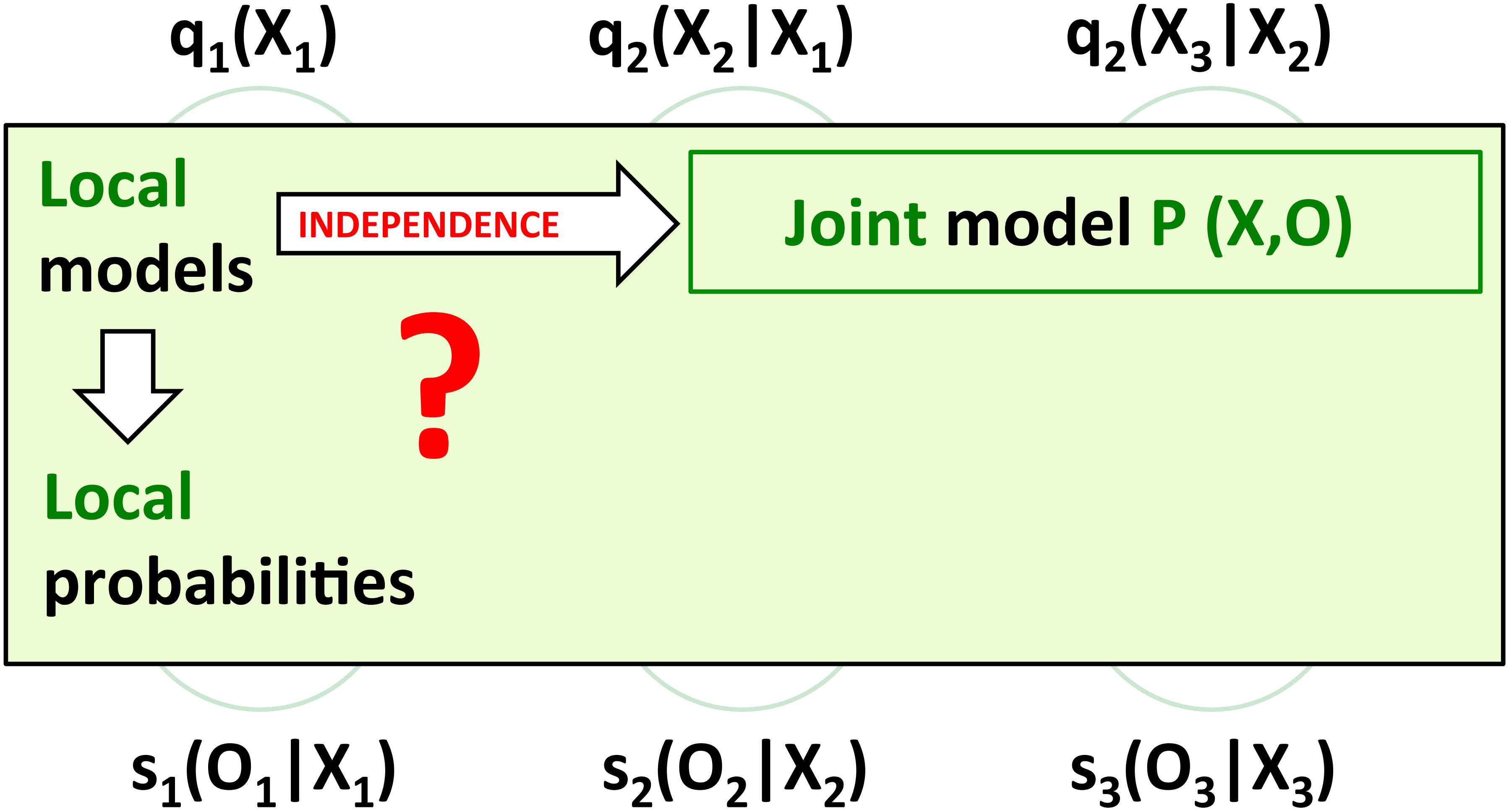
**Local models**

INDEPENDENCE

**Joint model P (X,O)**

$S_1\ (O_1|X_1)$ $S_2\ (O_2|X_2)$ $S_3\ (O_3|X_3)$

# Precise hidden Markov model



$q_1(X_1)$ $q_2(X_2|X_1)$ $q_2(X_3|X_2)$

**Local models**

INDEPENDENCE

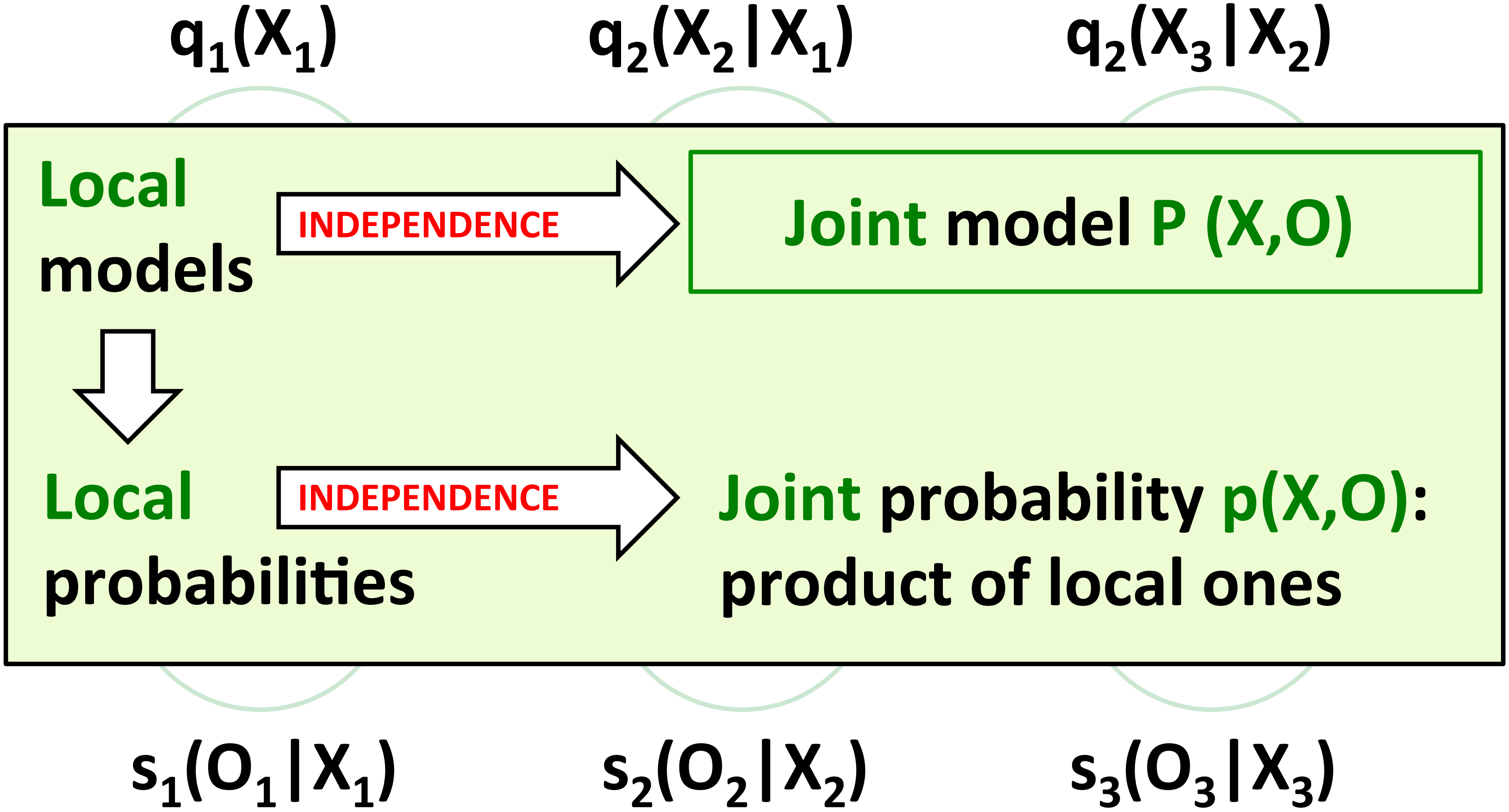
**Joint model P (X,O)**

**Local probabilities**

?

$s_1(O_1|X_1)$ $s_2(O_2|X_2)$ $s_3(O_3|X_3)$

# Precise hidden Markov model



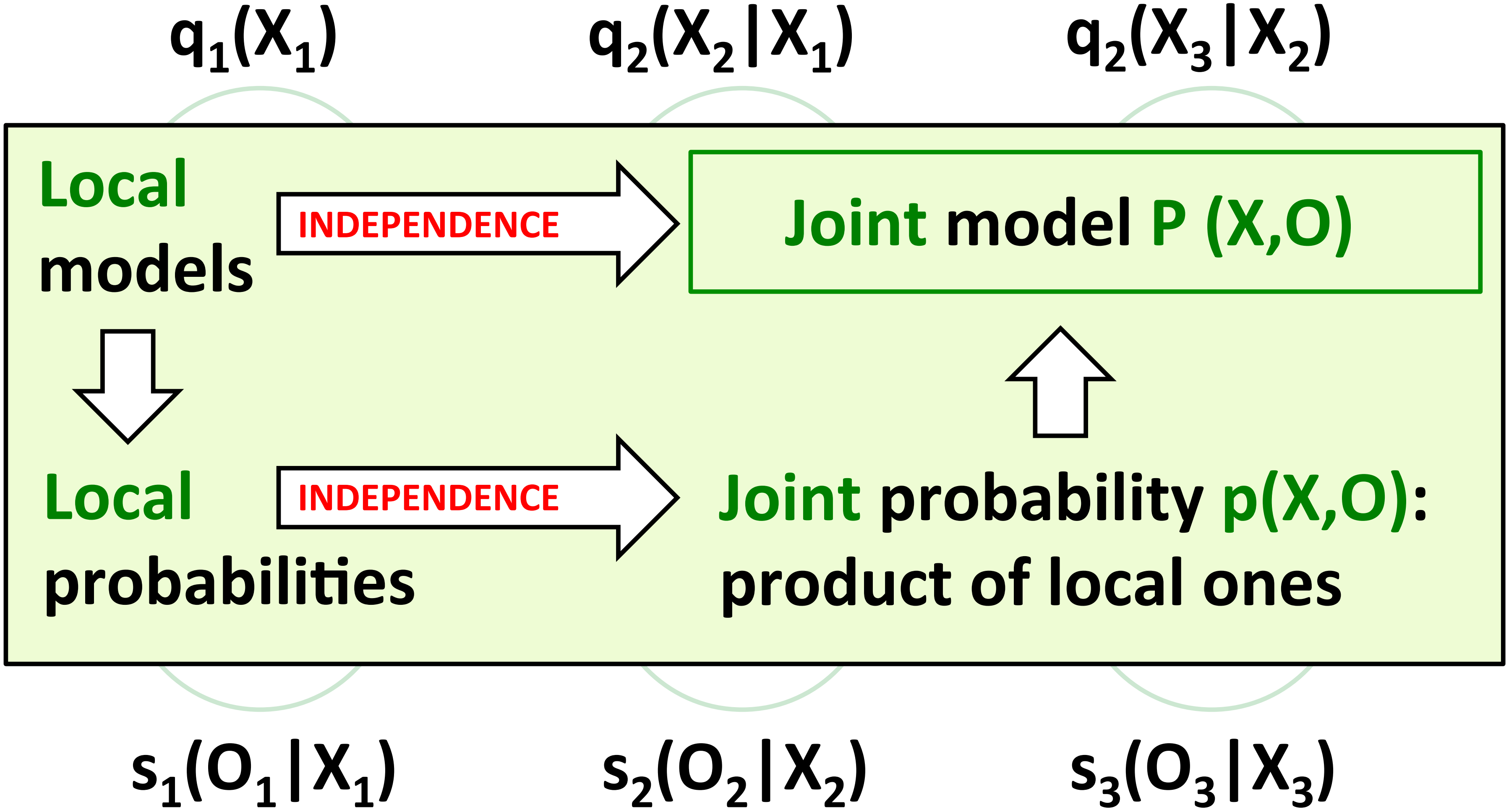
$q_1(X_1)$ $q_2(X_2|X_1)$ $q_2(X_3|X_2)$

**Local** **models** → INDEPENDENCE → **Joint model P (X,O)**

**Local probabilities** → INDEPENDENCE → **Joint probability p(X,O): product of local ones**

$s_1(O_1|X_1)$ $s_2(O_2|X_2)$ $s_3(O_3|X_3)$

# Precise hidden Markov model



$q_1(X_1)$ $q_2(X_2|X_1)$ $q_2(X_3|X_2)$

Local models

INDEPENDENCE

Joint model P (X,O)

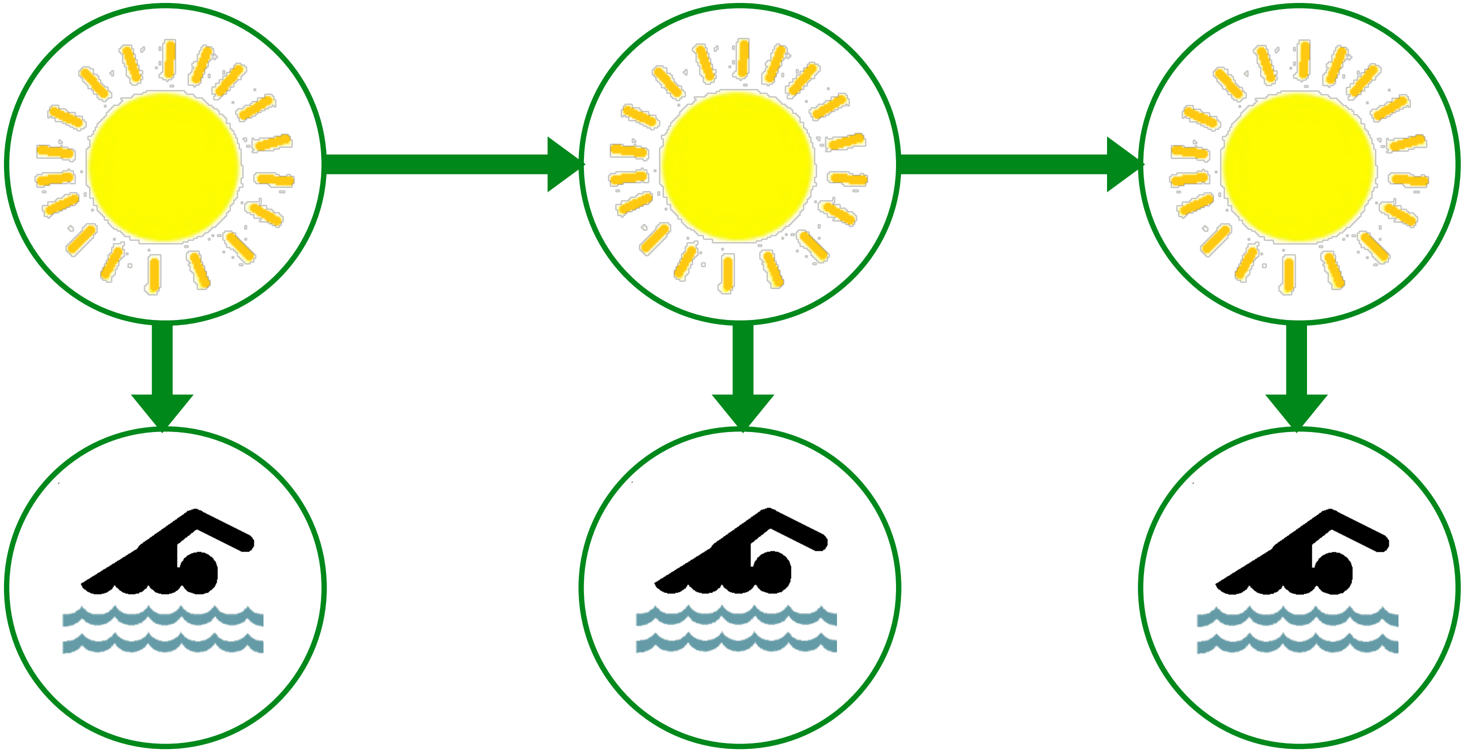
Local probabilities

INDEPENDENCE

Joint probability p(X,O): product of local ones

$s_1(O_1|X_1)$ $s_2(O_2|X_2)$ $s_3(O_3|X_3)$

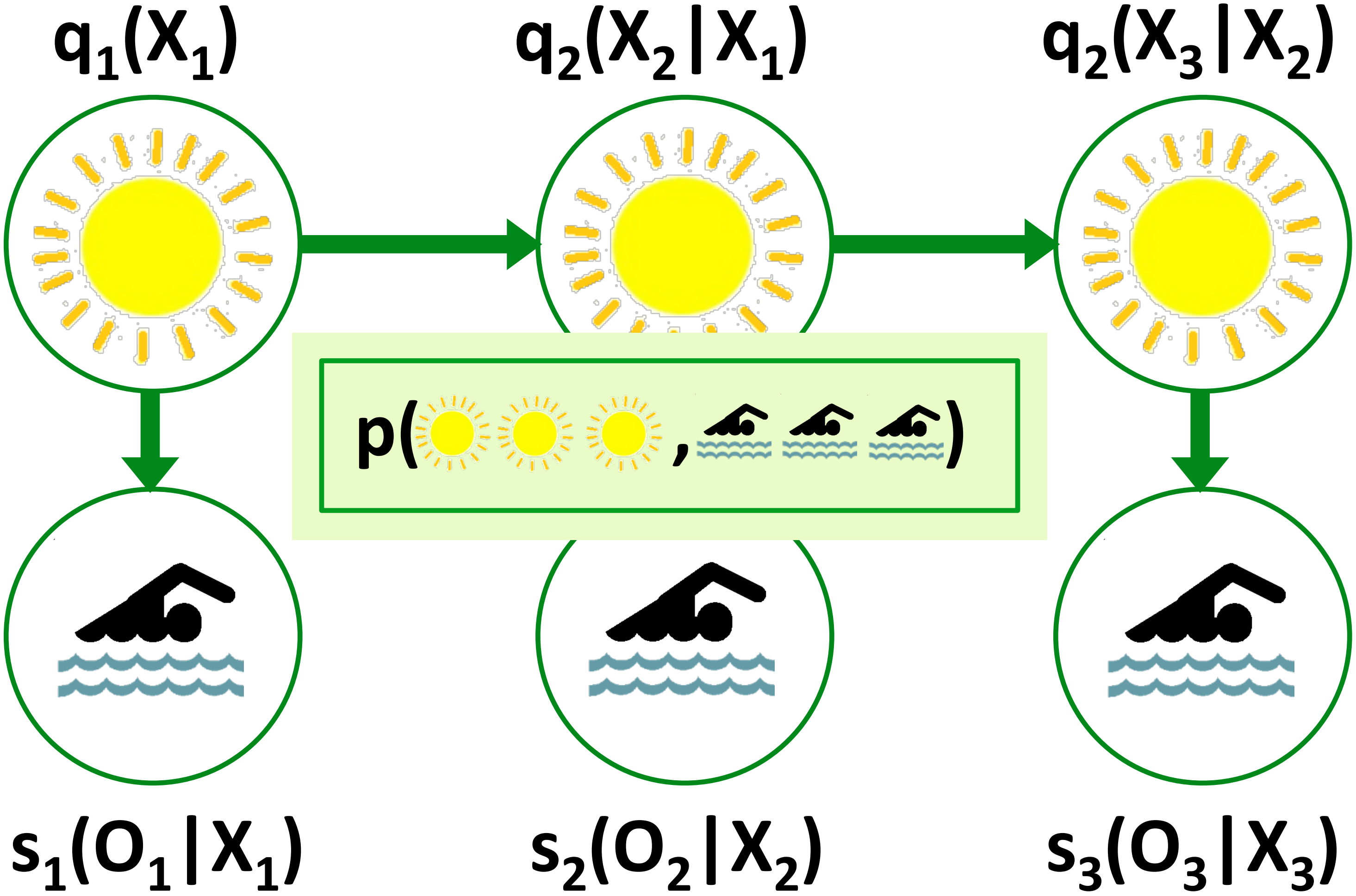
# Precise hidden Markov model



$q_1(X_1)$ $q_2(X_2|X_1)$ $q_2(X_3|X_2)$

$s_1(O_1|X_1)$ $s_2(O_2|X_2)$ $s_3(O_3|X_3)$

# Precise hidden Markov model

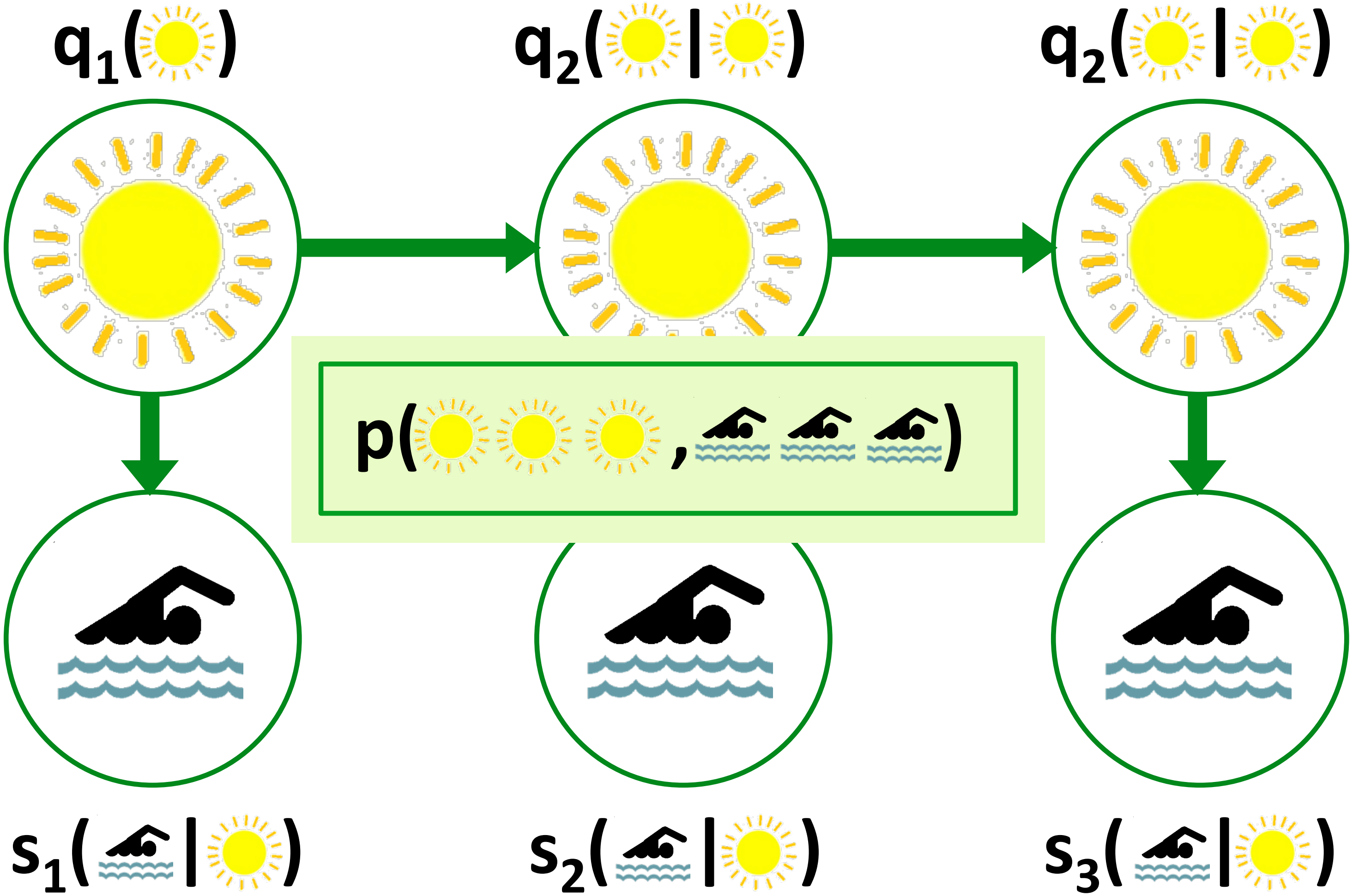


$q_1(X_1)$ $q_2(X_2|X_1)$ $q_2(X_3|X_2)$

p(☀☀☀, 🏊🏊🏊)

$s_1(O_1|X_1)$ $s_2(O_2|X_2)$ $s_3(O_3|X_3)$

# Precise hidden Markov model



$q_1$(☀)

$q_2$(☀|☀)

$q_2$(☀|☀)

p(☀☀☀,🏊🏊🏊)

$s_1$(🏊|☀)

