

Exponentially-fitted methods applied to fourth-order boundary value problems

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Abstract

Fourth-order boundary value problems are solved by means of exponentially-fitted methods of different orders. These methods, which depend on a parameter, can be constructed following a six-step flow chart of Ixaru and Vanden Berghe. Special attention is paid to the expression of the error term and to the choice of the parameter in order to make the error as small as possible. Some numerical examples are given to illustrate the practical implementation issues of these methods.

Key words: Boundary value problems; Exponential fitting; Error term; Frequency evaluation

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1 Introduction

We consider the numerical solution of the following fourth-order boundary value problem :

$$y^{(4)} + f(t)y = g(t), \quad a \leq t \leq b \quad (1.1)$$

subject to the boundary conditions

$$y(a) = A_1, y''(a) = A_2, y(b) = B_1, y''(b) = B_2. \quad (1.2)$$

Hereby f and g are continuous functions in $[a, b]$ and A_1, A_2, B_1 and B_2 are real constants. The unicity of the solution of this problem is guaranteed if $f(t) \geq 0$ and $f(t) \not\equiv 0$ by a theorem of Usmani [6].

In general, the analytical solution of (1.1), (1.2) can not be determined, and numerical techniques have to be applied. Several papers (see [1]-[18] and citations therein) have already been devoted to the development of such methods for equation (1.1) subject to various kinds of boundary conditions, among which also (1.2).

One possibility to construct suitable methods is to use finite differences, as Usmani did. In [13] it was shown that the formulae upon which Usmani's methods are based can also be obtained using polynomial interpolation. In that paper, modified methods were proposed which are based on a mixed type of interpolating functions, i.e. functions of the form

$$\alpha \cos \omega t + \beta \sin \omega t + \sum_{i=0}^{n-2} \gamma_i t^i$$

where ω is a free parameter which can be fitted to the problem at hand.

As explained in [19], such methods can also be obtained in a much simpler way, using a six step flow chart. In the next section, we will follow this procedure to construct exponentially fitted methods.

2 Derivation of the methods

For $N \geq 5$, we define $t_p := a + ph$ for $p = 0, 1, \dots, N + 1$ whereby $h := (b - a)/(N + 1)$ and we denote the approximate value of the solution $y(t_p)$ in each knot point as y_p .

The finite difference schemes we will construct follow from a central formula and a begin and end formula, respectively. The central formula takes the form

$$\begin{aligned} y_{p-2} + a_1 y_{p-1} + a_0 y_p + a_1 y_{p+1} + y_{p+2} = \\ h^4 \left(b_2 (y_{p+2}^{(4)} + y_{p-2}^{(4)}) + b_1 (y_{p+1}^{(4)} + y_{p-1}^{(4)}) + b_0 y_p^{(4)} \right). \end{aligned} \quad (2.3)$$

The begin formula reads

$$\begin{aligned} c_1 y_0 + c_2 y_1 + c_3 y_2 + y_3 = \\ d_1 h^2 y_0'' + h^4 (d_2 y_0^{(4)} + d_3 y_1^{(4)} + d_4 y_2^{(4)} + d_5 y_3^{(4)} + d_6 y_4^{(4)} + d_7 y_5^{(4)}). \end{aligned}$$

Due to the symmetry of the problem, the end formula can be derived from the begin formula by rewriting the latter in a backward form starting from t_{N+1} .

As explained by Ixaru and Vanden Berghe [19], to construct EF methods, one can follow a six-step procedure. In this section, we consider the first five steps. The last step, which deals with the error of the method, is the subject of the next section.

First consider the construction of the central formula:

- Step i : with $\mathbf{a} := [a_0, a_1, b_0, b_1, b_2]$ we define the operator $\mathcal{L}[h, \mathbf{a}]$ as

$$\begin{aligned} \mathcal{L}[h, \mathbf{a}]y(t) := & y(t - 2h) + a_1 y(t - h) + a_0 y(t) + a_1 y(t + h) + y(t + 2h) \\ & - h^4 \left(b_2 (y^{(4)}(t + 2h) + y^{(4)}(t - 2h)) + b_0 y^{(4)}(t) + b_1 (y^{(4)}(t + h) + y^{(4)}(t - h)) \right). \end{aligned}$$

- Step ii : we determine the maximum value of M such that the algebraic system $\{L_m^*(\mathbf{a}) = 0 | m = 0, 1, 2, \dots, M - 1\}$ with $L_m^*(\mathbf{a}) = h^{-m} \mathcal{L}[h, \mathbf{a}]x^m|_{x=0}$ can be solved.

