Optimal Control of Linear Systems with Quadratic Cost and Imprecise Forward Irrelevant Input Noise

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Preliminaries We consider a finite-state, discrete-time stationary linear system with a deterministic (known) initial state $X_0 = x_0$. For all $k \in \{0, \ldots, n\}$, the dynamics of the system is described by

$$X_{k+1} = aX_k + bu_k(X_{0:k}) + W_k.$$

In this expression, a and b are real-valued parameters and the state X_k and noise W_k at time k are real-valued random variables. The control input u_k at time k is also real-valued and is taken to be some function of the previous states $x_{0:k}$. We call a tuple of control input functions $u_{0:n} \coloneqq (u_0, u_1, \ldots, u_n)$ a *control policy.* We measure the performance of such a control policy by means of the associated linear quadratic cost

$$\eta \left[u_{0:n} \middle| x_0 \right] \coloneqq \sum_{k=0}^{n} r u_k(X_{0:k})^2 + qX_{k+1}^2,$$

where r is a strictly positive real number and q is a non-negative real number.

The Precise Case If the uncertain noise terms W_k are modelled by means of a probability measure, an optimal control policy is usually required to minimise the expected value of the cost. Under some relatively weak technical assumptions, there will be a unique control policy $\hat{u}_{0:n}$ that satisfies this optimality criterion. If the noise is white—if the noise terms at different time instants are uncorrelated—then for all $k \in \{0, \ldots, n\}$, this optimal control policy is given by

$$\hat{u}_k(x_{0:k}) \coloneqq -\tilde{r}_k b(m_{k+1} a x_k + h_k), \tag{1}$$

where the parameters $\tilde{r}_k$, m_{k+1} and h_k are derived from the initial conditions $m_{n+1} \coloneqq q$ and $h_{n+1} \coloneqq 0$ and, for all $k \in \{0, \ldots, n\}$, the recursive expressions $\tilde{r}_k \coloneqq (r + b^2 m_{k+1})^{-1}$, $m_k \coloneqq q + a^2 \tilde{r}_k r m_{k+1}$ and

$$h_k \coloneqq m_{k+1} \mathrm{E}\left(W_k\right) + a\tilde{r}_{k+1} r h_{k+1}, \tag{2}$$

where $\mathrm{E}\left(W_k\right)$ is the expected value of W_k. In general, if the noise is not white, computing the optimal control policy $\hat{u}_{0:n}$ is intractable.

The Imprecise Case Our contribution consists in studying a generalised version of this problem, where the noise is described by an imprecise uncertainty model—a set of probability measures—and the optimal control policies are those that are E-admissible—that minimise the expected value of the cost for at least one element of this set. We show that if the model for the noise is *forward irrelevant* [1]—an imprecise notion of independence—then the corresponding set of optimal control policies is again characterised by Eq. (1). The only difference is that h_k is not given by Eq. (2), but instead takes values in some interval. If $a \geq 0$, then for all $k \in \{0, \ldots, n\}$, the exact lower and upper bounds of this interval are

$$\underline{h}_k = m_{k+1} \underline{\mathrm{E}}(W_k) + a\tilde{r}_{k+1} r \underline{h}_{k+1},$$
$$\overline{h}_k = m_{k+1} \overline{\mathrm{E}}(W_k) + a\tilde{r}_{k+1} r \overline{h}_{k+1},$$

with $\overline{h}_{n+1} = \underline{h}_{n+1} \coloneqq 0$, and $\underline{\mathrm{E}}(W_k)$ and $\overline{\mathrm{E}}(W_k)$ the lower and upper expectations of W_k. If $a \leq 0$, $\underline{h}_{k+1}$ and $\overline{h}_{k+1}$ switch places. At first sight, these bounds might seem to follow trivially from Eq. (2), but this is *not* the case, because the optimisation ranges over a forward irrelevant set of probability measures, almost *none* of whose members corresponds to white noise.

In this way, for any time k and state history $x_{0:k}$, we obtain an interval of optimal control inputs. Nevertheless, in a practical control situation, a single control input has to be chosen. The most obvious or lazy choice is to apply the control input which, amongst the ones in the interval, has the lowest absolute value. In future work, we would like to investigate how this type of lazy control performs in practice.

Keywords. Linear system, quadratic cost, optimal control, imprecise noise, forward irrelevance.

References

[1] Gert de Cooman and Enrique Miranda. Forward irrelevance. *Journal of Statistical Planning and Inference*, 139(2):256 – 276, 2009.