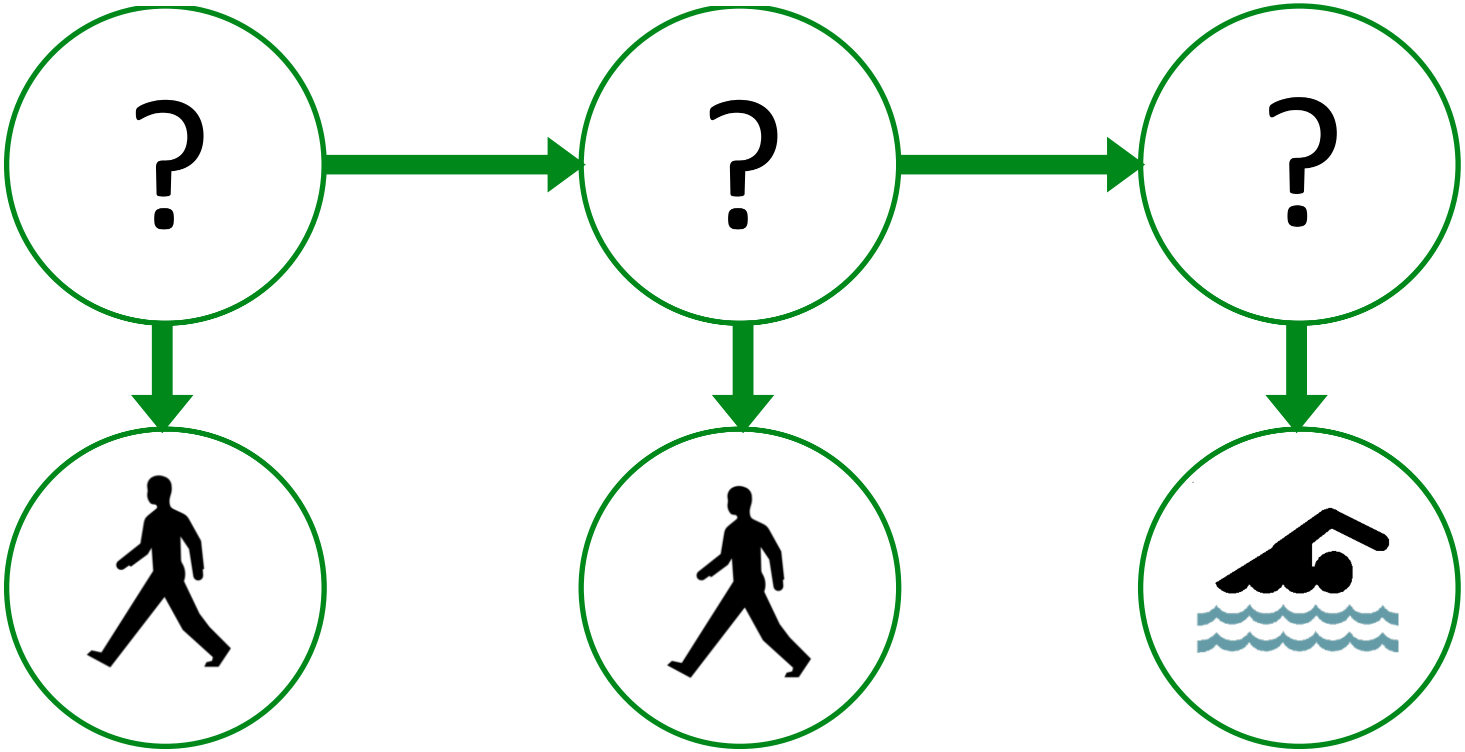
$s_2$(🏊|☀)

$s_3$(🏊|☀)

# State sequence estimation in imprecise hidden Markov models

# State sequence estimation

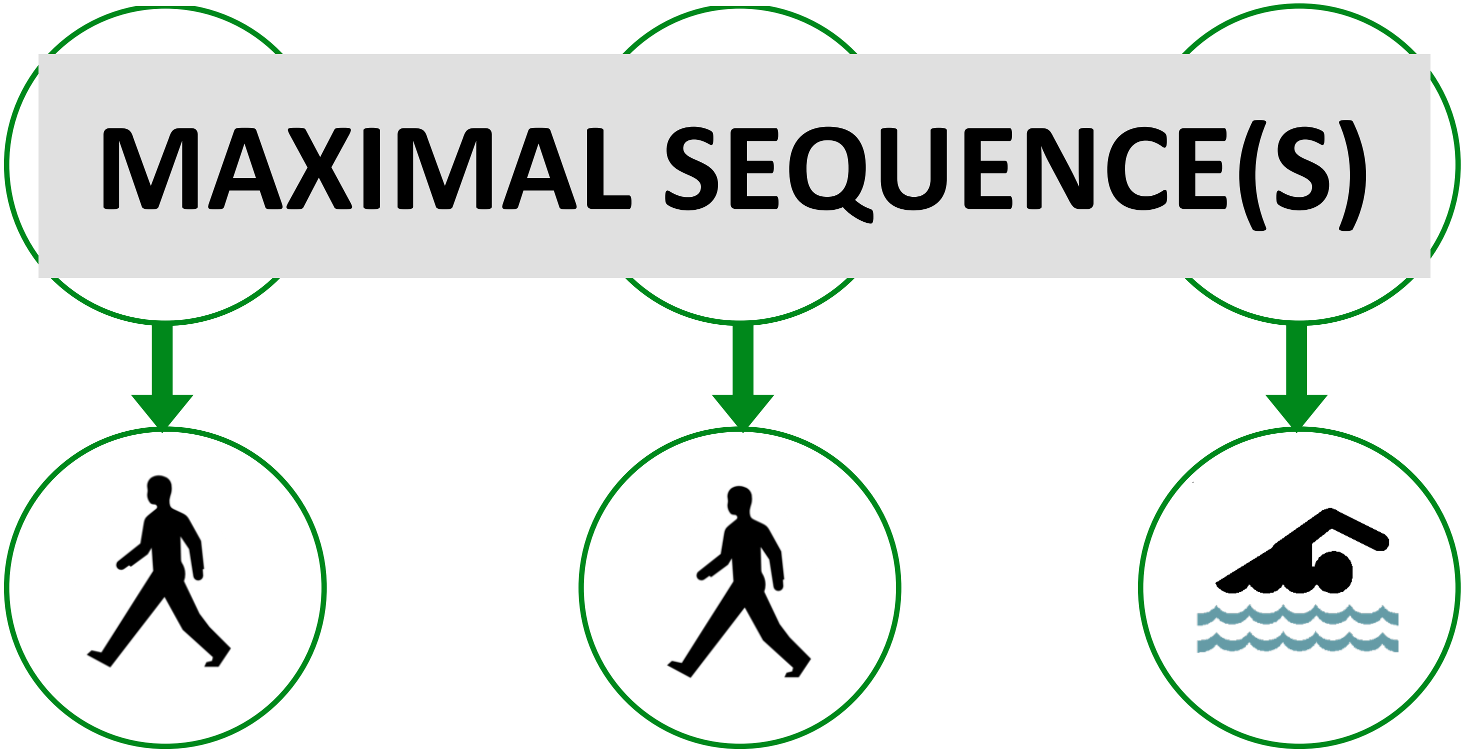
A sequence of hidden state variables



A sequence of observable output variables

# State sequence estimation

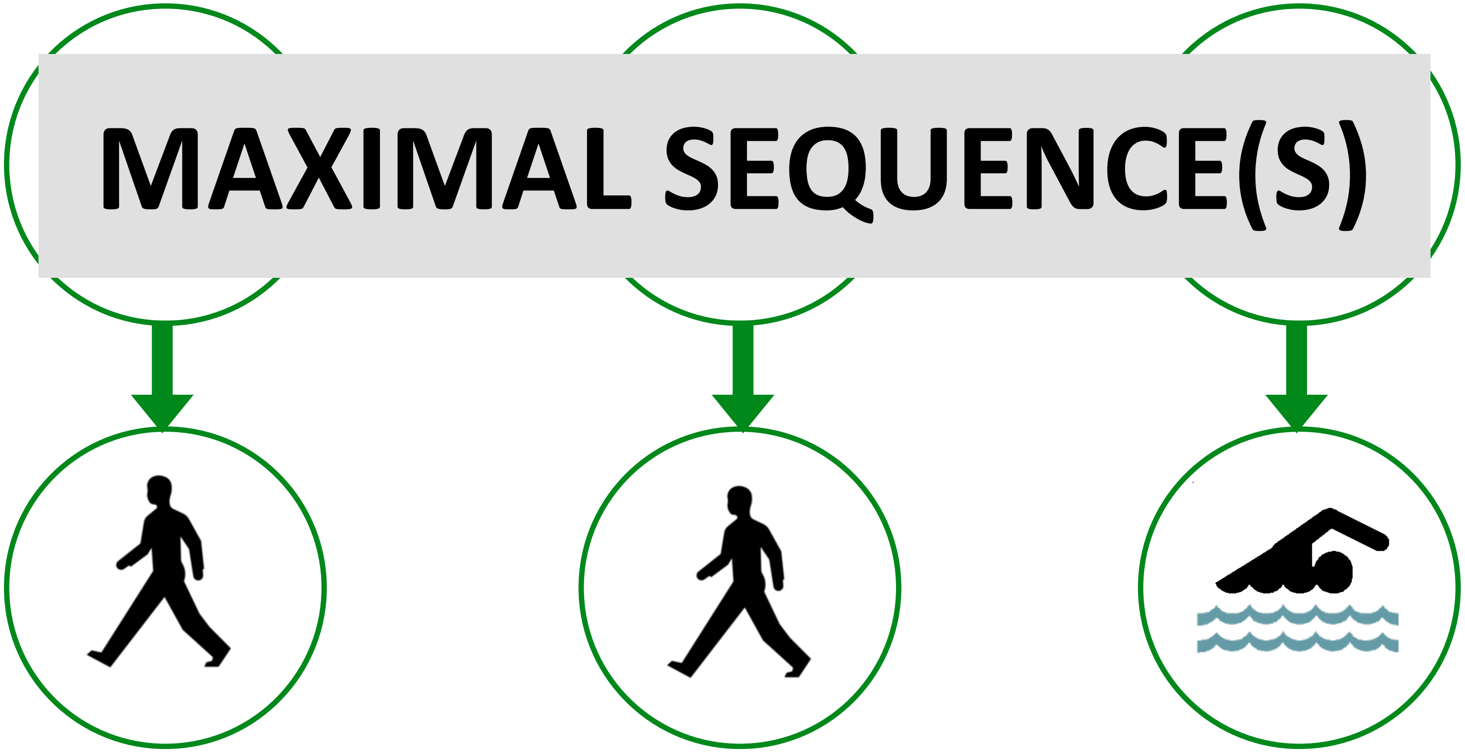
How to estimate the hidden state sequence ?



MAXIMAL SEQUENCE(S)

# State sequence estimation

**Highest conditional probability p(X| 🚶 🚶 🏊)!**



**MAXIMAL SEQUENCE(S)**

# State sequence estimation

**Highest conditional probability** p(X| 🚶 🚶 🏊)!

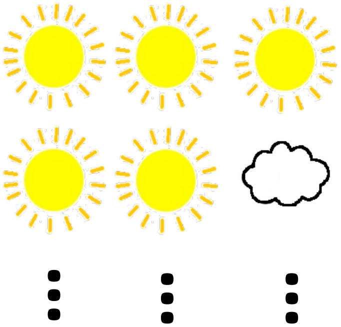
$$p(X \mid \text{🚶 🚶 🏊}) = \frac{p(X, \text{🚶 🚶 🏊})}{p(\text{🚶 🚶 🏊})}$$

# State sequence estimation

**Highest conditional probability $p(X|\text{🚶 🚶 🏊})$!**

$$p(X|\text{🚶 🚶 🏊}) = \frac{p(X,\text{🚶 🚶 🏊})}{p(\text{🚶 🚶 🏊})}$$

⇩



# State sequence estimation

**Highest conditional probability p(X| 🚶 🚶 🏊)!**

**(or equally high!)**

$$p(X|\text{🚶 🚶 🏊}) = \frac{p(X,\text{🚶 🚶 🏊})}{p(\text{🚶 🚶 🏊})}$$

⇩

☀ ☀ ☀

☀ ☀ ☁

⋮ ⋮ ⋮

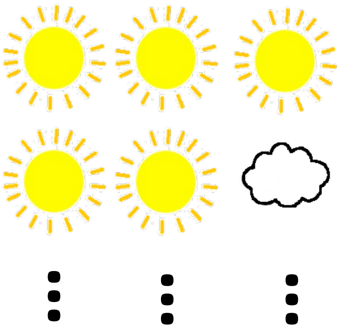
# State sequence estimation

Highest conditional probability p(X| 🚶 🚶 🏊 )!

(or equally high!)

$$p(X|\text{🚶 🚶 🏊}) = \frac{p(X,\text{🚶 🚶 🏊})}{p(\text{🚶 🚶 🏊})} \neq 0$$

⇩



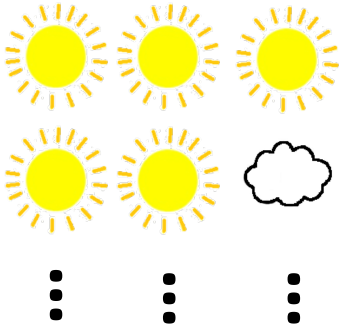
# State sequence estimation

**Highest conditional probability p(X| 🚶 🚶 🏊)!**

**(or equally high!)**

p(X| 🚶 🚶 🏊) = p(X, 🚶 🚶 🏊) / p(🚶 🚶 🏊) ≠ 0

⇩



⇩

**What do we do if this is not fullfilled?**

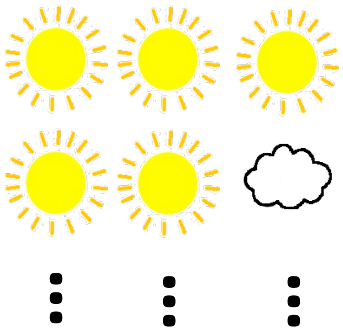
# State sequence estimation

**Highest conditional probability** $p(X \mid \text{🚶 🚶 🏊})$!

**(or equally high!)**

$$p(X \mid \text{🚶 🚶 🏊}) = \frac{p(X, \text{🚶 🚶 🏊})}{p(\text{🚶 🚶 🏊})} \neq 0$$

⇩



**How to make our criterion easier to check and include the zero-case at the same time?**

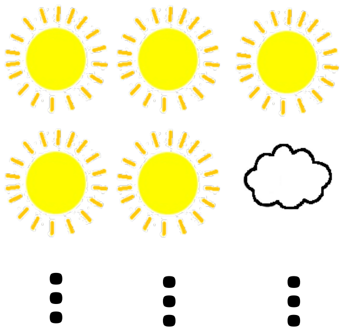
# State sequence estimation

Highest unconditional probability p(X, 🚶 🚶 🏊 )!

(or equally high!)

$$p(X\,|\,\text{🚶 🚶 🏊}) = \frac{p(X,\text{🚶 🚶 🏊})}{p(\text{🚶 🚶 🏊})} \neq 0$$

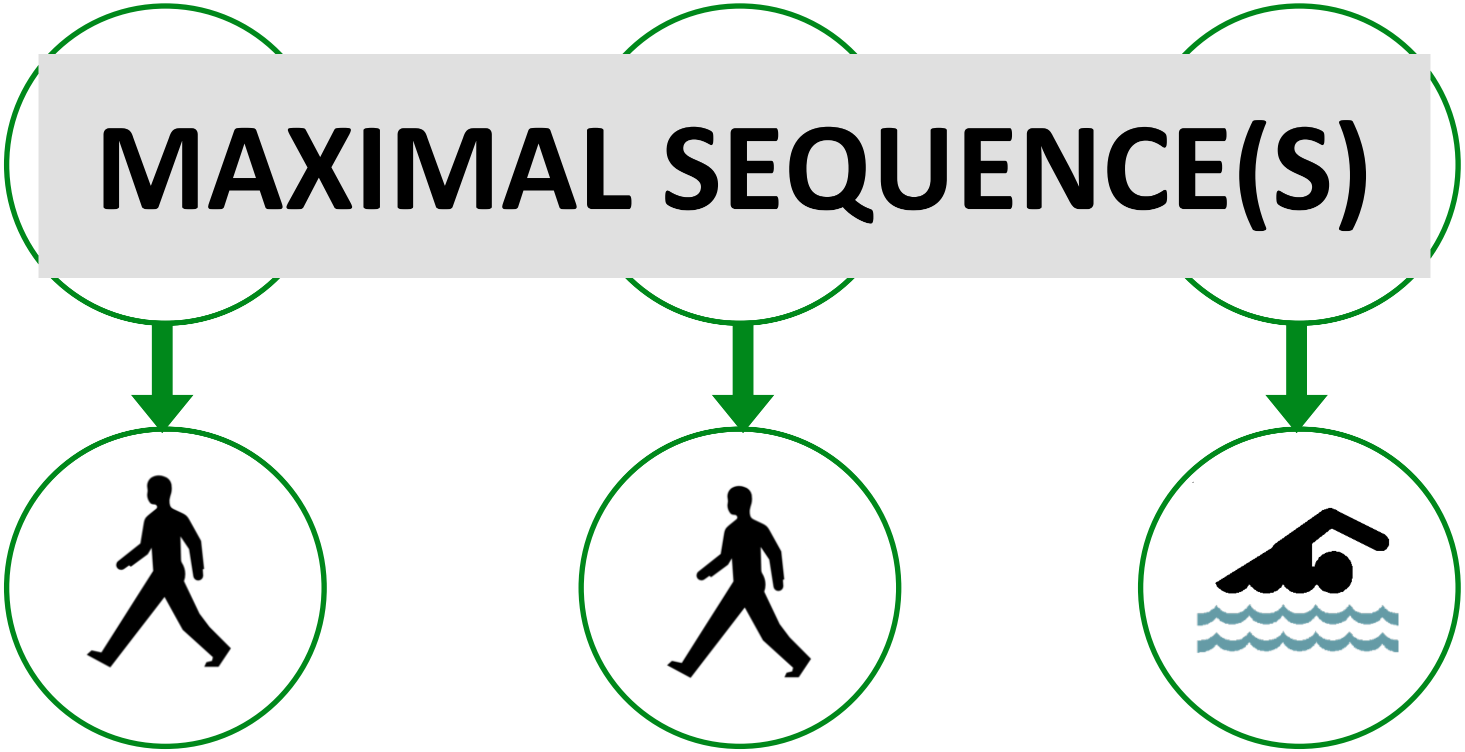
⇩



How to make our criterion easier to check and include the zero-case at the same time?

# State sequence estimation

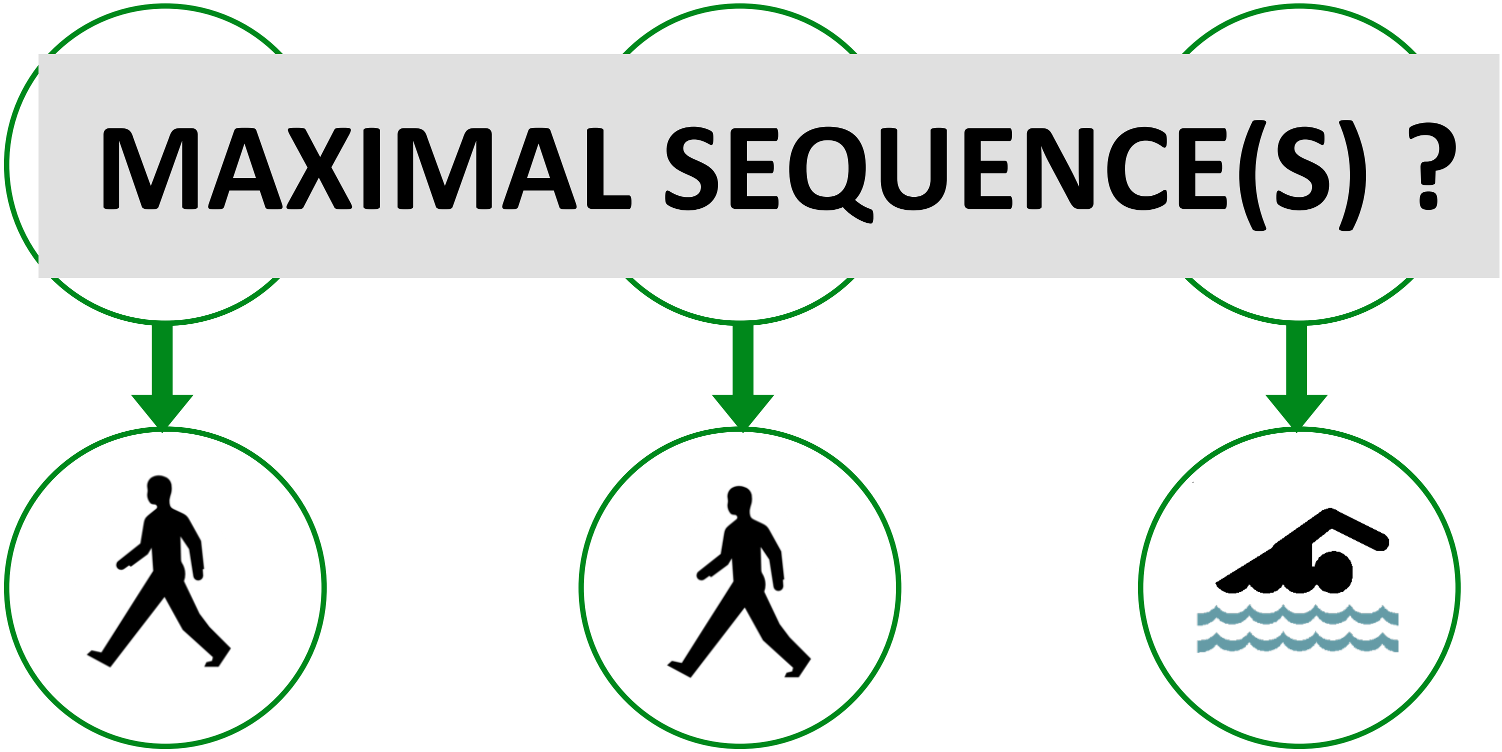
**Highest unconditional probability p(X, 🚶 🚶 🏊 )!**



**MAXIMAL SEQUENCE(S)**

# State sequence estimation

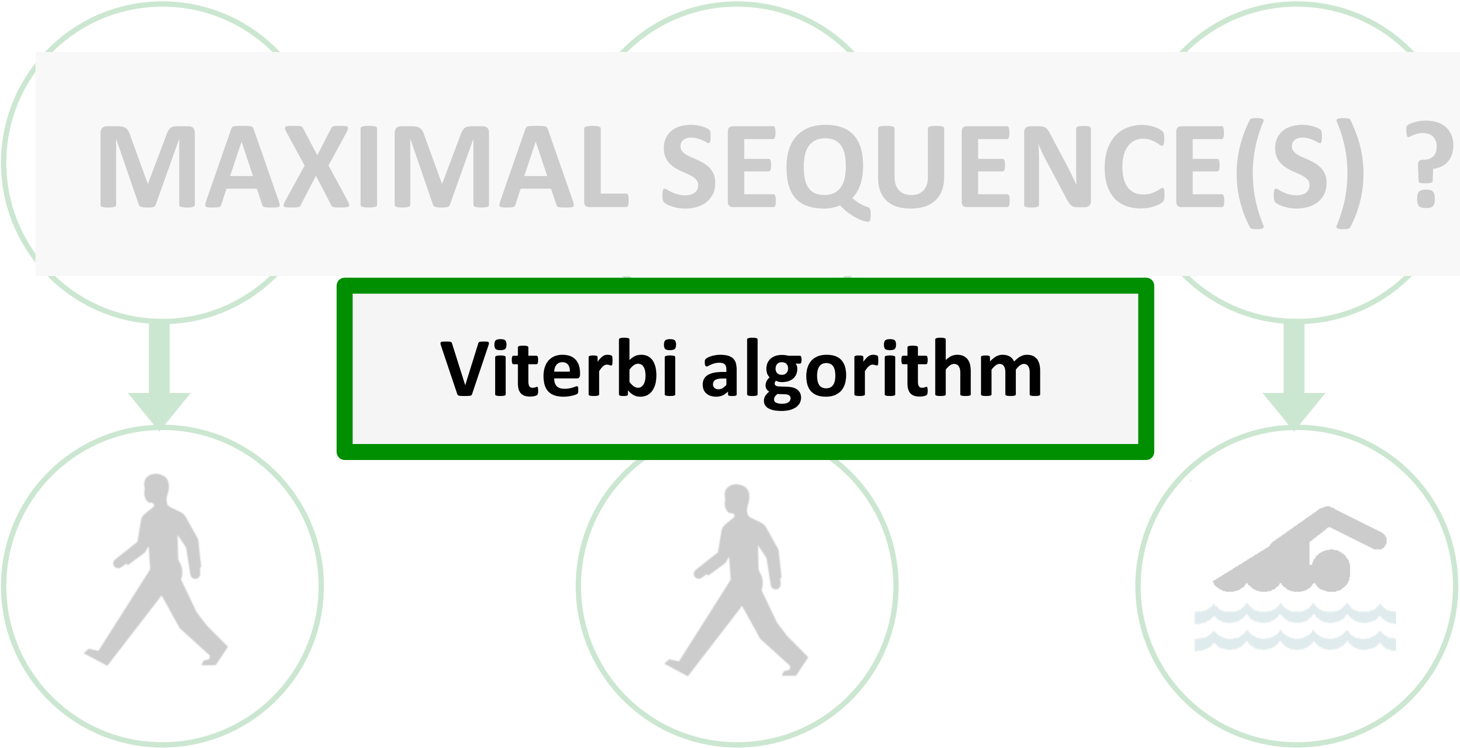
Highest unconditional probability p(X, 🚶 🚶 🏊 )?