Due to the symmetry $L_{2k+1}^* = 0$ for any integer value of k . Further, we find

$$\begin{aligned}
L_0^*(\mathbf{a}) &= 2 + 2a_1 + a_0 \\
L_2^*(\mathbf{a}) &= 8 + 2a_1 \\
L_4^*(\mathbf{a}) &= 32 + 2a_1 - 48b_2 - 48b_1 - 24b_0 \\
L_6^*(\mathbf{a}) &= 128 + 2a_1 - 2880b_2 - 720b_1 \\
L_8^*(\mathbf{a}) &= 512 + 2a_1 - 53760b_2 - 3360b_1 \\
L_{10}^*(\mathbf{a}) &= 2048 + 2a_1 - 645120b_2 - 10080b_1
\end{aligned}$$

such that $M = 10$ and the solution of the corresponding system is

$$a_1 = -4, \quad a_0 = 6, \quad b_0 = \frac{79}{120}, \quad b_1 = \frac{31}{180}, \quad b_2 = -\frac{1}{720}. \quad (2.4)$$

We also note that, if one assumes that $b_2 = 0$, $M = 8$ and the solution is

$$a_1 = -4, \quad a_0 = 6, \quad b_0 = \frac{2}{3}, \quad b_1 = \frac{1}{6}. \quad (2.5)$$

When one assumes that $b_2 = b_1 = 0$, $M = 6$ and

$$a_1 = -4, \quad a_0 = 6, \quad b_0 = 1. \quad (2.6)$$

- Step *iii*: to construct EF methods, we start from $E_0^*(\pm z, \mathbf{a}) := \exp(\mp \mu t) \mathcal{L}[h, \mathbf{a}] \exp(\pm \mu t)$ where $z := \mu h$ and we build $G^+(Z, \mathbf{a}) := (E_0^*(z, \mathbf{a}) + E_0^*(-z, \mathbf{a}))/2$ and $G^-(Z, \mathbf{a}) := (E_0^*(z, \mathbf{a}) - E_0^*(-z, \mathbf{a}))/2z$ where $Z = z^2$. Due to the symmetry, one then finds $G^-(Z, \mathbf{a}) \equiv 0$ and

$$G^+(Z, \mathbf{a}) = 2\eta_{-1}(4Z) + 2a_1\eta_{-1}(Z) + a_0 - 2Z^2b_2\eta_{-1}(4Z) - b_0Z^2 - 2Z^2b_1\eta_{-1}(Z)$$

where the functions $\eta_{-1}(Z)$ and $\eta_0(Z)$ are defined as

$$\eta_{-1}(Z) = \begin{cases} \cos(|Z|^{1/2}) & \text{if } Z < 0, \\ \cosh(Z^{1/2}) & \text{if } Z \geq 0, \end{cases} \quad \eta_0(Z) = \begin{cases} \sin(|Z|^{1/2})/|Z|^{1/2} & \text{if } Z < 0, \\ 1 & \text{if } Z = 0, \\ \sinh(Z^{1/2})/Z^{1/2} & \text{if } Z > 0. \end{cases}$$

Further we also compute the derivatives $G^{\pm(m)}(Z, \mathbf{a})$ with respect to Z . Defining

$$\eta_n(Z) := \frac{1}{Z}[\eta_{n-2}(Z) - (2n-1)\eta_{n-1}(Z)], \quad n = 1, 2, 3, \dots \quad (2.7)$$

differentiation gives

$$\eta'_n(Z) = \frac{1}{2}\eta_{n+1}(Z), \quad n = 1, 2, 3, \dots$$

which makes $G^{\pm(m)}(Z, \mathbf{a})$ easy to compute.

- Step *iv* : We consider a reference set of M functions :

$$\{1, t, t^2, \dots, t^K\} \cup \{\exp(\pm\mu t), t \exp(\pm\mu t), t^2 \exp(\pm\mu t), \dots, t^P \exp(\pm\mu t)\}$$

where $K + 2P = M - 3$. The reference set can be characterized by the couple (K, P) . The set in which there is no classical (i.e. polynomial) component is identified by $K = -1$ while the set in which there is no exponential fitting component is identified by $P = -1$. For the case $M = 10$ e.g., six choices are possible : $(9, -1), (7, 0), \dots, (-1, 4)$. For the cases $M = 8$ and $M = 6$ the (P, K) values are resp. varying from $(7, -1)$ down to $(-1, 3)$ and from $(5, -1)$ down to $(-1, 2)$.

- Step *v* : solve the algebraic system

$$L_k^*(\mathbf{a}) = 0, \quad 0 \leq k \leq K, \quad G^{\pm(p)}(Z, \mathbf{a}) = 0, \quad 0 \leq p \leq P. \quad (2.8)$$

- Step *vi* : The error term (see next section).

As an example, we consider the case $M = 6$. The coefficients are rewritten in terms of $A := \eta_{-1}(Z)$ and $B := \eta_0(Z)$.

- (i) $(K, P) = (5, -1)$: $a_1 = -4, \quad a_0 = 6, \quad b_0 = 1,$
- (ii) $(K, P) = (3, 0)$: $a_1 = -4, \quad a_0 = 6, \quad b_0 = \frac{4(A-1)^2}{Z^2},$
- (iii) $(K, P) = (1, 1)$

$$\begin{aligned} a_1 &= \frac{4(2A^2 - 2 - BZA)}{BZ - 4A + 4}, \\ a_0 &= \frac{2ZB(4A - 1) - 8(2A + 1)(A - 1)}{BZ - 4A + 4}, \\ b_0 &= \frac{-4B(A - 1)^2}{Z(BZ - 4A + 4)}, \end{aligned}$$