MAXIMAL SEQUENCE(S) ?

# State sequence estimation

Highest unconditional probability p(X, 🚶 🚶 🏊 )?

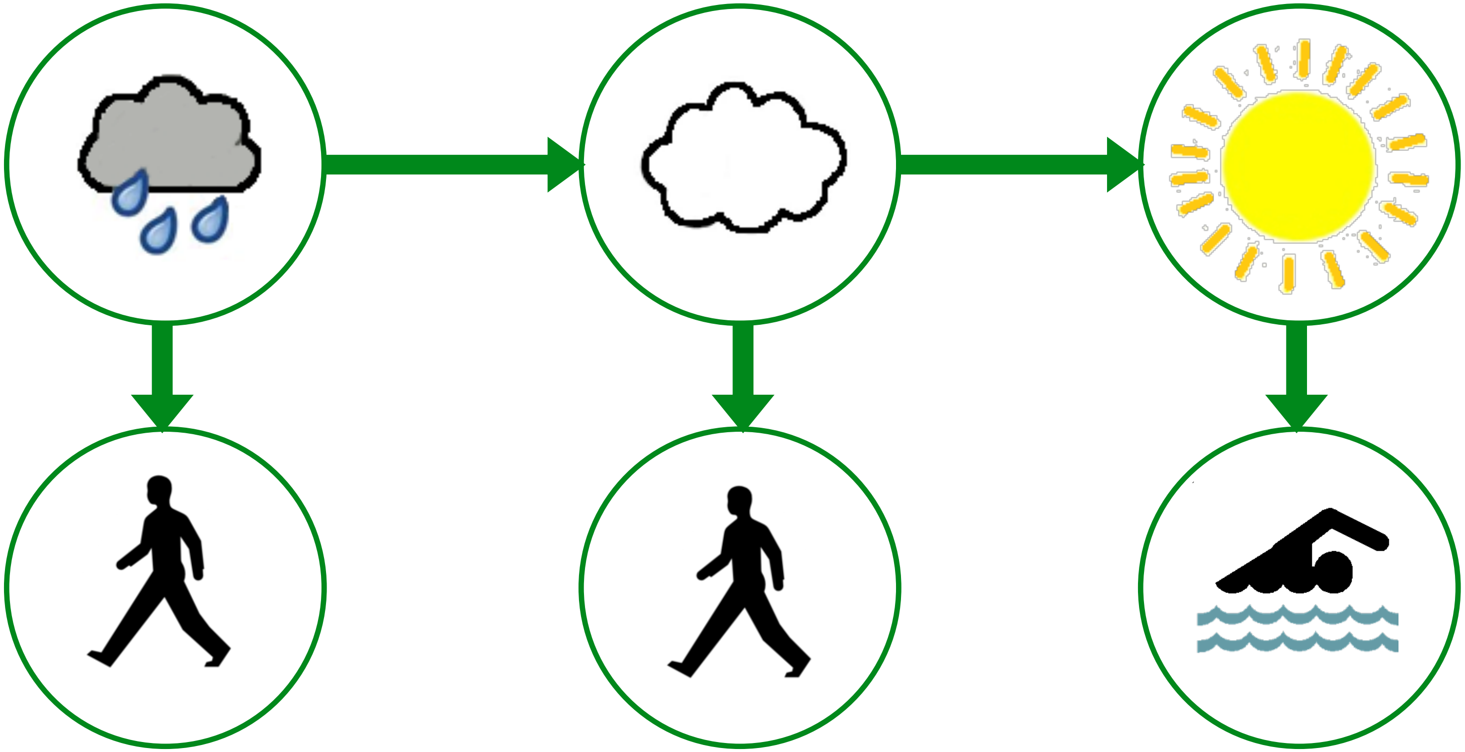


MAXIMAL SEQUENCE(S) ?

**Viterbi algorithm**

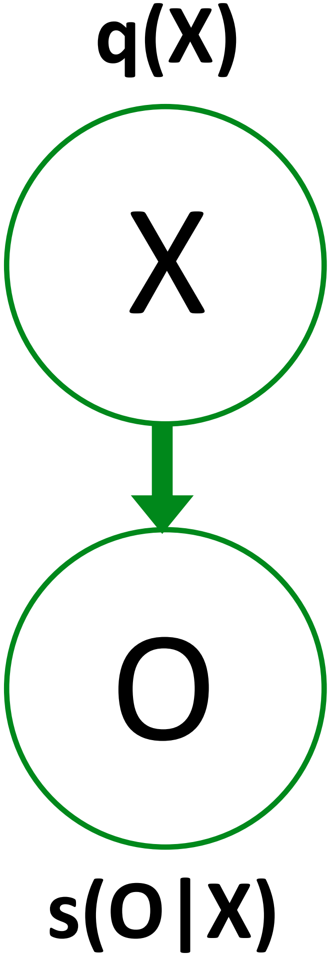
# State sequence estimation

**Highest unconditional probability p(X, 🚶 🚶 🏊 )?**

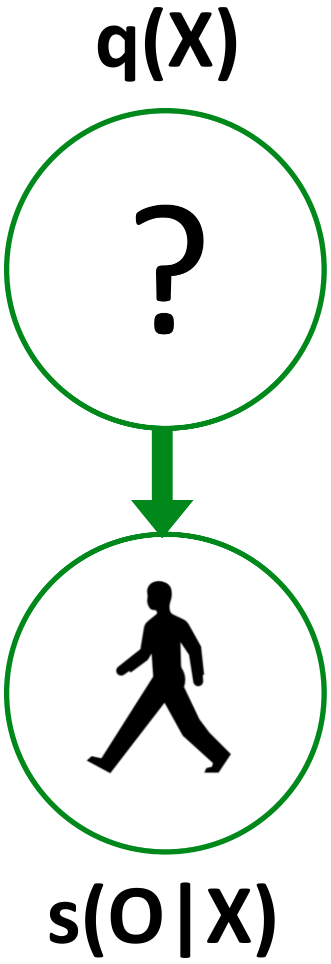


**EXERCISE!**

# Exercise (part 1)

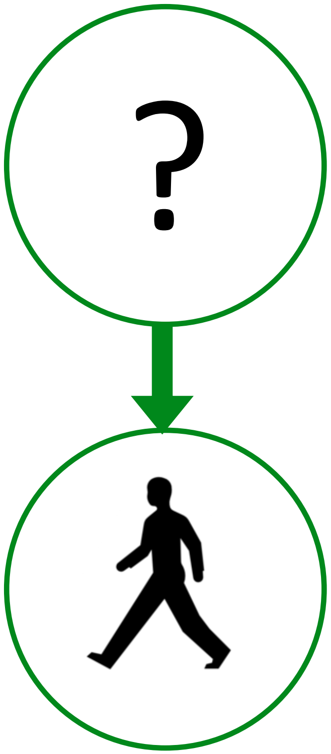


# Exercise (part 1)



## Estimate the (hidden) state!

# Exercise (part 1)



q(X)

?

s(O|X)

## Estimate the (hidden) state!

q(☀) = 70%

q(☁) = 20%

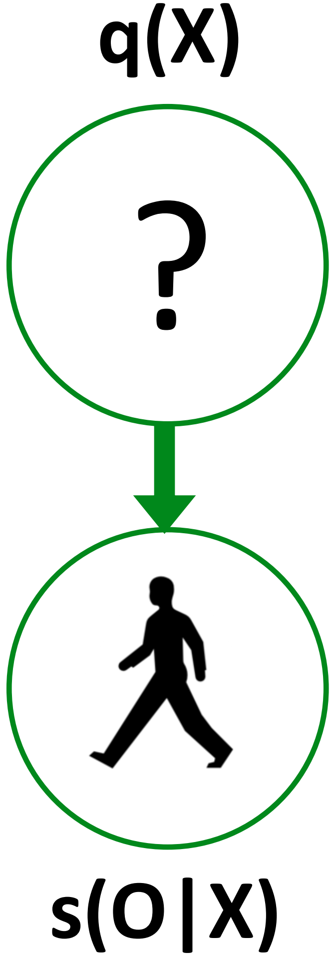
q(🌧) = 10%

s( 🚶 | ☀ ) = 20%

s( 🚶 | ☁ ) = 60%

s( 🚶 | 🌧 ) = 10%

# Exercise (part 1)



## Estimate the (hidden) state!

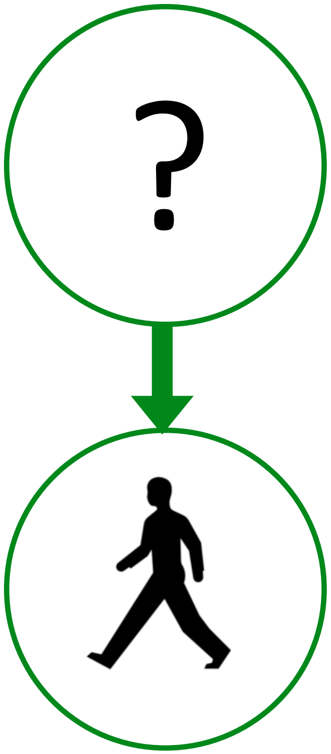
q(☀) = 7/10      s( 🚶 | ☀) = 2/10

q(☁) = 2/10      s( 🚶 | ☁) = 6/10

q(🌧) = 1/10      s( 🚶 | 🌧) = 1/10

# Exercise (part 1)

q(X)



s(O|X)

## Estimate the (hidden) state!

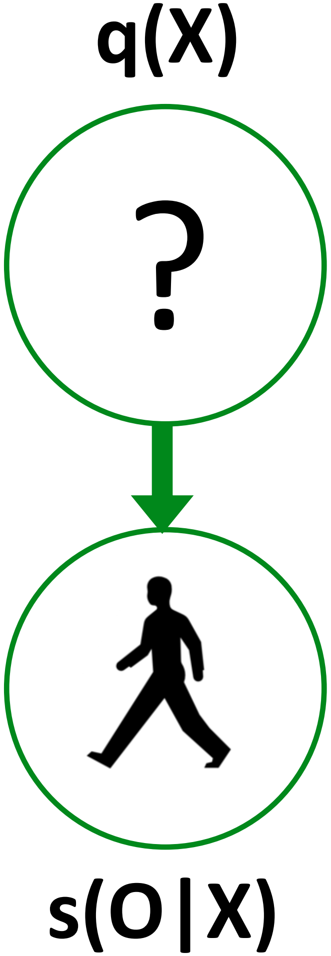
q(☀) = 7/10 s(🚶|☀) = 2/10

q(☁) = 2/10 s(🚶|☁) = 6/10

q(🌧) = 1/10 s(🚶|🌧) = 1/10

**Which state(s) X has (have) the highest unconditional probability p(X,🚶)?**

# Exercise (part 1)



## Estimate the (hidden) state!

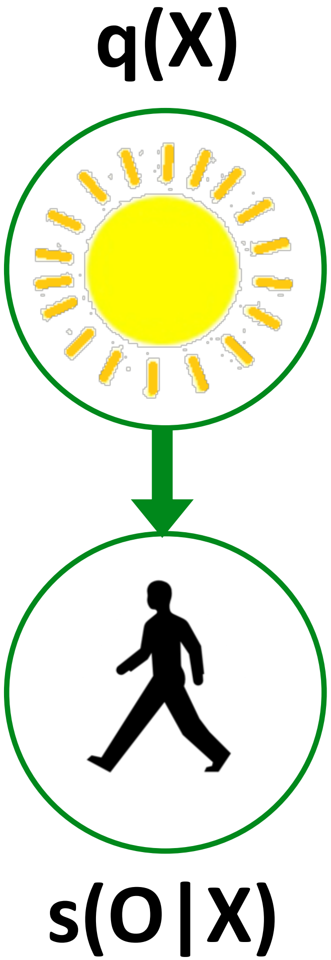
p(☀,🚶) = 0.14 = 14/100

p(☁,🚶) = 0.12 = 12/100

p(🌧,🚶) = 0.01 = 1/100

**Which state(s) X has (have) the highest unconditional probability p(X,🚶)?**

# Exercise (part 1)



q(X)

s(O|X)

## Estimate the (hidden) state!

p(☀, 🚶) = 0.14 = 14/100

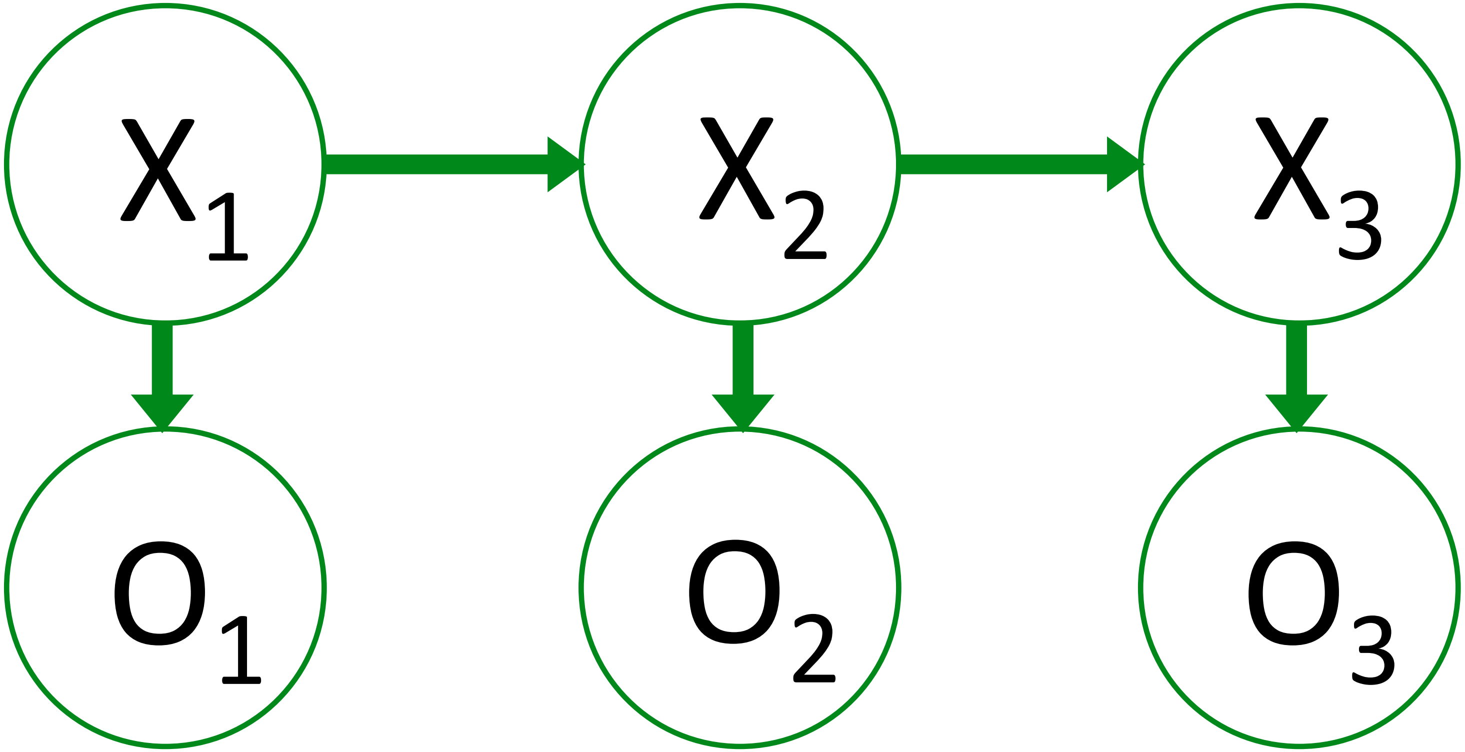
p(☁, 🚶) = 0.12 = 12/100

p(🌧, 🚶) = 0.01 = 1/100

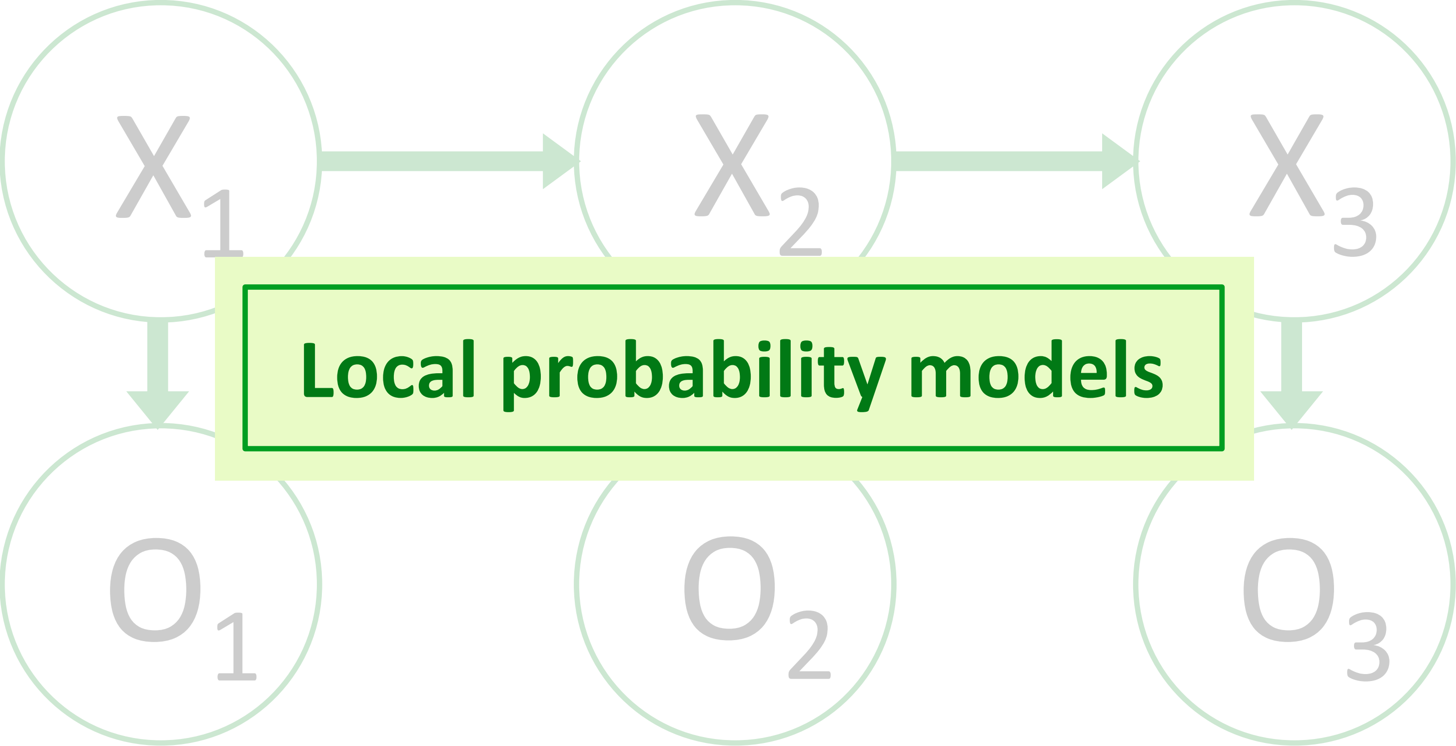
**Which state(s) X has (have) the highest unconditional probability p(X, 🚶)?** ⇒ ☀