- (iv) $(K, P) = (-1, 2)$

$$a_1 = \frac{4(2A^2 - 1 - 3BA)}{3B - A},$$

$$a_0 = \frac{2(3B(2A^2 + 1) + BZ(A^2 - 1) - 3A(2A^2 - 1))}{3B - A},$$

$$b_0 = \frac{2B(A^2 - 1)}{Z(3B - A)}.$$

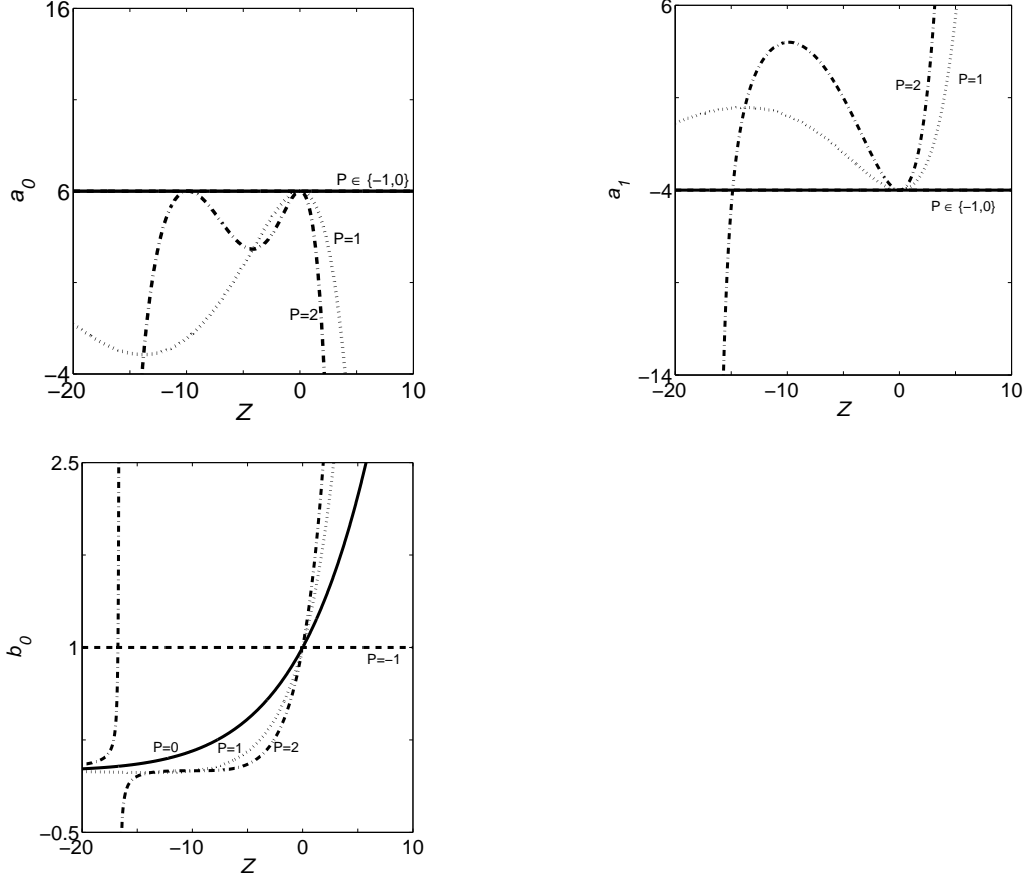


Fig. 1. The coefficients a_0 , a_1 and b_0 as a function of Z in the case $M = 6$. The graphs for these coefficients as functions of Z are depicted in Figure 1. In the same way, the coefficients for the cases $M = 8$ and $M = 10$ can be computed.

Following the same procedure the expressions for the coefficients of the begin formula can be obtained for the same reference set. Let us consider the case $M = 6$ again. For each value of P , it turns out that

$$c_1 = a_1 + 2 \quad c_2 = a_0 - 1 \quad c_3 = a_1 \quad d_3 = b_0.$$

The expressions for the values of the other coefficients in the begin formula

are

$$\begin{aligned}
\text{(i) } (K, P) = (5, -1) : & \quad d_1 = -1, \quad d_2 = -\frac{1}{12}, \\
\text{(ii) } (K, P) = (3, 0) : & \quad d_1 = -1, \quad d_2 = \frac{2 - 2A + Z}{Z^2}, \\
\text{(iii) } (K, P) = (1, 1) : & \quad d_1 = \frac{4 - 4A + ZB}{Z}, \quad d_2 = \frac{2A - 2 - ZB}{Z^2}, \\
\text{(iv) } (K, P) = (-1, 2) : & \quad d_1 = \frac{A - 3B}{2}, \quad d_2 = \frac{B - A}{4Z}.
\end{aligned}$$

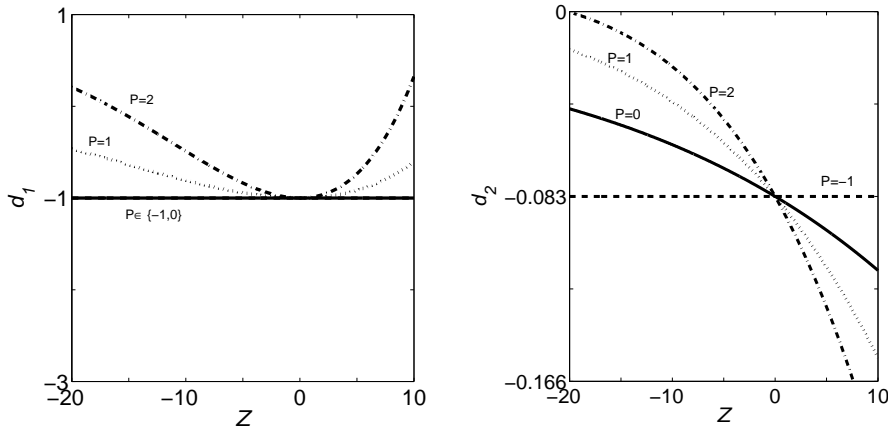


Fig. 2. The coefficients d_1 and d_2 as functions of Z in the case $M = 6$.