# State sequence estimation in imprecise hidden Markov models

# Imprecise hidden Markov model

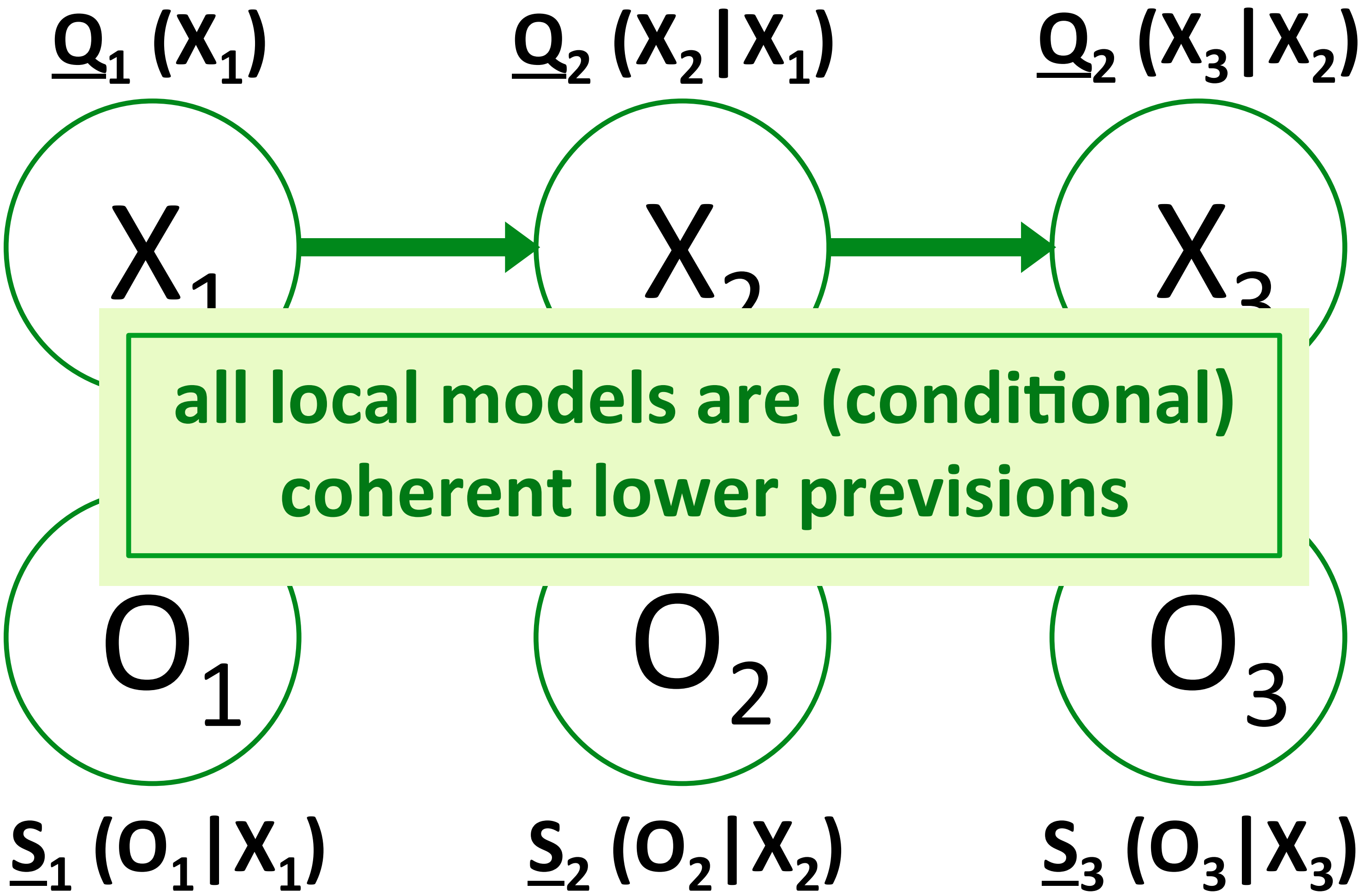


# Imprecise hidden Markov model



**Local probability models**

# Imprecise hidden Markov model



$\underline{Q}_1 (X_1)$ $\underline{Q}_2 (X_2|X_1)$ $\underline{Q}_2 (X_3|X_2)$

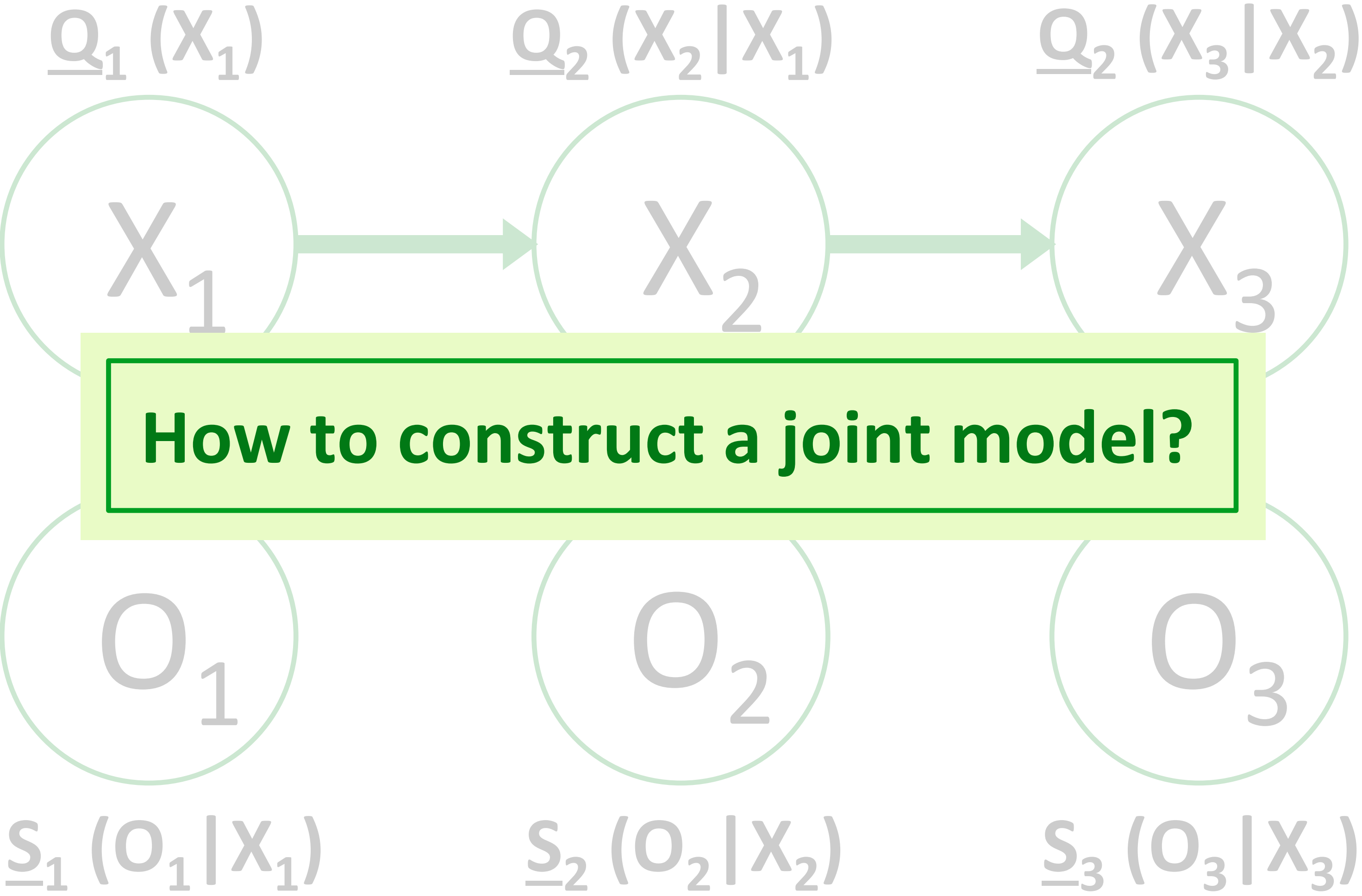
$X_1$ $X_2$ $X_3$

all local models are (conditional) coherent lower previsions

$O_1$ $O_2$ $O_3$

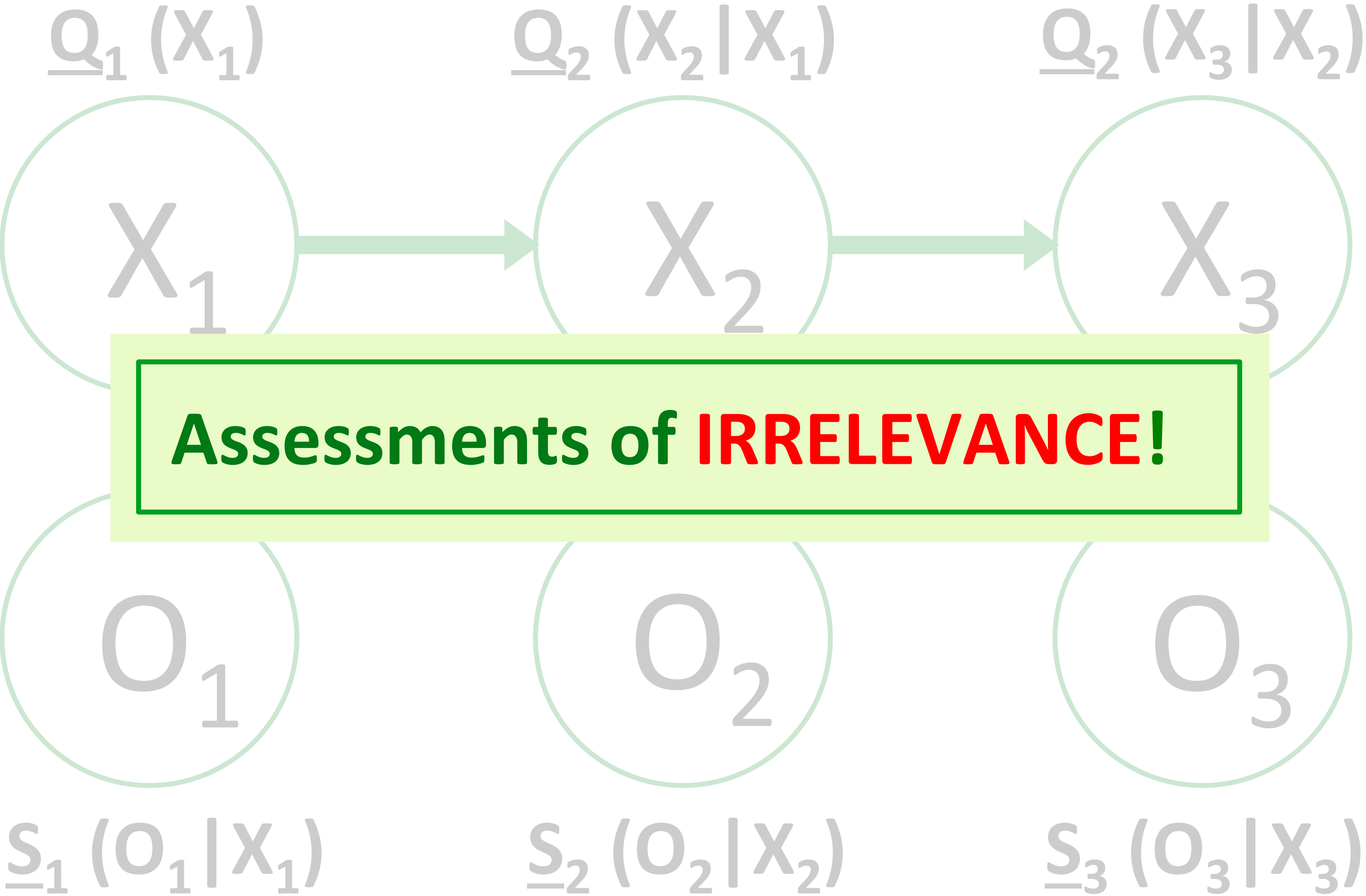
$\underline{S}_1 (O_1|X_1)$ $\underline{S}_2 (O_2|X_2)$ $\underline{S}_3 (O_3|X_3)$

# Imprecise hidden Markov model



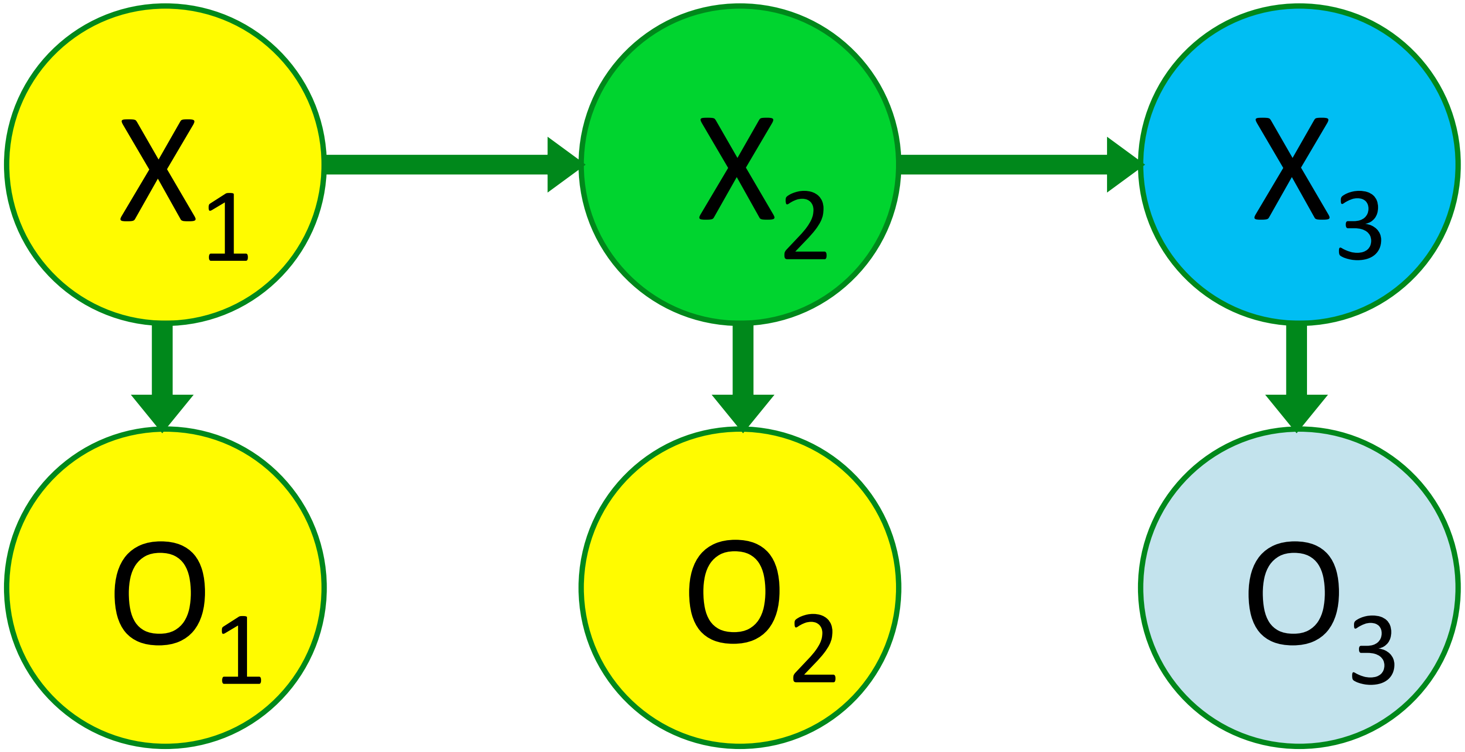
**How to construct a joint model?**

# Imprecise hidden Markov model



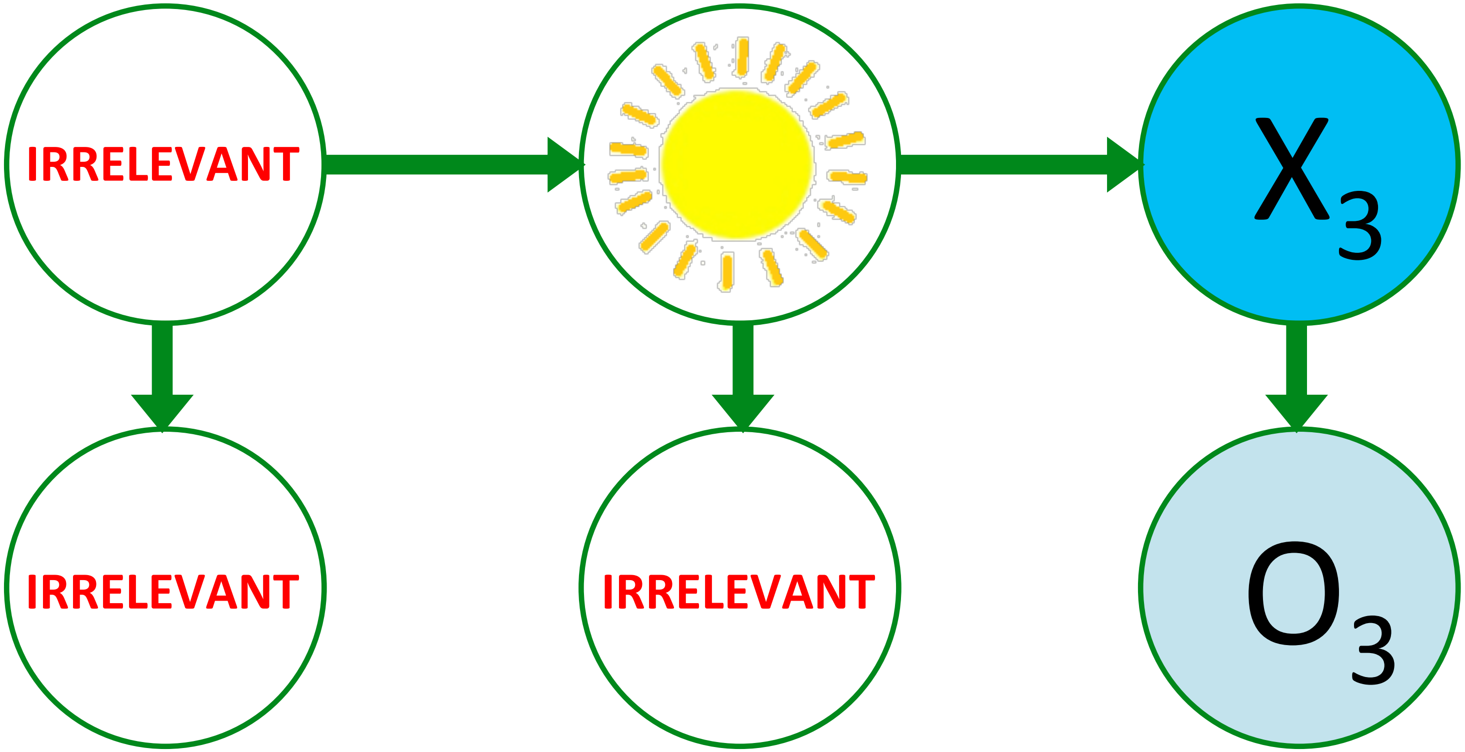
**Assessments of IRRELEVANCE!**

# Imprecise hidden Markov model



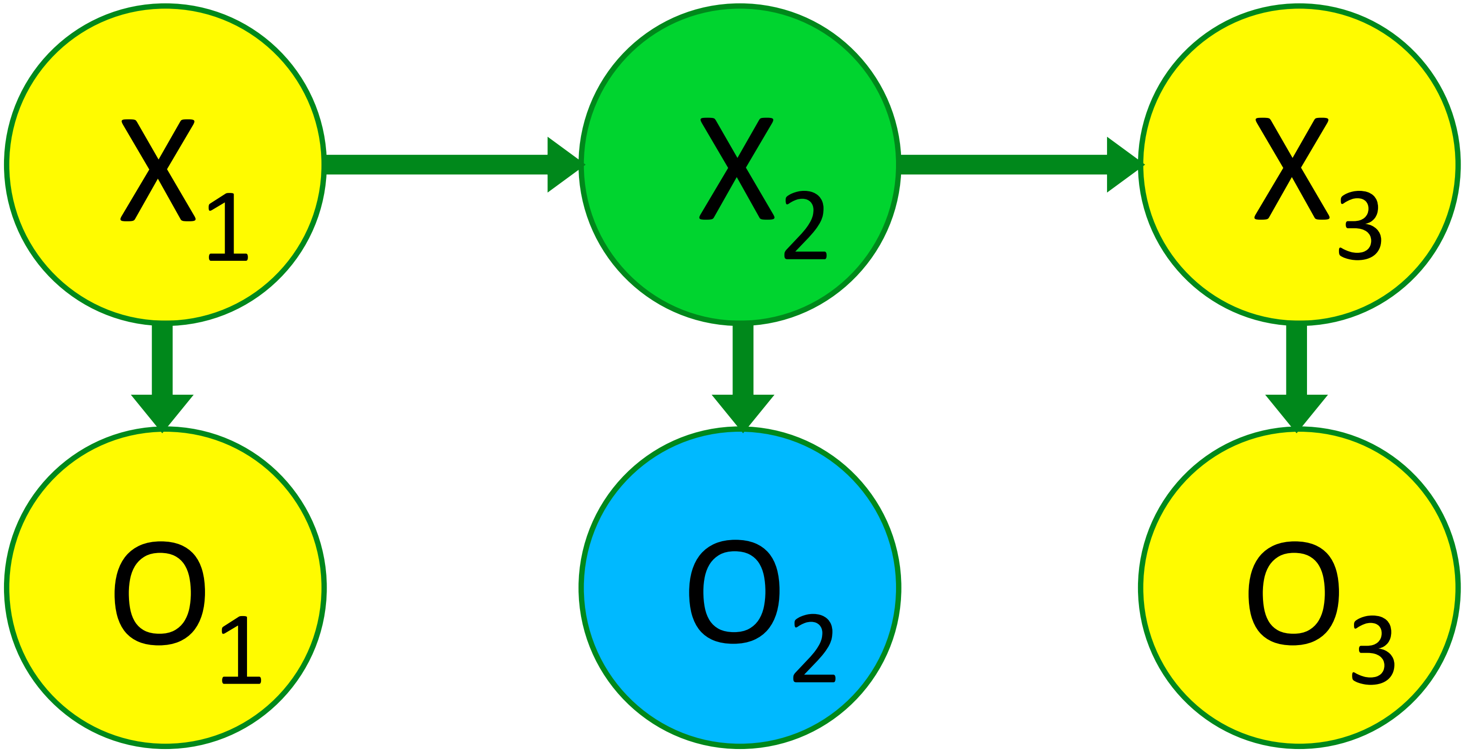
Conditional on its **mother variable**, the **non-parent non-descendants** of any **variable** in the tree are **epistemically irrelevant** to this **variable** and its **descendants**

# Imprecise hidden Markov model



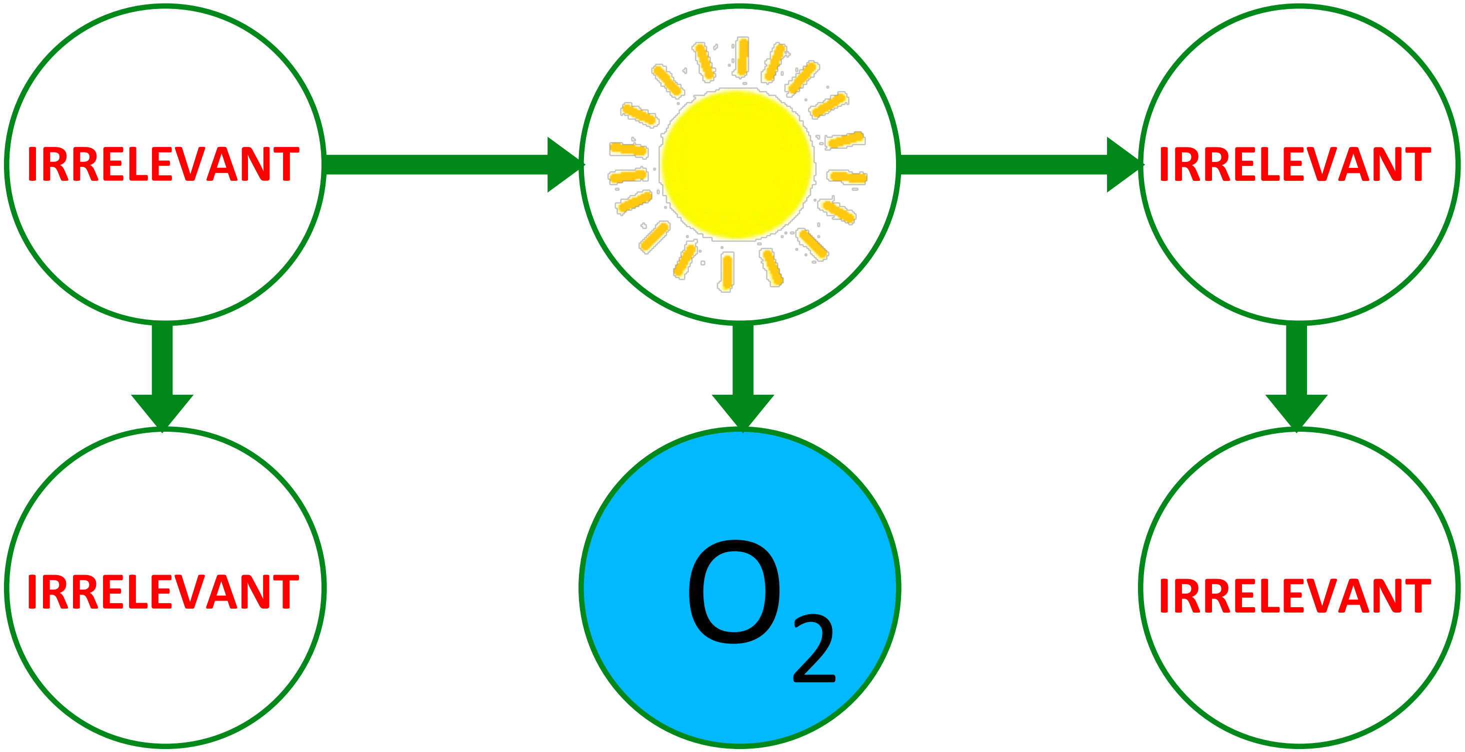
Conditional on its **mother variable**, the **non-parent non-descendants** of any **variable** in the tree are **epistemically irrelevant** to this **variable** and its **descendants**

# Imprecise hidden Markov model



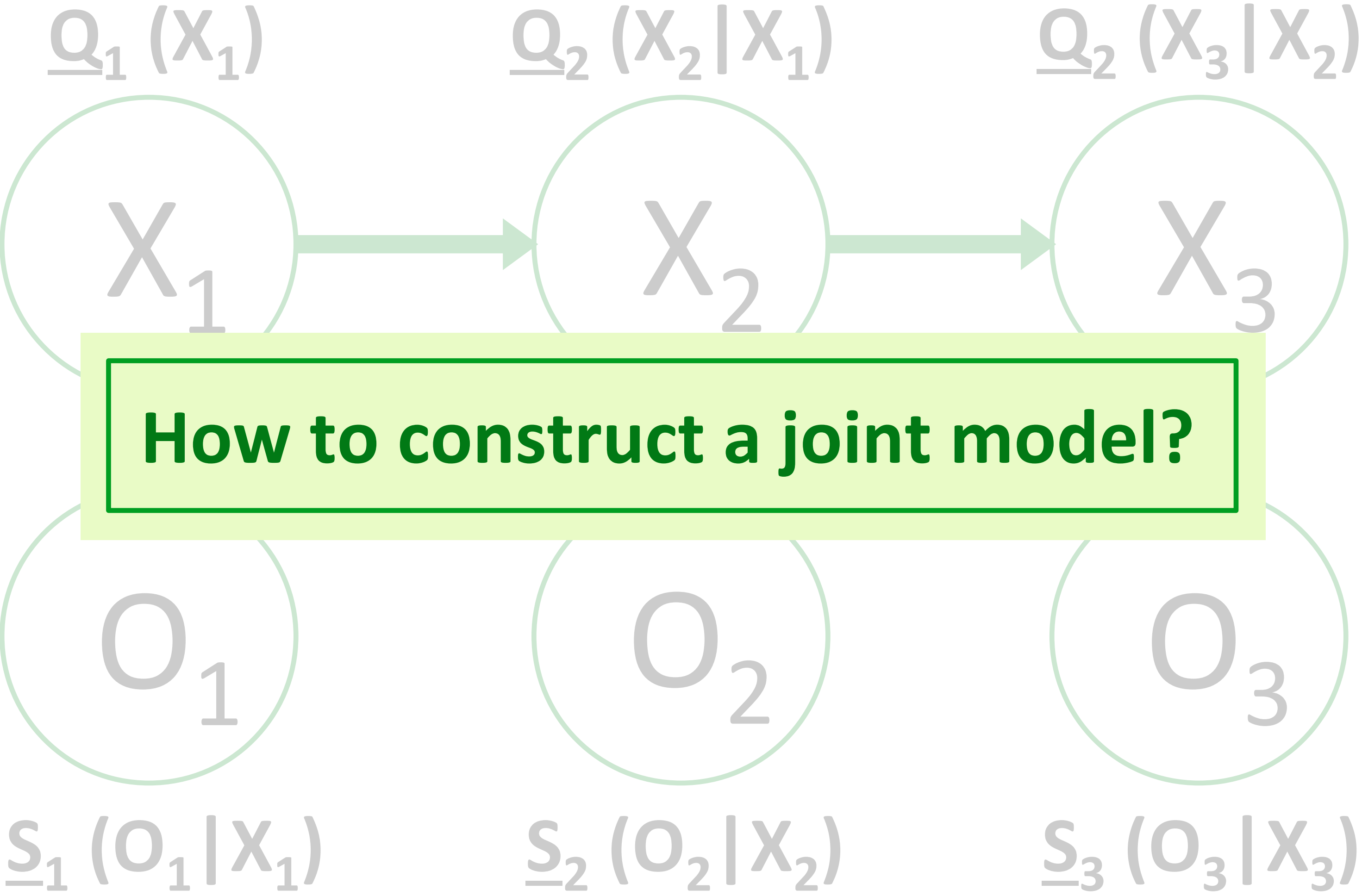
Conditional on its **mother variable**, the **non-parent non-descendants** of any **variable** in the tree are **epistemically irrelevant** to this **variable** and its **descendants**

# Imprecise hidden Markov model



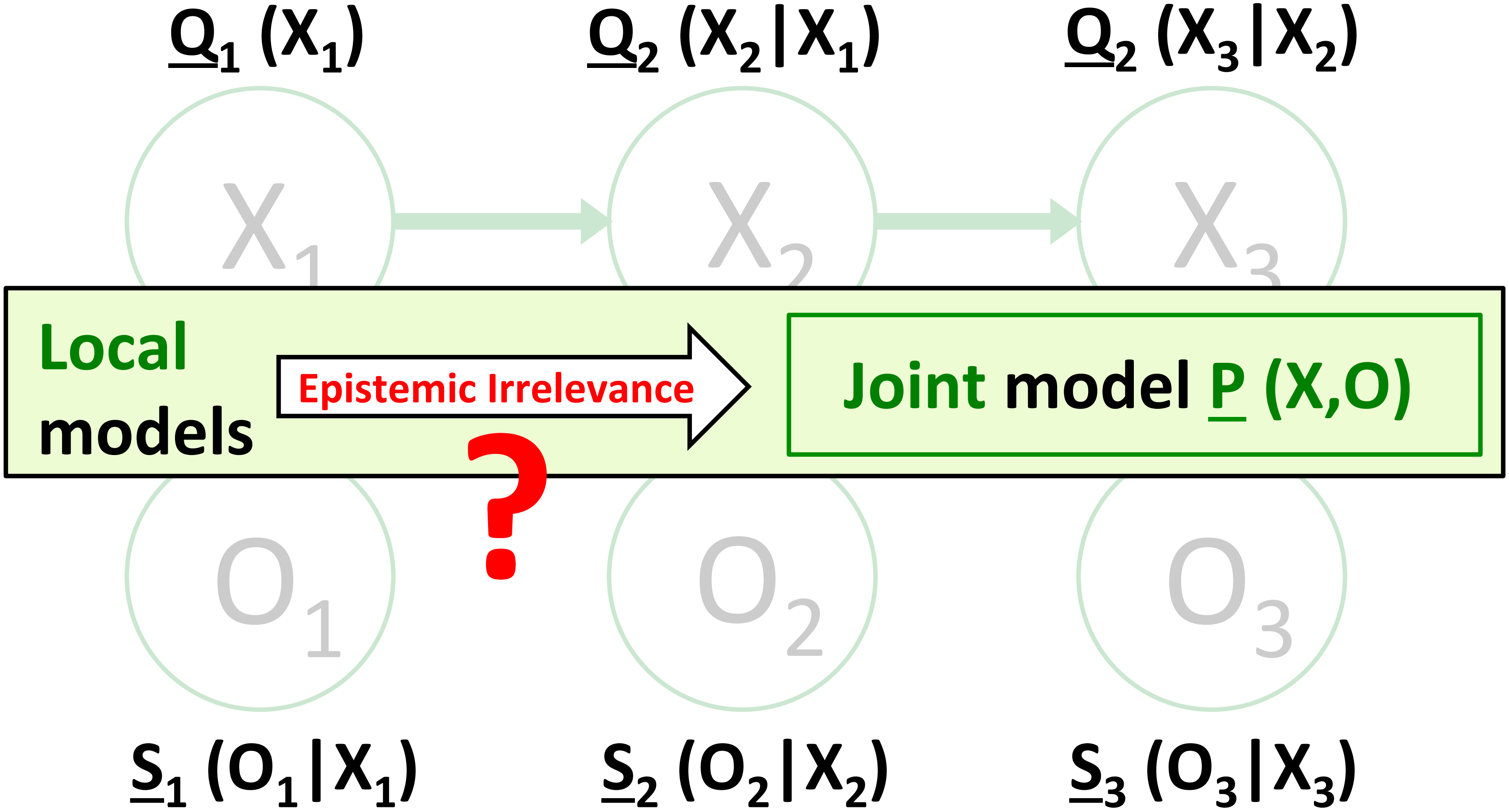
Conditional on its **mother variable**, the **non-parent non-descendants** of any **variable** in the tree are **epistemically irrelevant** to this **variable** and its **descendants**

# Imprecise hidden Markov model



**How to construct a joint model?**

# Imprecise hidden Markov model



$\underline{Q}_1\,(X_1)$ $\underline{Q}_2\,(X_2|X_1)$ $\underline{Q}_2\,(X_3|X_2)$

**Local models**

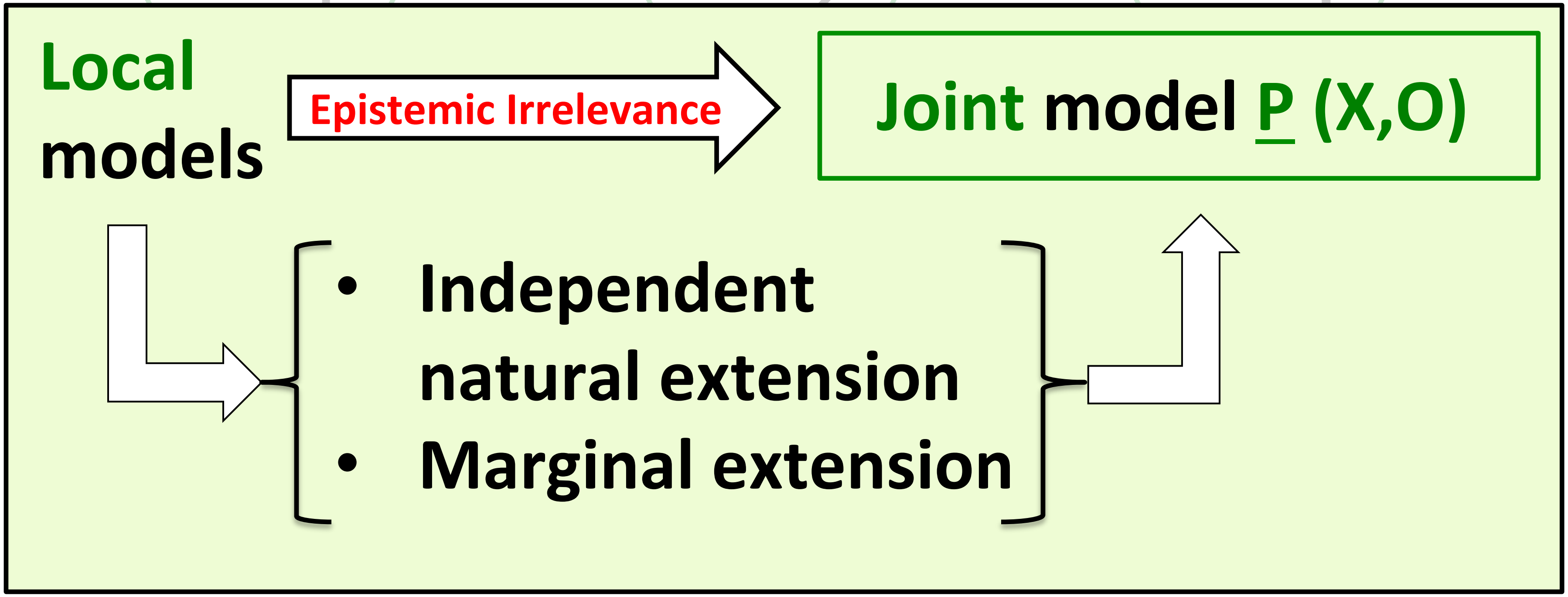
Epistemic Irrelevance

Joint model $\underline{P}\,(X,O)$

$\underline{S}_1\,(O_1|X_1)$ $\underline{S}_2\,(O_2|X_2)$ $\underline{S}_3\,(O_3|X_3)$

# Imprecise hidden Markov model

**Epistemic Irrelevance yields formulas that recursively construct a global model (Details: see lesson by Gert)**



**Local models**

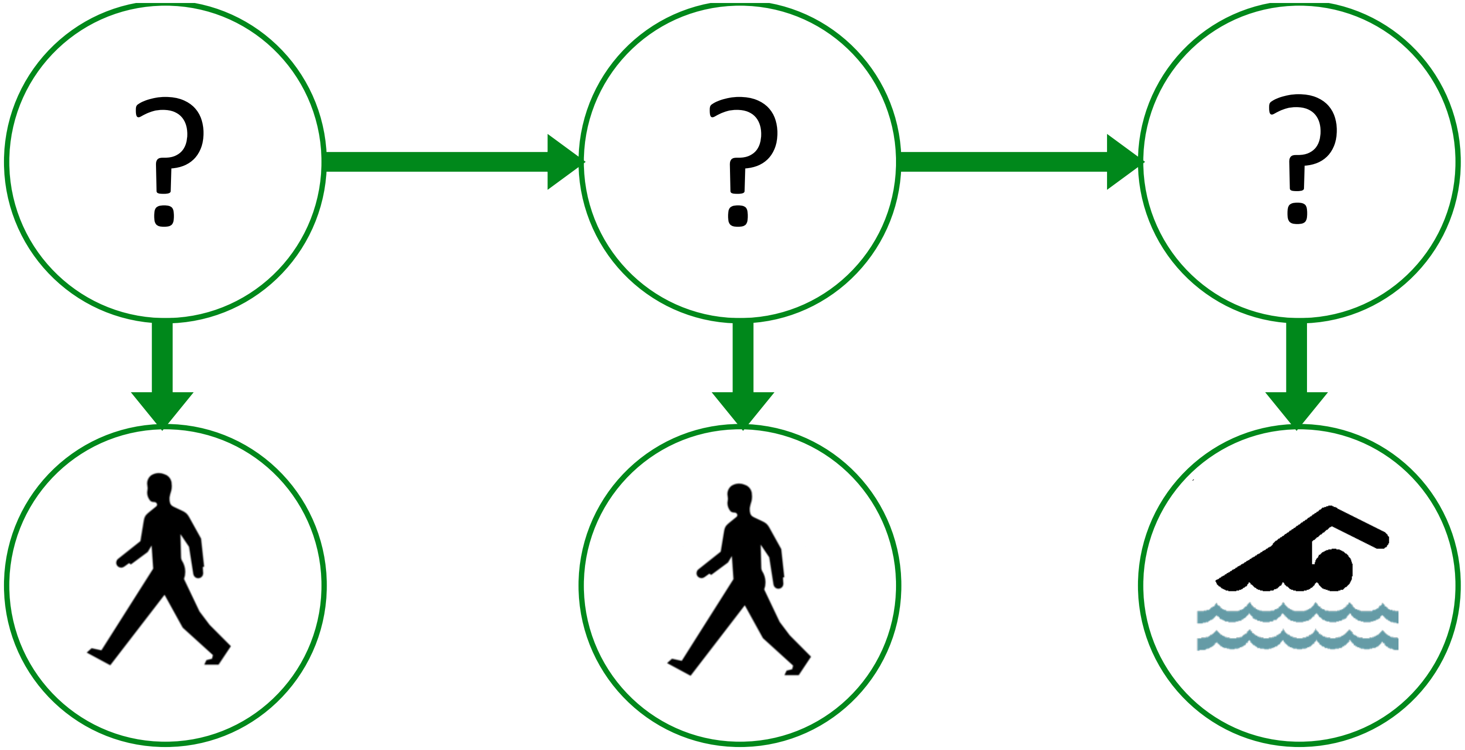
**Epistemic Irrelevance**

**Joint model $\underline{P}(X,O)$**

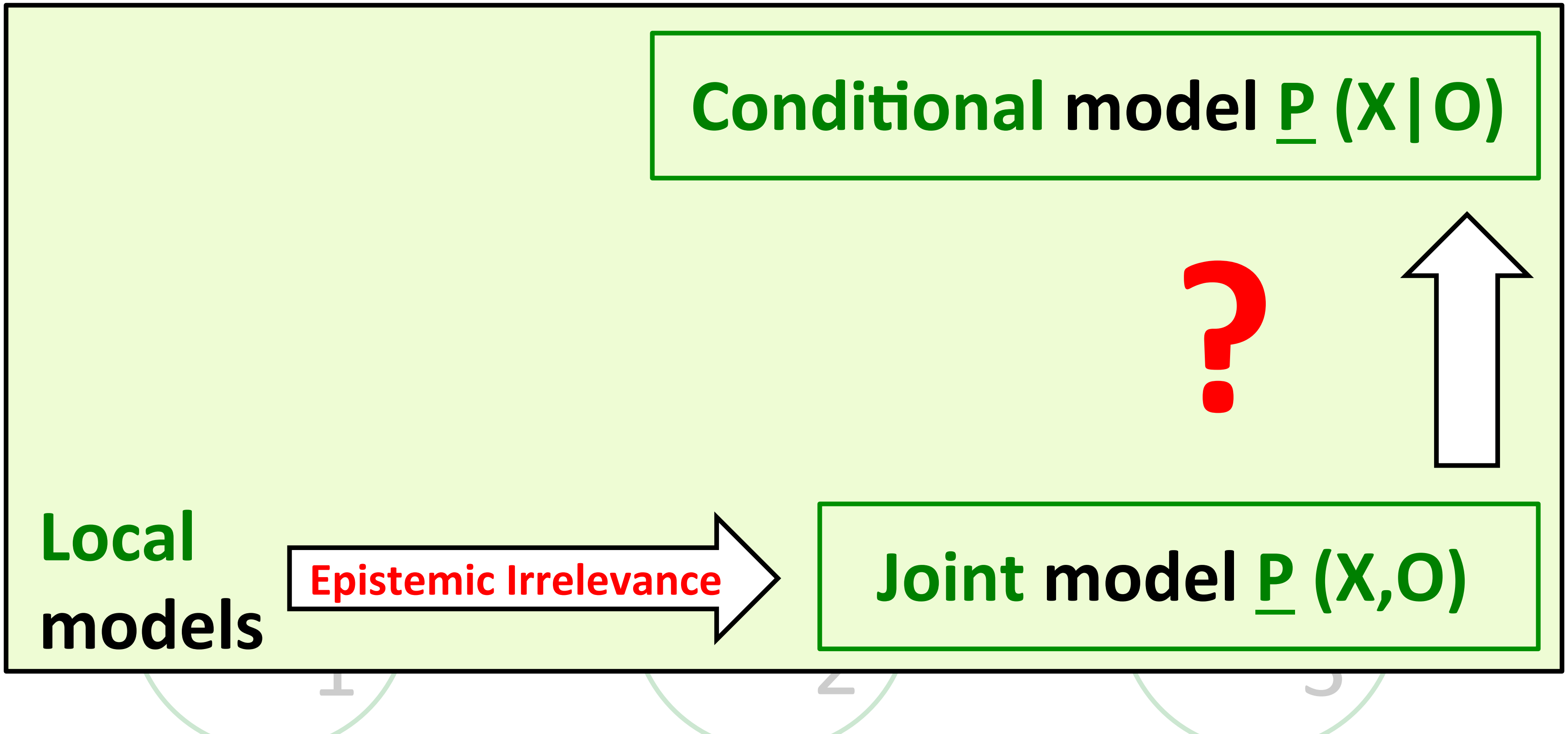
- **Independent natural extension**
- **Marginal extension**

# State sequence estimation in imprecise hidden Markov models

# (Imprecise) state sequence estimation



# (Imprecise) state sequence estimation



Conditional model $\underline{P}(X|O)$

?

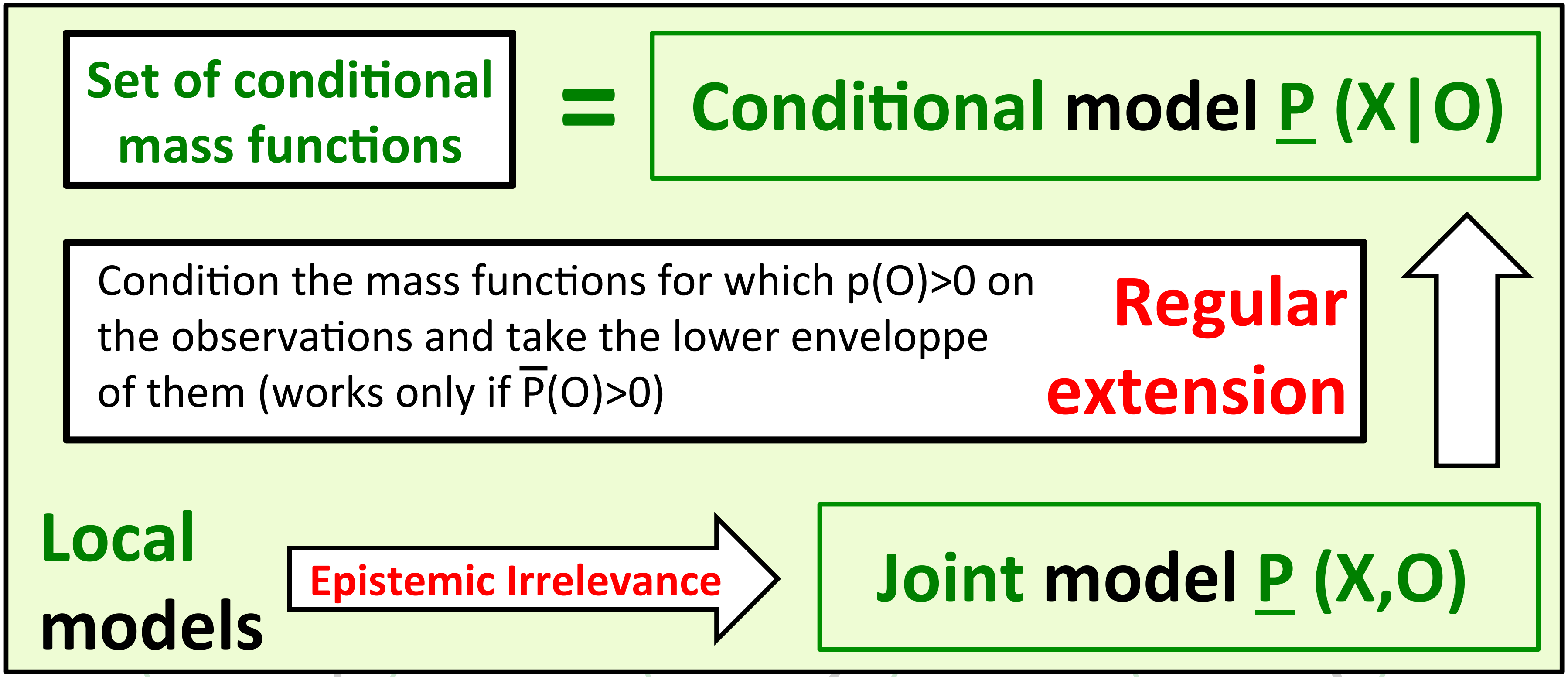
Local models

Epistemic Irrelevance

Joint model $\underline{P}(X,O)$

**Conditioning the joint model on the observations**

# (Imprecise) state sequence estimation



Set of conditional mass functions = Conditional model $\underline{P}(X|O)$

Condition the mass functions for which p(O)>0 on the observations and take the lower enveloppe of them (works only if $\overline{P}(O)>0$)

**Regular extension**

**Local models**

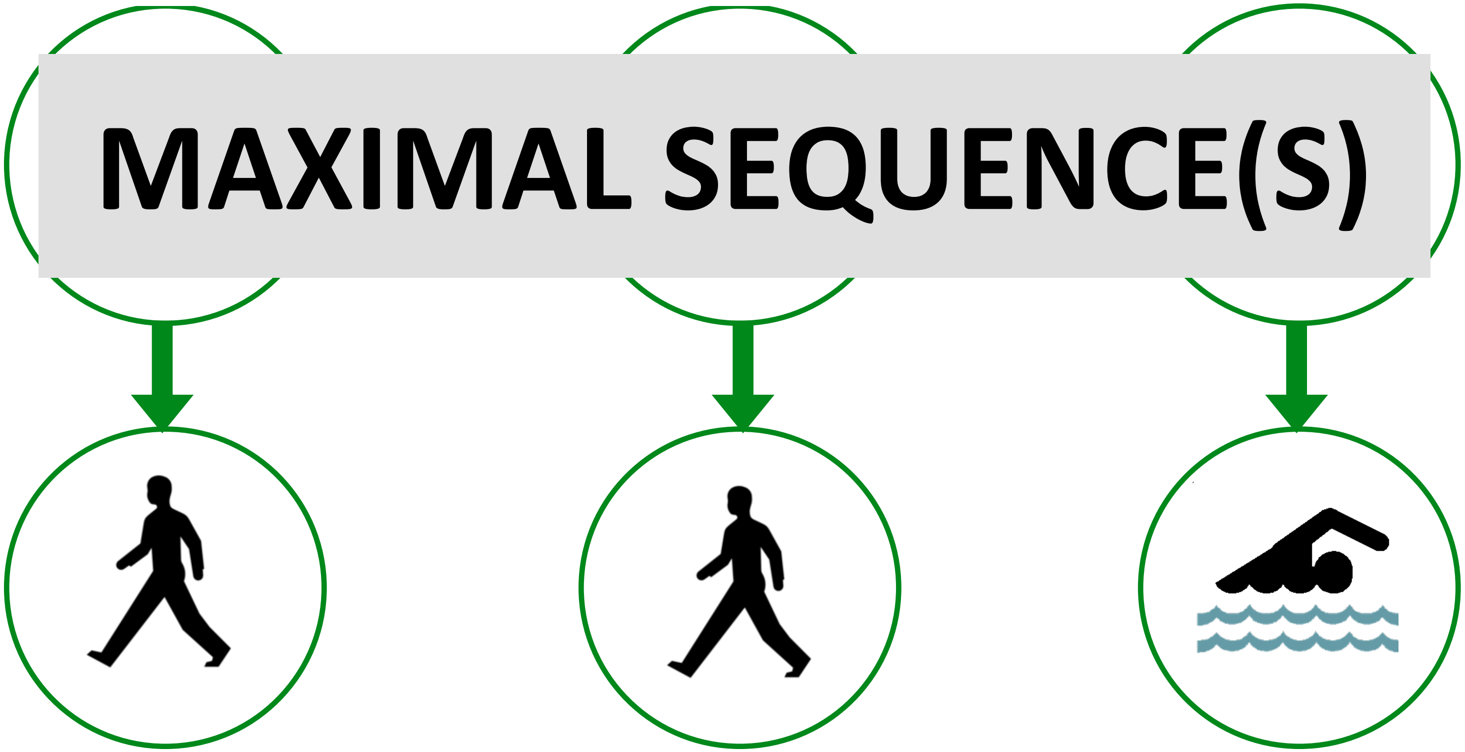
**Epistemic Irrelevance**

**Joint model $\underline{P}(X,O)$**

**Conditioning the joint model on the observations**

# (Imprecise) state sequence estimation

**How to estimate the state sequence?**



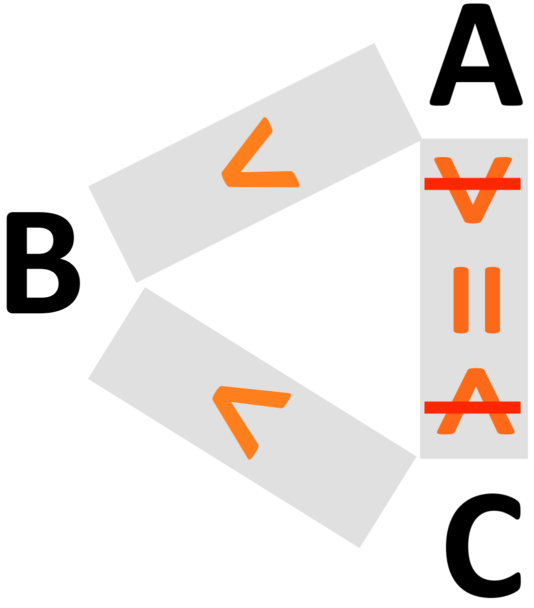
**MAXIMAL SEQUENCE(S)**

# (Imprecise) state sequence estimation

**PRECISE: total ordering**

**A > B if p(A|O) > p(B|O)**

**( O = 🚶 🚶 🏊 )**



**Maximal sequence(s):**

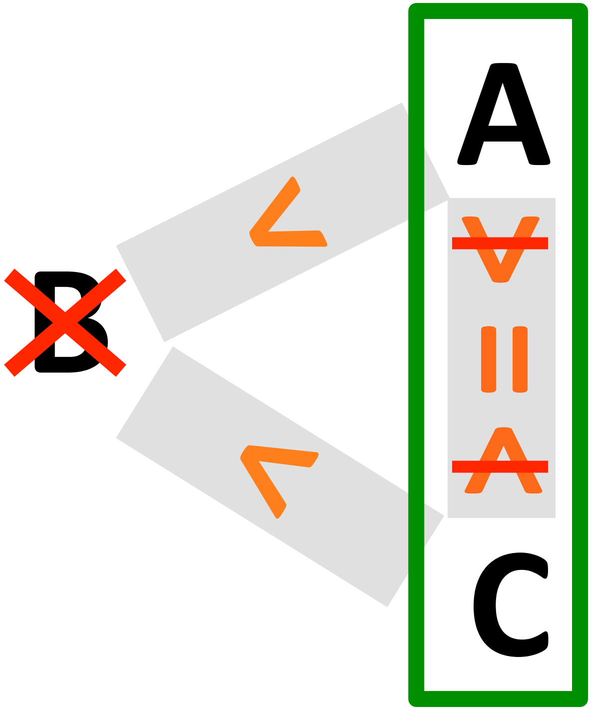
**the undominated sequence(s) in this total ordering**

# (Imprecise) state sequence estimation

**PRECISE: total ordering**

A > B if p(A|O) > p(B|O)

( O = 🚶 🚶 🏊 )



**Maximal sequence(s):**

the undominated sequence(s)
in this total ordering

# (Imprecise) state sequence estimation

**IMPRECISE ?**

**$A > B$ if ~~$p(A|O) > p(B|O)$~~ ?**

# (Imprecise) state sequence estimation

**IMPRECISE:**

$A > B$ if $p(A|O) > p(B|O)$ for all $p(X|O)$ in $\underline{P}(X|O)$

**Criterion of MAXIMALITY!**

**(hence the name: maximal sequences)**

# (Imprecise) state sequence estimation

**IMPRECISE:**

A > B if $p(A|O) > p(B|O)$ for all $p(X|O)$ in $\underline{P}(X|O)$

$\Leftrightarrow$ $P(I_A|O) > P(I_B|O)$ for all $P(X|O)$ in $\underline{P}(X|O)$

$\Leftrightarrow$ $P(I_A - I_B|O) > 0$ for all $P(X|O)$ in $\underline{P}(X|O)$

$\Leftrightarrow$ $\underline{P}(I_A - I_B|O) > 0$

**Always correct, but hard to calculate…**

**Can we use the joint directly?**

# (Imprecise) state sequence estimation

**IMPRECISE:**

$A > B$ if $p(A|O) > p(B|O)$ for all $p(X|O)$ in $\underline{P}(X|O)$

$\Updownarrow$ if $p(O)>0$ for all $p(X,O)$ in $\underline{P}(X,O)$

$(\underline{P}(O)>0)$

$p(A,O) > p(B,O)$ for all $p(X,O)$ in $\underline{P}(X,O)$

$\Leftrightarrow$ $P(I_A I_O) > P(I_B I_O)$ for all $P(X,O)$ in $\underline{P}(X,O)$

$\Leftrightarrow$ $P([I_A - I_B]\, I_O) > 0$ for all $P(X,O)$ in $\underline{P}(X,O)$

$\Leftrightarrow$ $\underline{P}([I_A - I_B]\, I_O) > 0$

# (Imprecise) state sequence estimation

**IMPRECISE:**

$A > B$ if $p(A|O) > p(B|O)$ for all $p(X|O)$ in $\underline{P}(X|O)$

$\Updownarrow$ if $p(O)>0$ for all $p(X,O)$ in $\underline{P}(X,O)$

($\underline{P}(O)>0$) necessary?

We want to allow local lower probabilities to be zero!

$\underline{P}([I_A - I_B]\, I_O) > 0$

# (Imprecise) state sequence estimation

**IMPRECISE:**

A > B if $p(A|O) > p(B|O)$ for all $p(X|O)$ in $\underline{P}(X|O)$

⇕

**For forward irrelevant HMMs:**

if all local upper probabilities are strictly positive (lower probabilities may be zero)

**DOES NOT HOLD IN GENERAL!**

$$\underline{P}([I_A - I_B]\, I_O) > 0$$

# (Imprecise) state sequence estimation

**IMPRECISE:**

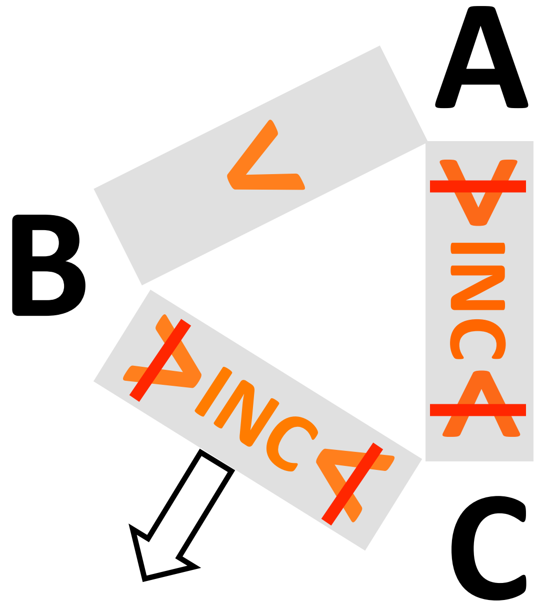
A > B if $\underline{P}([I_A - I_B]\; I_O) > 0$ !

# (Imprecise) state sequence estimation

**IMPRECISE: partial ordering**

**A > B if $\underline{P}([I_A - I_B]\ I_O) > 0$**

**Maximal sequence(s):**
**undominated sequence(s)**
**in this partial ordering**



B and C are
INCOMPARABLE

## X is maximal

⇕

**For all Y : Y $\not>$ X ⇔ For all Y : $\underline{P}([I_Y - I_X]\ I_O) \leq 0$**

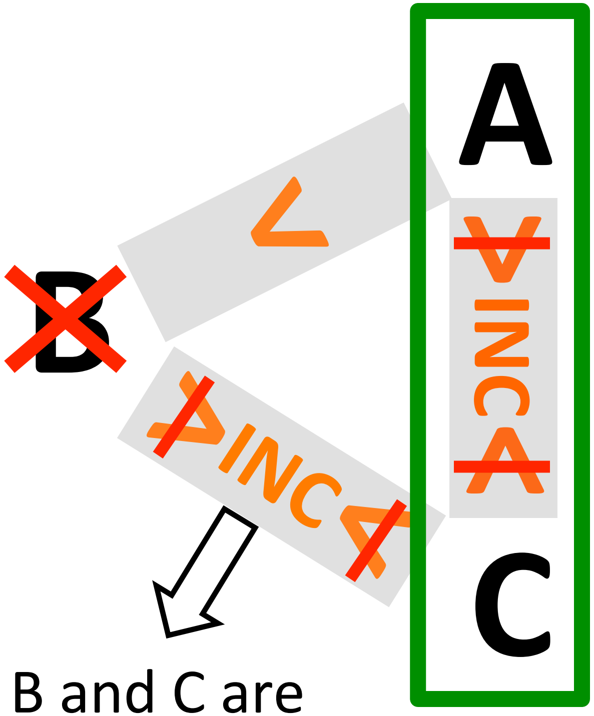
# (Imprecise) state sequence estimation

**IMPRECISE: partial ordering**

**$A > B$ if $\underline{P}([I_A - I_B]\ I_O) > 0$**

**Maximal sequence(s):**

**undominated sequence(s) in this partial ordering**



B and C are INCOMPARABLE

**X is maximal**

$$\Updownarrow$$

**For all Y : $Y \not> X$ $\Longleftrightarrow$ For all Y : $\underline{P}([I_Y - I_X]\ I_O) \leq 0$**

# (Imprecise) state sequence estimation

**X is maximal** ⟺ For all Y : $\underline{P}([I_Y - I_X]\, I_O) \leq 0$

**How can we determine the set of maximal sequences efficiently?**

# (Imprecise) state sequence estimation

**X is maximal** ⟺ **For all Y : $\underline{P}([I_Y - I_X]\, I_O) \leq 0$**

**How can we determine the set of maximal sequences efficiently?**

**EstiHMM:** an efficient algorithm to determine the maximal state sequences in an imprecise hidden Markov model

# (Imprecise) state sequence estimation

**How can we determine the set of maximal sequences efficiently?**

**Trick nr. 1**

Using the joint model instead of the conditional one

$$Y > X \ \text{ if } \ \underline{P}(I_Y - I_X | O) > 0 \quad \Longleftrightarrow \quad \underline{P}([I_Y - I_X]\, I_O) > 0$$

$$\Downarrow$$

In forward irrelevant HMMs with strictly positive local upper probabilities

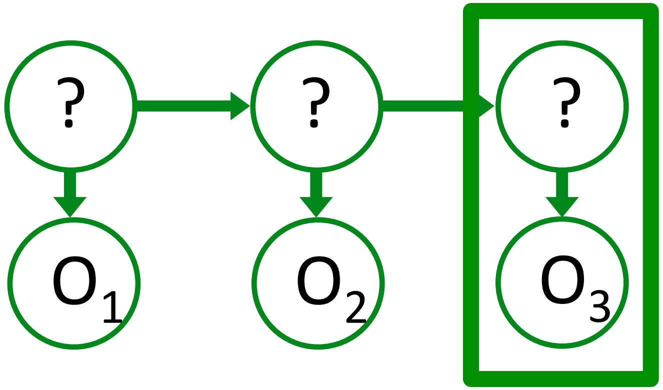
# (Imprecise) state sequence estimation

**How can we determine the set of maximal sequences efficiently?**

**Trick nr. 2**

**Working recursively**

**Principle of optimality (Bellman)**



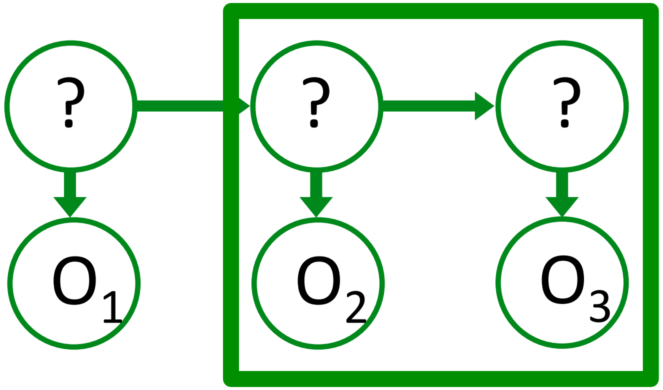
# (Imprecise) state sequence estimation

How can we determine the set of **maximal sequences** efficiently?

**Trick nr. 2**

Working recursively

Principle of optimality (Bellman)



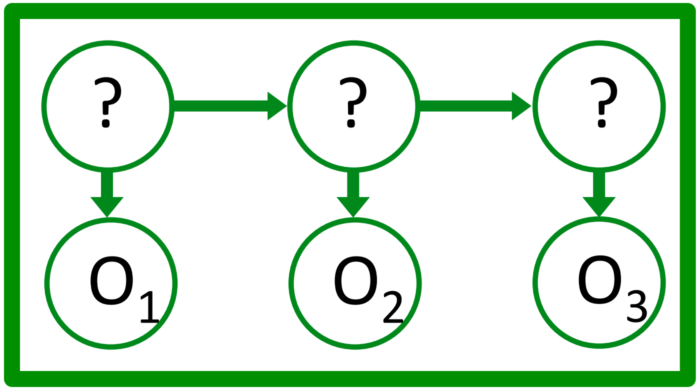
# (Imprecise) state sequence estimation

**How can we determine the set of maximal sequences efficiently?**

**Trick nr. 2**

**Working recursively**

**Principle of optimality (Bellman)**



# (Imprecise) state sequence estimation

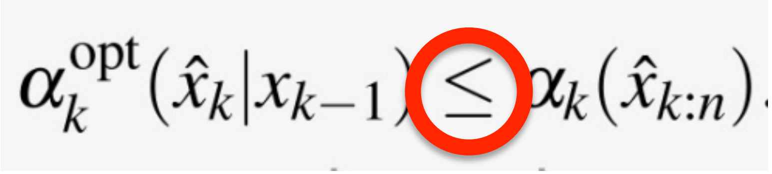
**How can we determine the set of maximal sequences efficiently?**

**Trick nr. 3**

Reformulating the criterion of maximality

X is maximal ⇔ For all Y : $\underline{P}(I_O[I_Y - I_X]) \leq 0$

⇔



$$\alpha_k^{\text{opt}}(\hat{x}_k|x_{k-1}) \leq \alpha_k(\hat{x}_{k:n}).$$

# (Imprecise) state sequence estimation

**How can we determine the set of maximal sequences efficiently?**

**Trick nr. 4**

## Storing solutions efficiently

**6 (possibly) maximal sequences for a binary HMM of length 8:**

Two state values: **0** or **1**

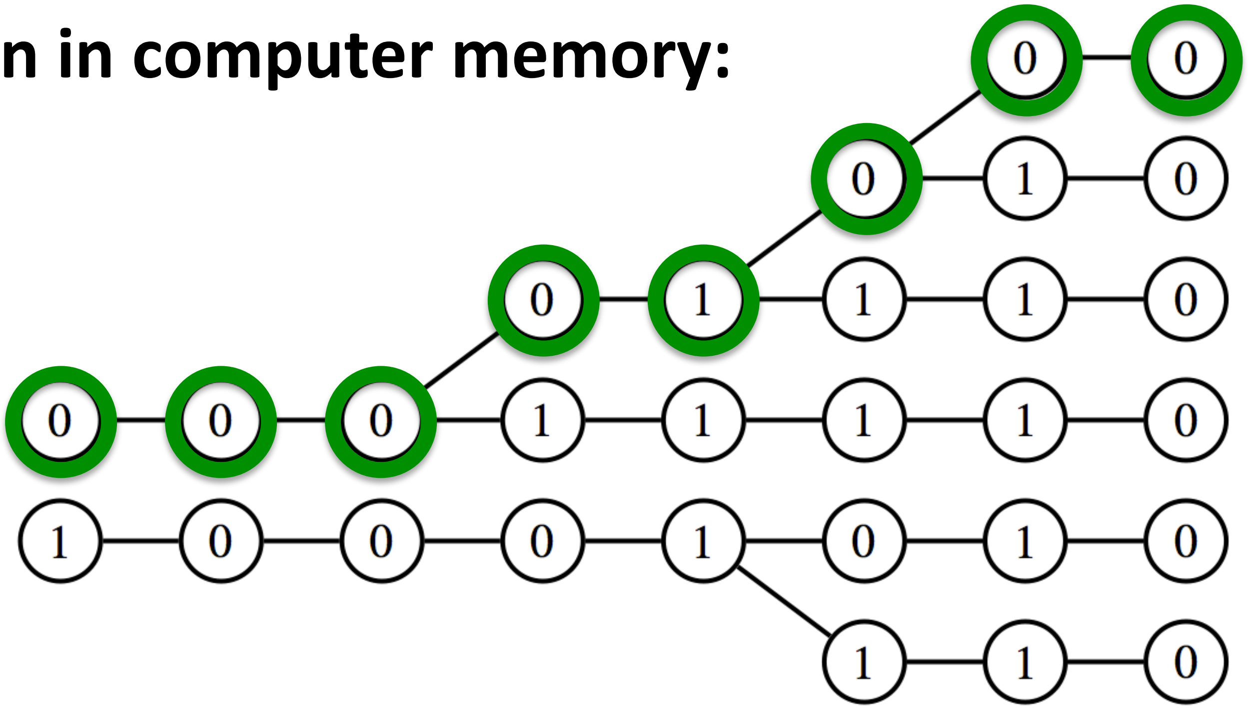
$$\{00001000, 00001010, 00001110, 00011110, 10001010, 10001110\}$$

# (Imprecise) state sequence estimation

**Trick nr. 4** Storing solutions efficiently

$\{00001000, 00001010, 00001110, 00011110, 10001010, 10001110\}$

**Representation in computer memory:**



# (Imprecise) state sequence estimation

**Trick nr. 4** Storing solutions efficiently

$$\{00001000, 00001010, 00001110, 00011110, 10001010, 10001110\}$$

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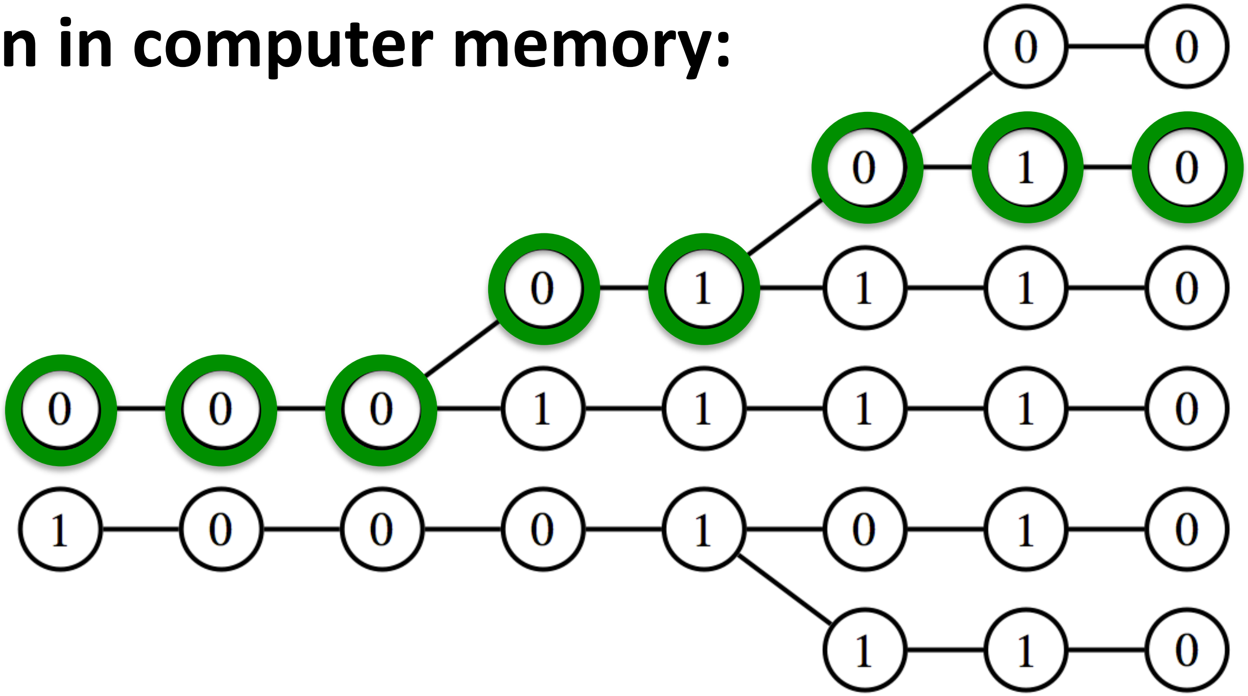

# (Imprecise) state sequence estimation

**Trick nr. 4** Storing solutions efficiently

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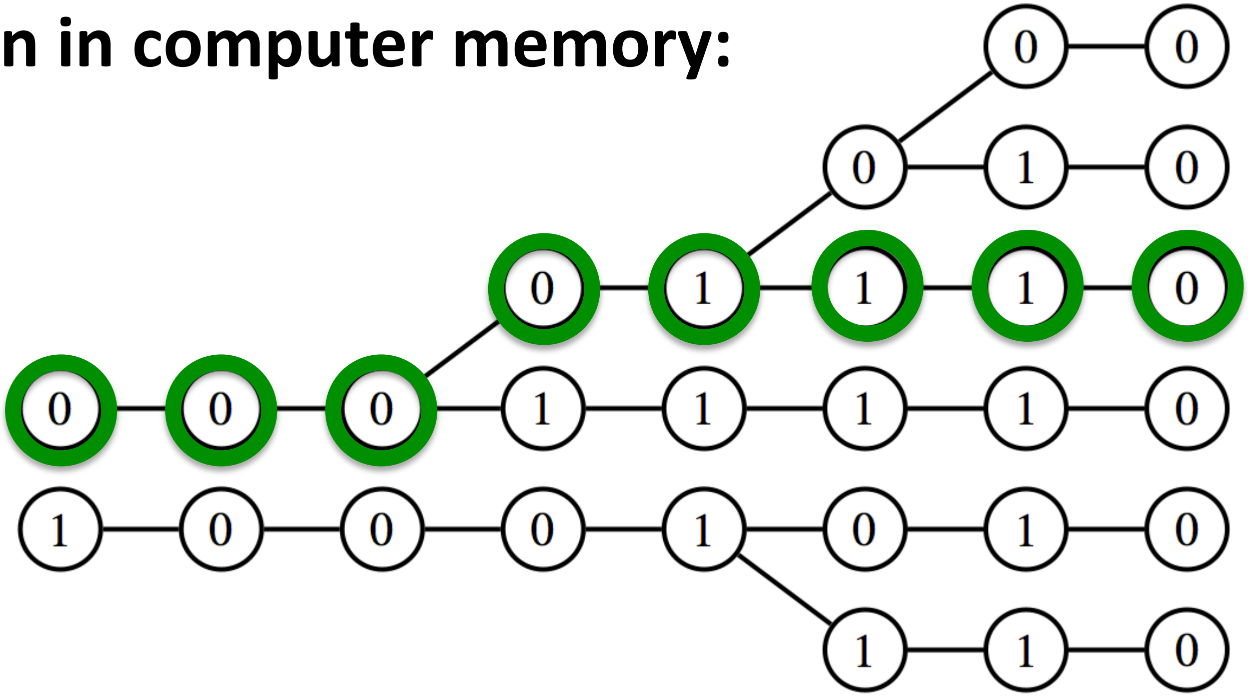

# (Imprecise) state sequence estimation

**Trick nr. 4** Storing solutions efficiently

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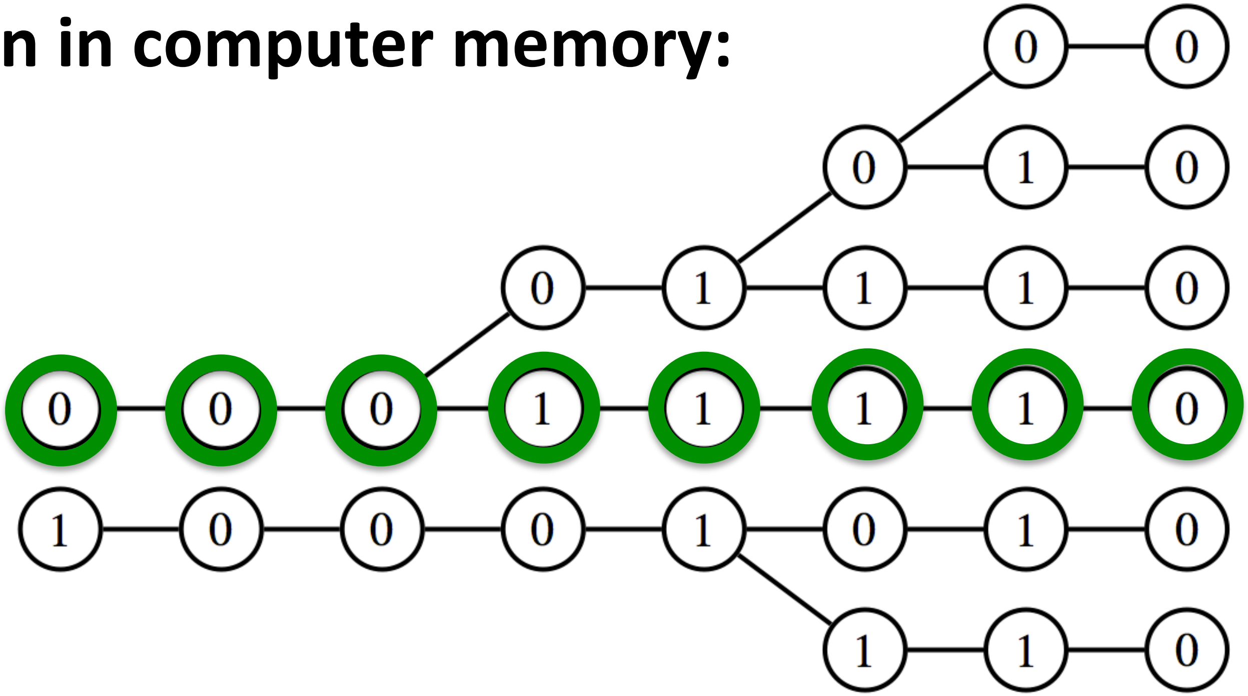
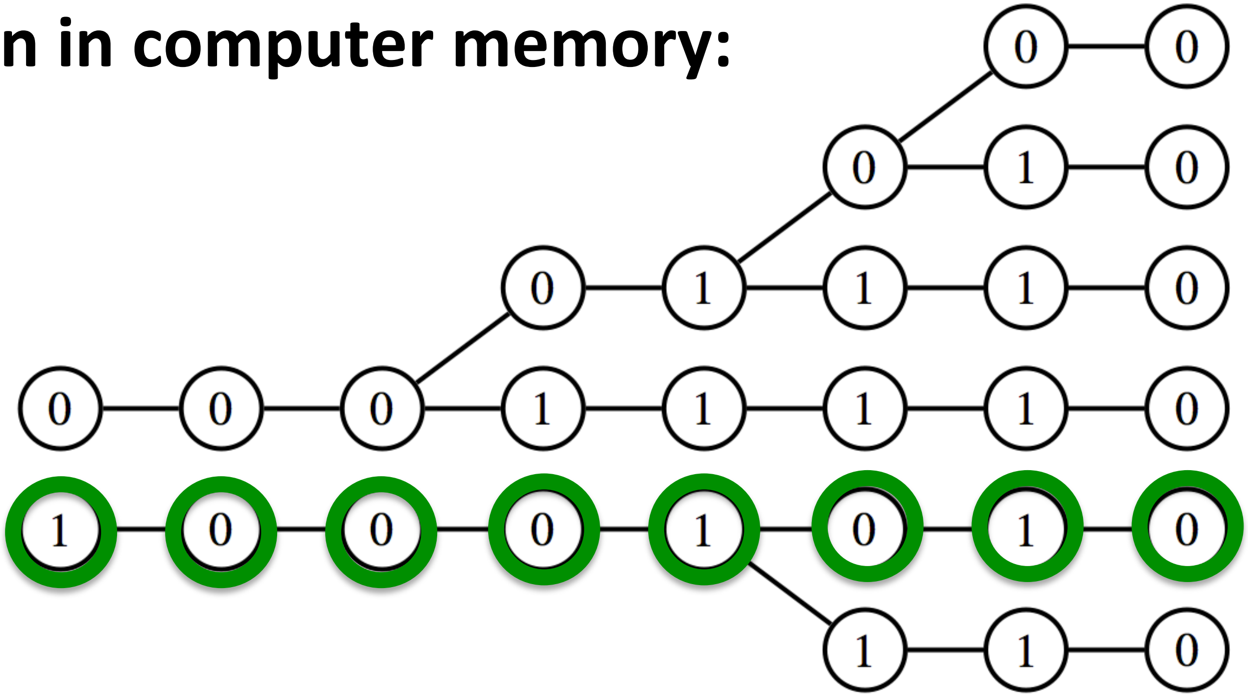


# (Imprecise) state sequence estimation

## **Trick nr. 4** Storing solutions efficiently

$\{00001000, 00001010, 00001110, 00011110, 10001010, 10001110\}$

**Representation in computer memory:**

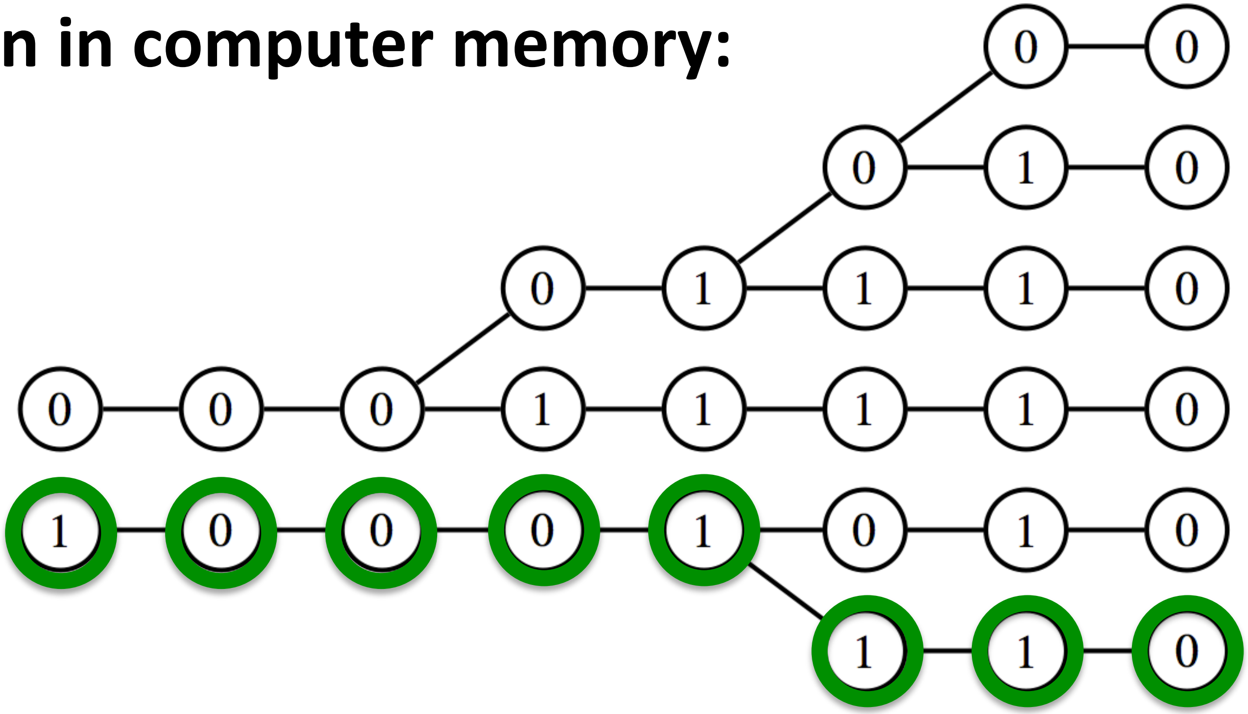


# (Imprecise) state sequence estimation

**Trick nr. 4** Storing solutions efficiently

$$\{00001000, 00001010, 00001110, 00011110, 10001010, 10001110\}$$

**Representation in computer memory:**



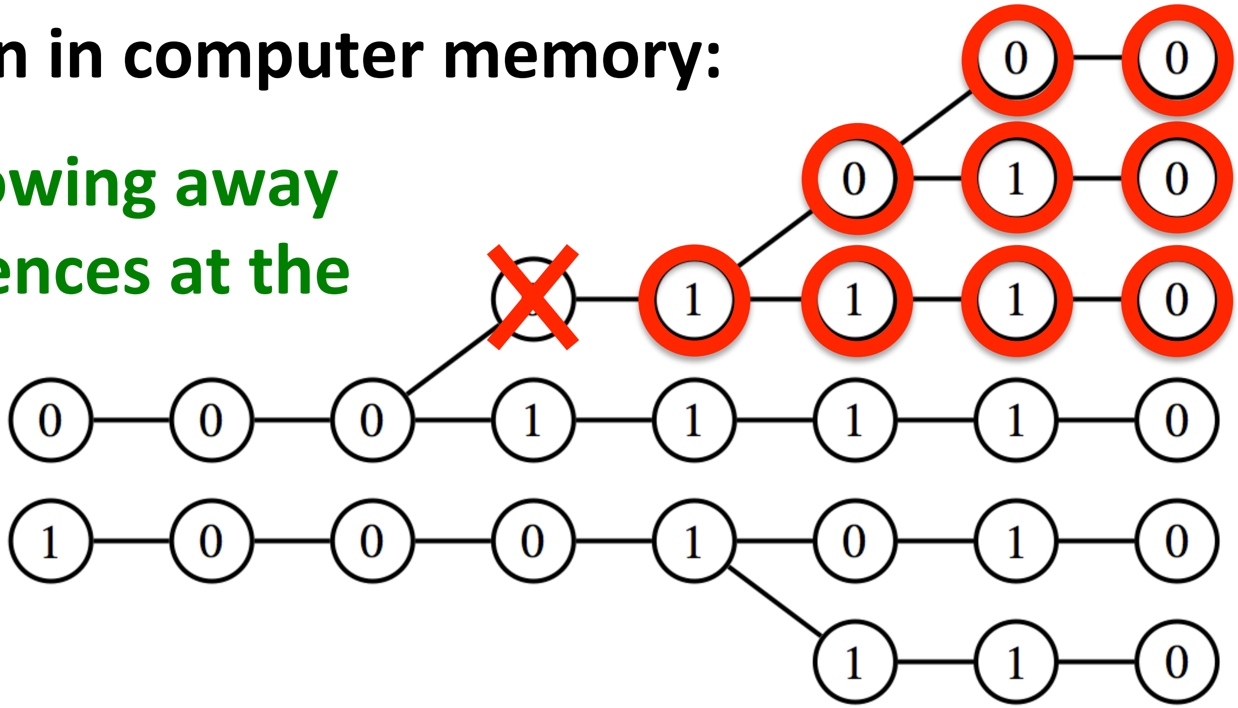
# (Imprecise) state sequence estimation

**Trick nr. 4** Storing solutions efficiently

$\{\cancel{00001000}, \cancel{00001010}, \cancel{00001110}, 00011110, 10001010, 10001110\}$

**Representation in computer memory:**

**Efficiently throwing away multiple sequences at the same time!**



# (Imprecise) state sequence estimation

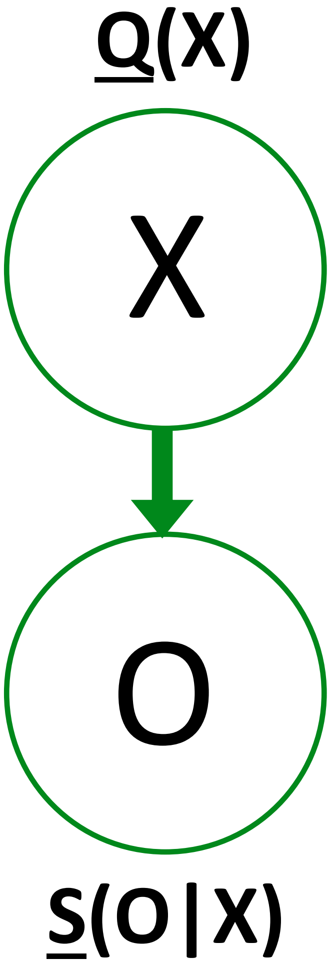
**EstiHMM:** an **efficient algorithm** to determine the maximal state sequences in an imprecise hidden Markov model

## Computational complexity

- **Linear in the number of maximal sequences!**
- **Quadratic in the length of the HMM**
- **Cubic in the number of possible states**

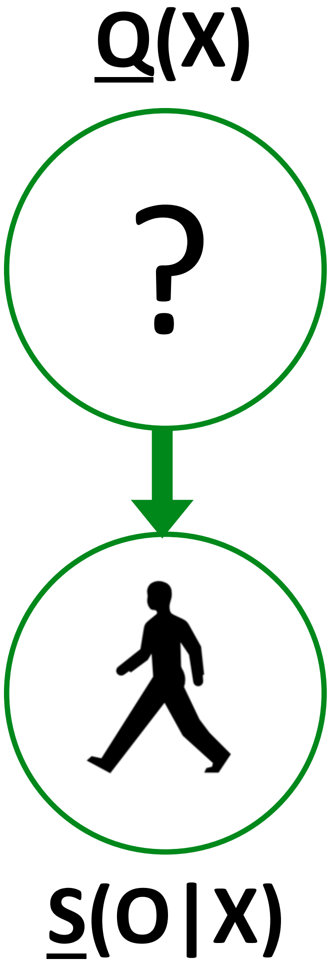
**EXERCISE!**

# Exercise (part 2)


$\underline{Q}(X)$

$\underline{S}(O|X)$

# Exercise (part 2)



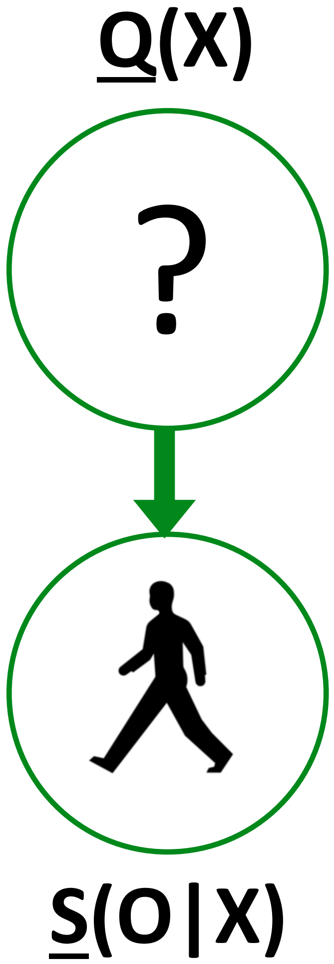
$\underline{Q}(X)$

?

$\underline{S}(O|X)$

Estimate the (hidden) state!

# Exercise (part 2)



## Estimate the (hidden) state!

For all gambles **f** on X:

**$\underline{Q}$(f)** = 0.9[0.7**f**(☀) + 0.2**f**(☁) + 0.1**f**(🌧)]
+ 0.1min{**f**(☀), **f**(☁), **f**(🌧)}

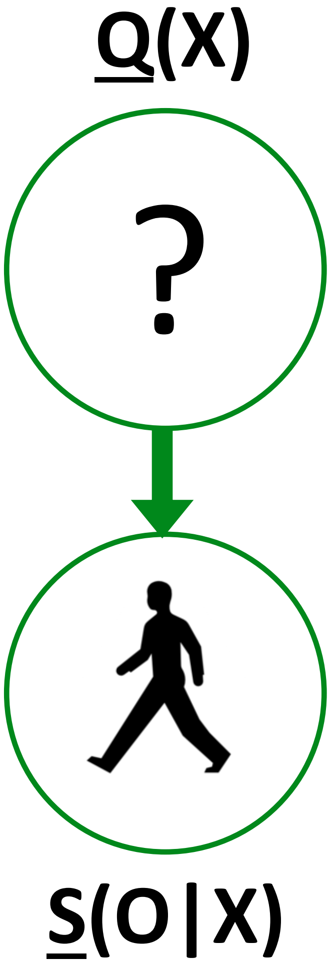
For all gambles **f** on O:

**$\underline{S}$(f|☀)** = 0.9[0.1**f**(📺) + 0.2**f**(🚶) + 0.7**f**(🏊)]
+ 0.1min{**f**(📺), **f**(🚶), **f**(🏊)}

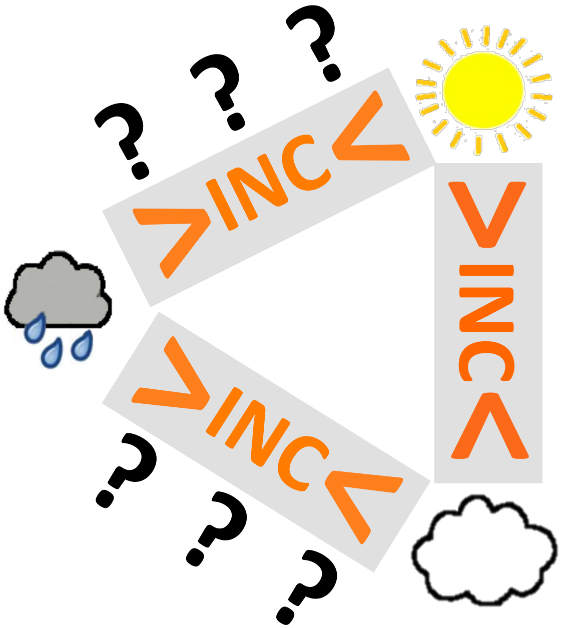
**$\underline{S}$(f|☁)** = 0.9[0.3**f**(📺) + 0.6**f**(🚶) + 0.1**f**(🏊)]
+ 0.1min{**f**(📺), **f**(🚶), **f**(🏊)}

**$\underline{S}$(f|🌧)** = 0.9[0.9**f**(📺) + 0.1**f**(🚶) + 0.0**f**(🏊)]
+ 0.1min{**f**(📺), **f**(🚶), **f**(🏊)}

# Exercise (part 2)

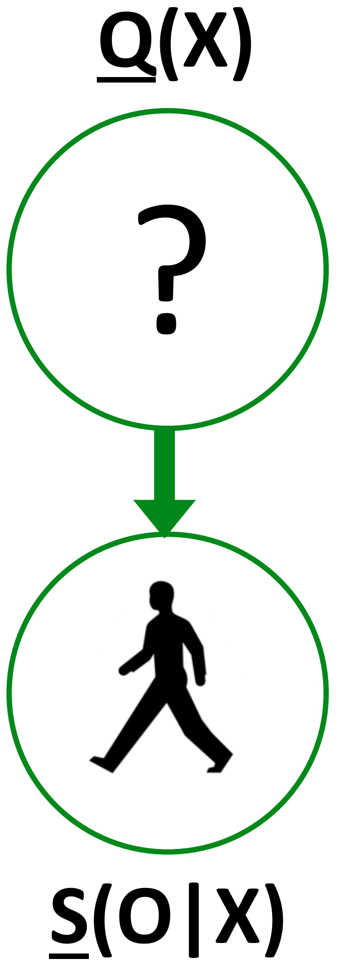


**Maximal estimates:** undominated estimates in the partial ordering $>$



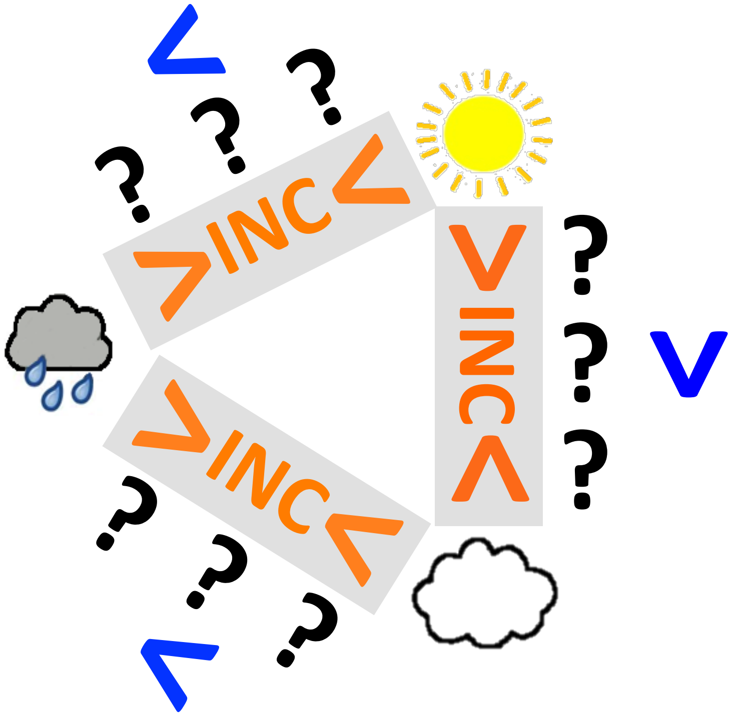
# Exercise (part 2)

**Maximal estimates:** undominated estimates in the partial ordering >



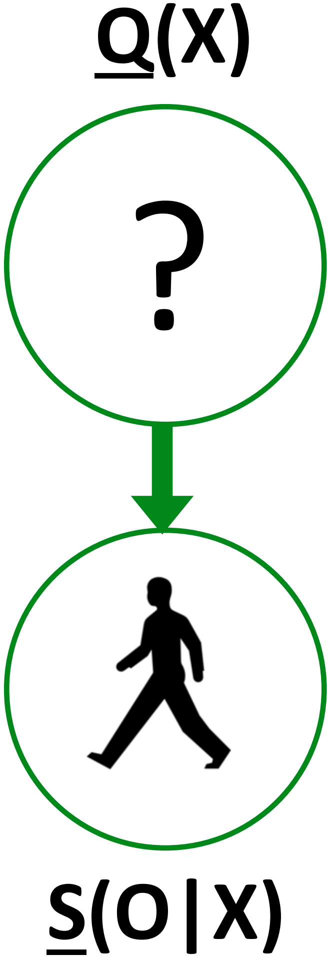
PRECISE

IMPRECISE



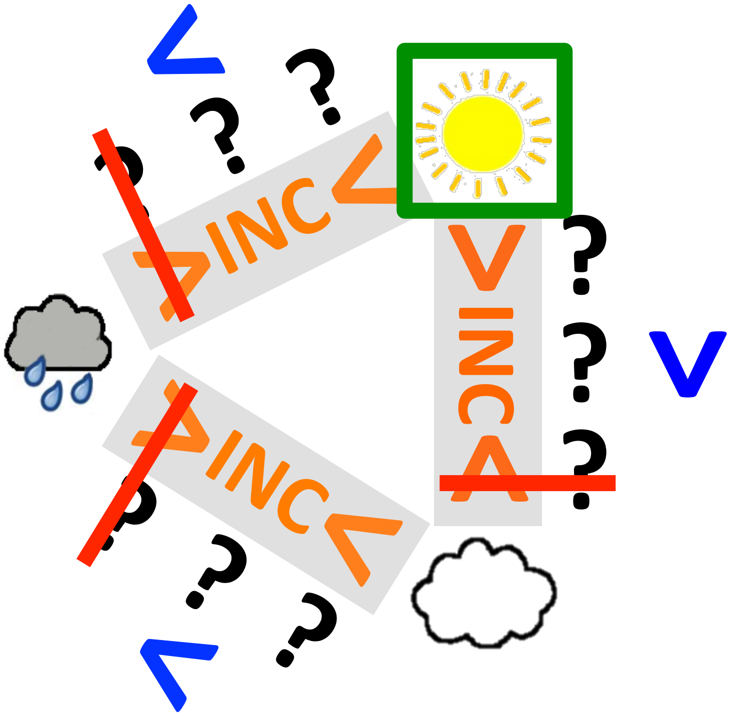
# Exercise (part 2)

**Maximal estimates:** undominated estimates in the partial ordering >



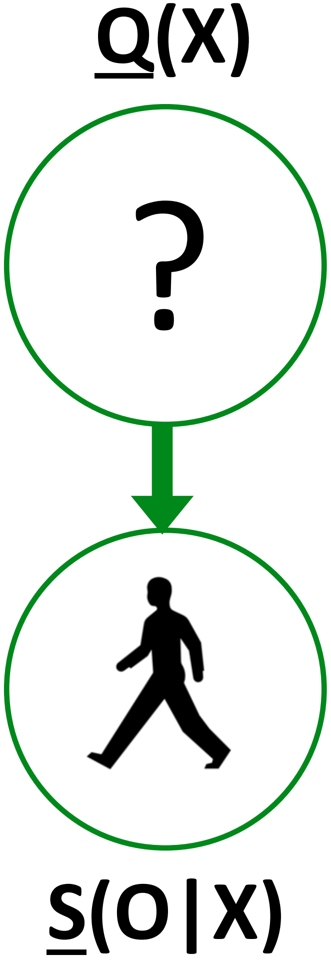
PRECISE

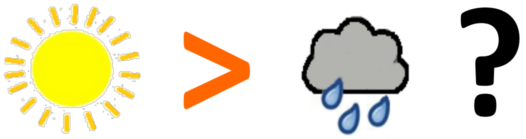
IMPRECISE

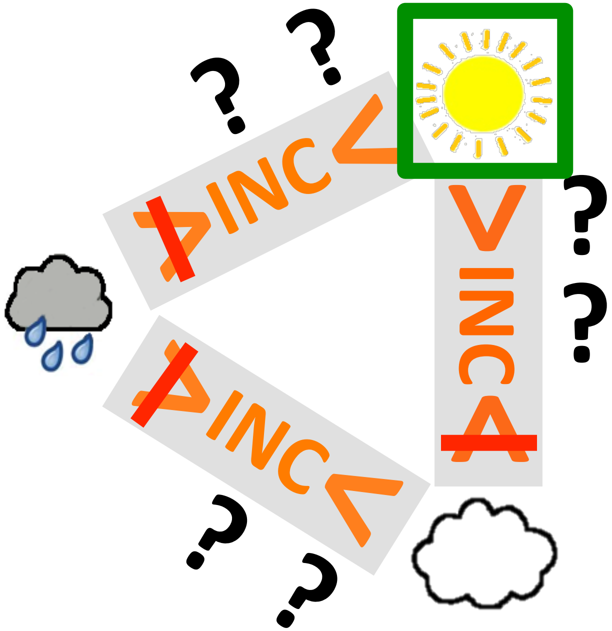


# Exercise (part 2)

**Maximal estimates:** undominated estimates in the partial ordering >

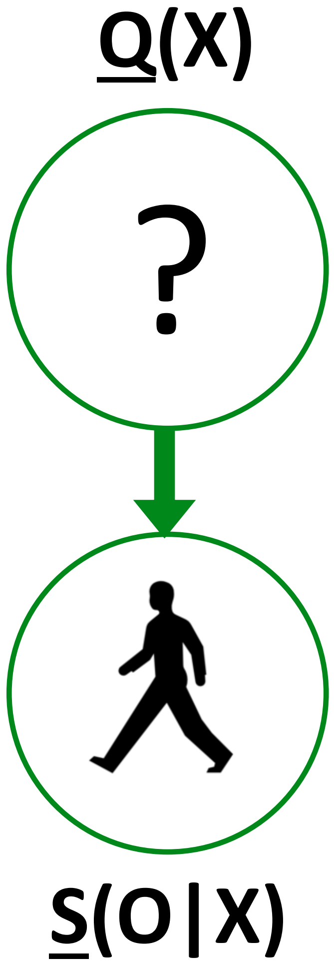






# Exercise (part 2)

**Maximal estimates:**
**undominated estimates in the partial ordering >**



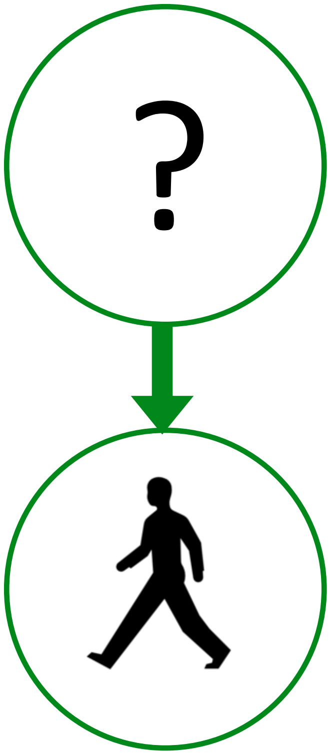
☀ > 🌧 ?

$\underline{P}([I_{☀} - I_{🌧}] \mid I_{🚶}) > 0$ ?

# Exercise (part 2)

**Maximal estimates:** undominated estimates in the partial ordering >

$\underline{Q}(X)$



$\underline{S}(O|X)$

☀ > 🌧 ?

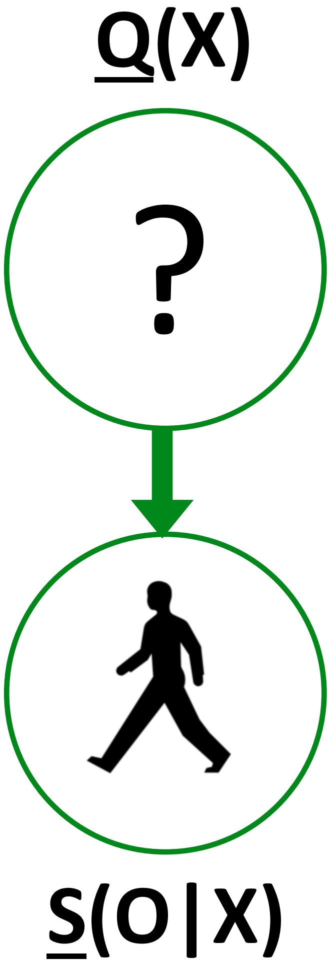
$\underline{P}([I_{☀} - I_{🌧}]\, I_{🚶}) > 0$ ?

|| (marginal extension)

$$\underline{Q}(\, \underbrace{\underline{S}([I_{☀} - I_{🌧}]\, I_{🚶} | X)}_{\text{function of } X!} \,) = ?$$

# Exercise (part 2)

**Maximal estimates:** undominated estimates in the partial ordering >



☀ > 🌧 ✓

$\underline{P}([I_{☀} - I_{🌧}]\, I_{🚶}) > 0$ ✓

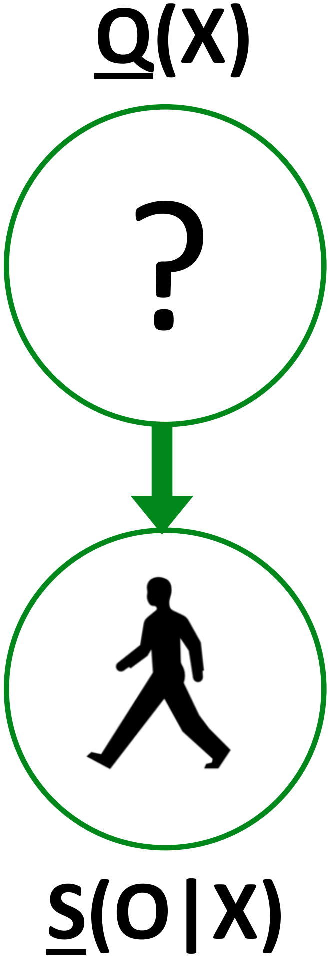
|| (marginal extension)

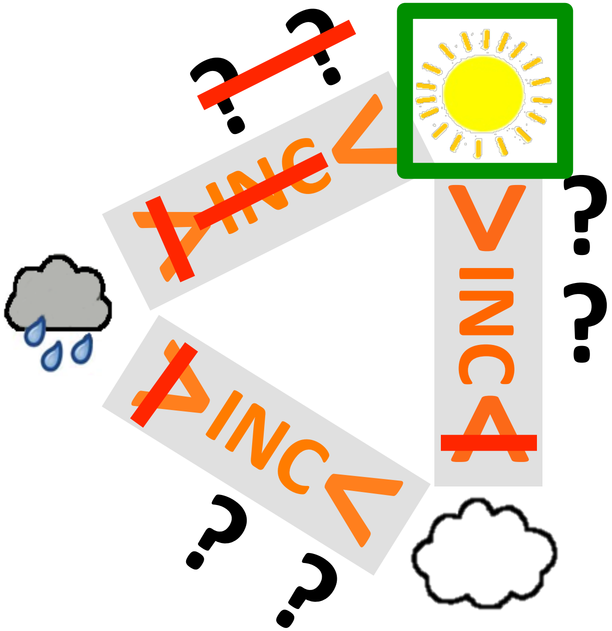
$$\underline{Q}(\, \underbrace{\underline{S}([I_{☀} - I_{🌧}]\, I_{🚶} \,|\, X)}_{\text{function of X!}} \,) = 0.0773$$

# Exercise (part 2)

**Maximal estimates:**

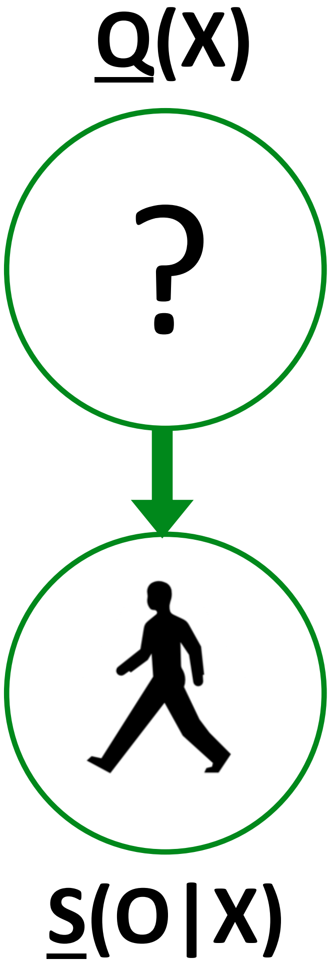
**undominated estimates in the partial ordering >**

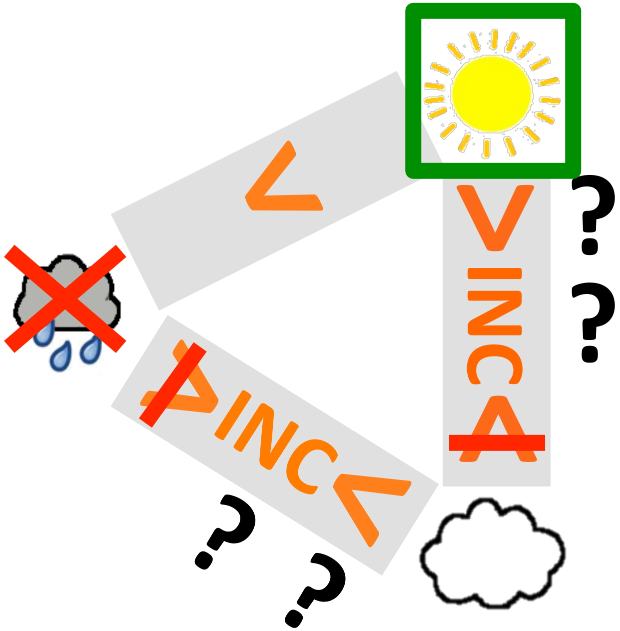




# Exercise (part 2)

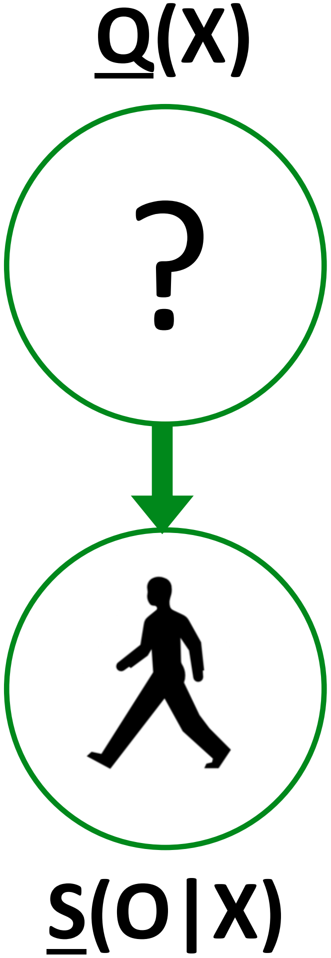
**Maximal estimates:**
**undominated estimates in the partial ordering >**

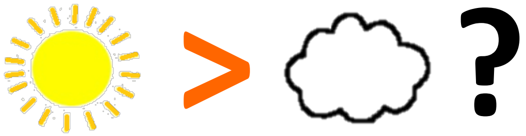


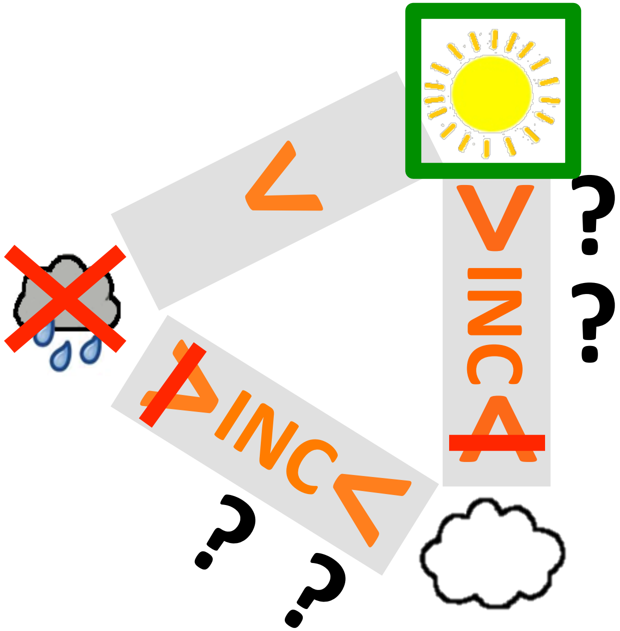


# Exercise (part 2)

**Maximal estimates:**
**undominated estimates in the partial ordering >**



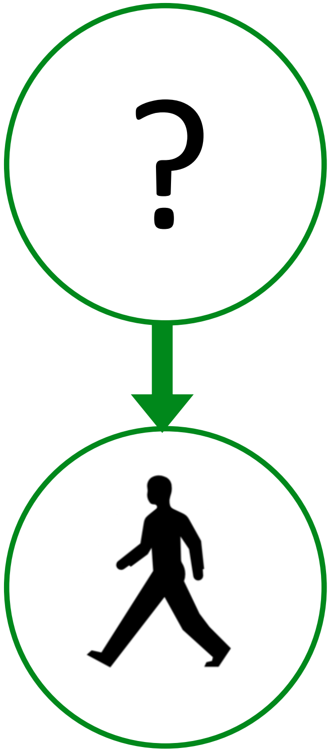




# Exercise (part 2)

**Maximal estimates:** undominated estimates in the partial ordering $>$

$\underline{Q}(X)$



$\underline{S}(O|X)$

☀ $>$ ☁ ?

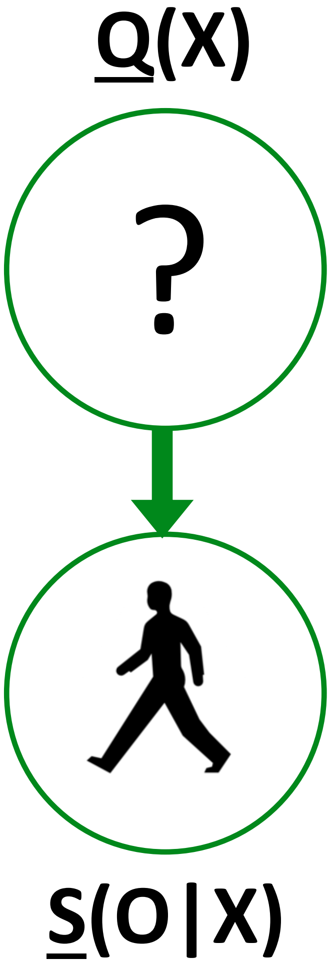
$\underline{P}([I_{☀} - I_{☁}]\, I_{🚶}) > 0$ ?

$\|$ (marginal extension)

$\underline{Q}(\, \underbrace{\underline{S}([I_{☀} - I_{☁}]\, I_{🚶} | X)}_{\text{function of X!}} \,) = ?$

# Exercise (part 2)

**Maximal estimates:** undominated estimates in the partial ordering >



☀ > ☁ ✗

$\underline{P}([I_{\text{☀}} - I_{\text{☁}}]\, I_{\text{🚶}}) > 0$ ✗

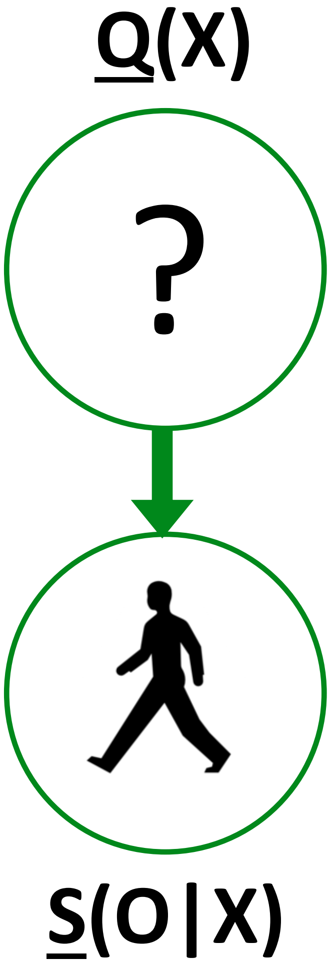
‖ (marginal extension)

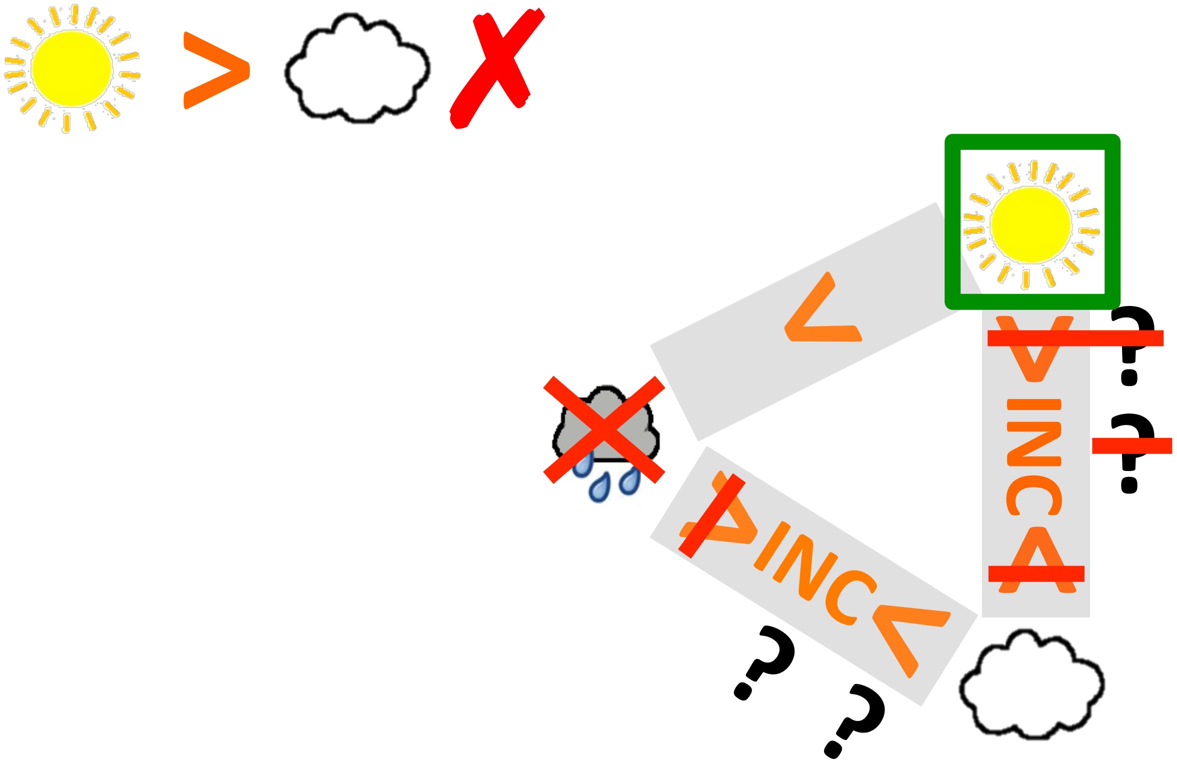
$$\underline{Q}(\,\underbrace{\underline{S}([I_{\text{☀}} - I_{\text{☁}}]\, I_{\text{🚶}}\,|X)}_{\text{function of } X!}\,) = -0.0658$$

# Exercise (part 2)

**Maximal estimates:**

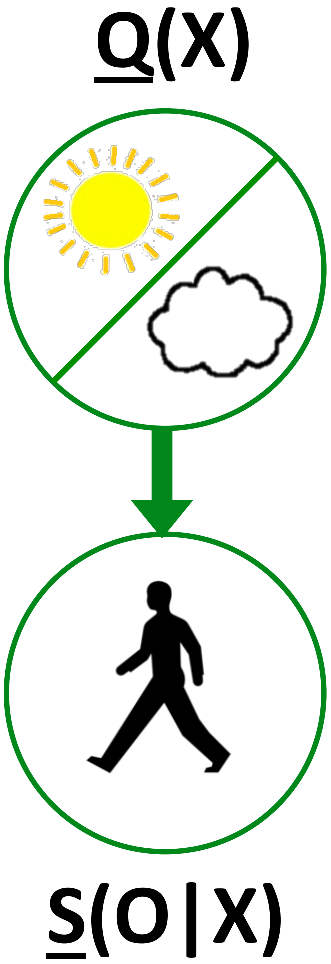
**undominated estimates in the partial ordering >**



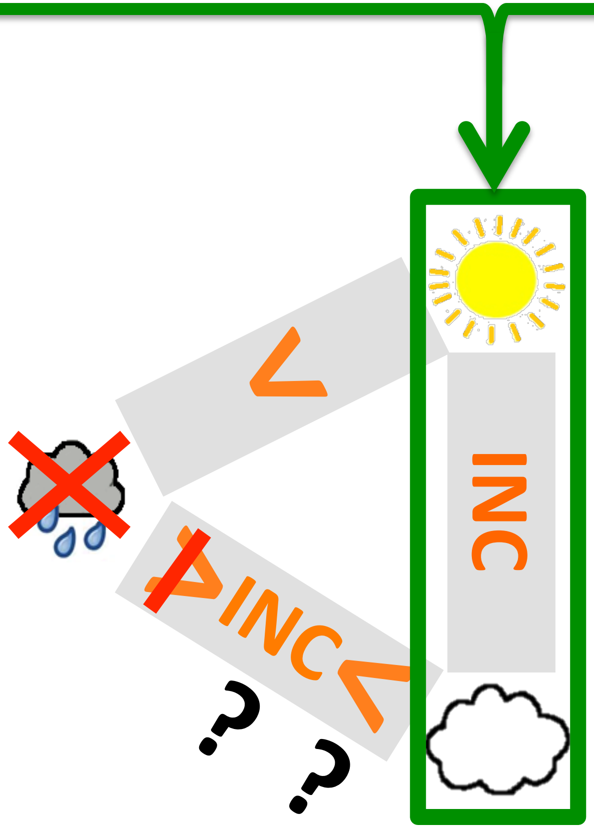


# Exercise (part 2)

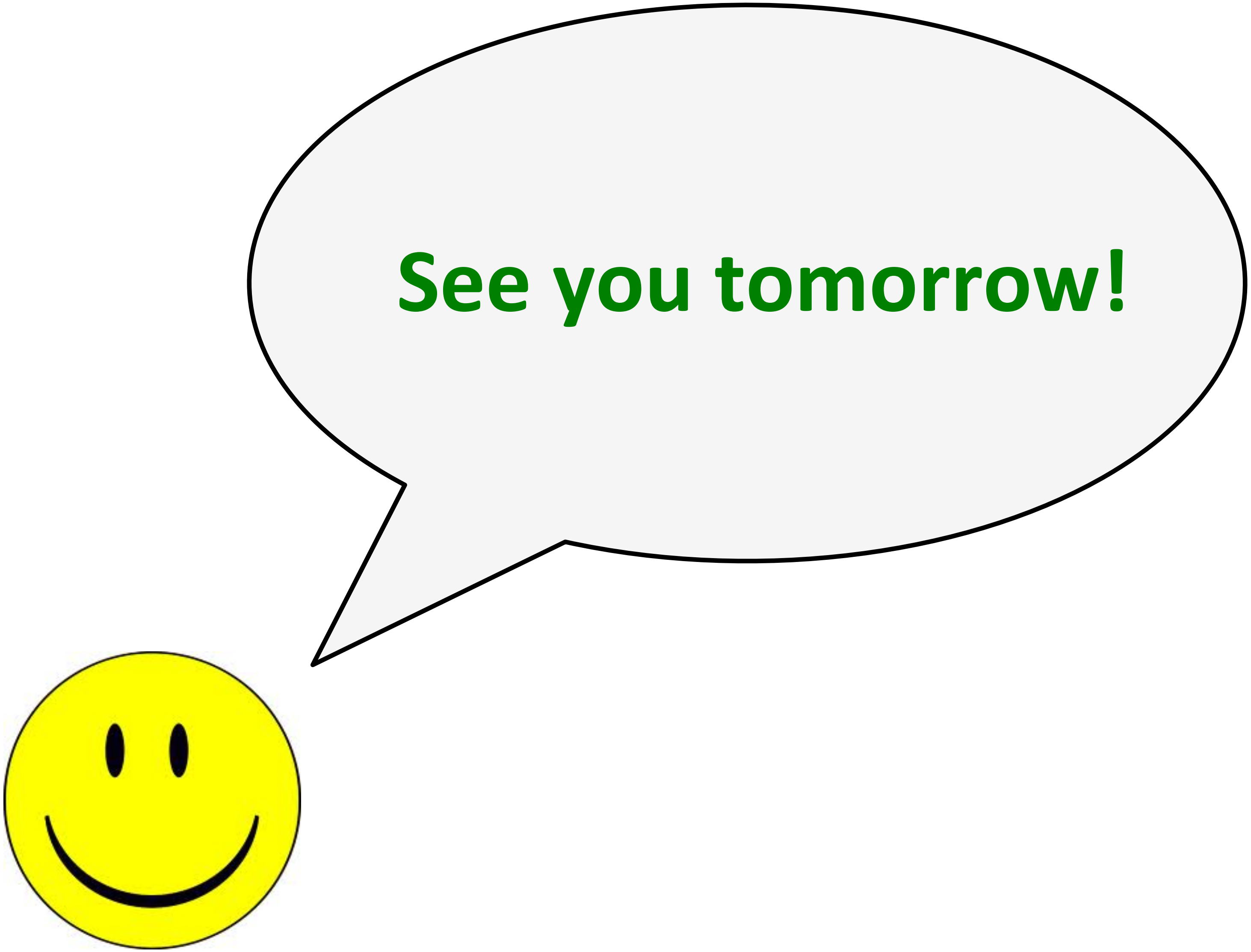
**Maximal estimates:** undominated estimates in the partial ordering >



✓



APPLICATIONS ?



**See you tomorrow!**