In general, the coefficients are Z dependent and in the trigonometric case, some singularities may arise. For sufficiently small values of $|Z|$ however, the coefficients are continuous, well behaved functions of Z , as is shown in Figure 1 and Figure 2. These pictures also clearly show that, for $0 \ll |Z| < 1$ the coefficients approximate their classical values. In order to avoid numerical instabilities the best way to compute the coefficients in that case is by means of a truncated Taylor series development. For larger values of $|Z|$, the values of the coefficients deviate from their constant (classical) values and their behaviour also becomes unpredictable. In the following sections, we will see that we are often only able to compute an approximate value for the parameter Z , and hence we can only compute approximate values for the coefficients a_0, a_1, \dots . When the coefficients are slowly varying functions of Z (as they are in the neighbourhood $Z = 0$), also approximate values of Z will still give quite good results. In the neighbourhood of singularities however, this is no longer the case. Therefore, from a practical point of view, the region around $Z = 0$ is by far the most interesting one, since it is typically for such values of Z that EF rules are applied.

3 Error analysis

We follow the approach of Coleman and Ixaru [20], who adapted a theory developed by Ghizzetti and Ossicini [21] to the EF framework. This approach was also taken in [22].

The result is as follows : the error $E[y]$ of a linear functional $\mathcal{L}[h, \mathbf{a}]$ defined over an interval $[\alpha, \beta]$ and applied to y is given by

$$E[y] = \int_{\alpha}^{\beta} \Phi(t) L[y](t) dx,$$

where $L \equiv D^{K+1}(D^2 - \mu^2)^{P+1}$ (in the exponential case) and where the kernel function $\Phi(t)$ is a function which is in the null space of L .

If $y \in \mathbb{C}^m(\alpha, \beta)$ and if the kernel $\Phi(t)$ is of constant sign in $] \alpha, \beta[$, the second mean-value theorem for integrals gives

$$E[y] = L[y](\zeta) \int_{\alpha}^{\beta} \Phi(t) dx \tag{3.9}$$

for some $\zeta \in] \alpha, \beta[$.

If Φ does not have a constant sign, we can rewrite $\Phi(t) = \Phi_+(t) + \Phi_-(t)$ where $\Phi_{\pm}(t) = \pm \max(0, \pm\Phi(t))$, such that, if $y \in \mathbb{C}^m(\alpha, \beta)$, the second mean-value theorem for integrals gives

$$E[y] = L[y](\zeta_+) \int_{\alpha}^{\beta} \Phi_+(t) dx + L[y](\zeta_-) \int_{\alpha}^{\beta} \Phi_-(t) dx.$$

Let us now check whether an expression of the form (3.9) holds. For the central formula, this would mean that

$$\text{lte} = h^M \Phi_{K,P}(Z) D^{K+1} (D^2 - \mu^2)^{P+1} y(\eta_p)$$

where $\Phi_{K,P}$ is some function with $\Phi_{K,P}(0) = L_M^*(\mathbf{a})/M!$ and $\eta_p \in (t_p - 2h, t_p + 2h)$. Due to the symmetry of the linear functional, it is sufficient to check the sign of $\varphi(t') := \Phi(t_p + t'h)/h^{M-1}$ for $t' \in [0, 2]$. This is done in the contour plots in Figure 3 for the case $M = 6$. In the case $P = 0$ the function φ has a constant sign for all values of Z for which it is defined, for the cases $P = 1$ and $P = 2$, there clearly is a bound on the values of Z for which (3.9) holds.

For the begin formula, similar contour plots are made, but now we check the sign of the kernel function for $t' \in [-1, 2]$. The results for $M = 6$ are depicted in Figure 4. The same conclusions hold as for the central formula.

If we summarize the results obtained so far for both the central and begin formula, we can conclude that for sufficiently small values of $|Z|$ one can quite easily compute (approximations of) the coefficients of the EF methods and that the error of such methods can be expressed in a closed form. In the next section, we will use this result to determine a suitable value for the parameter Z . However, since the error is expressed in terms of the unknown point ζ , the actual expression which will be used, is the series expansion of the error.

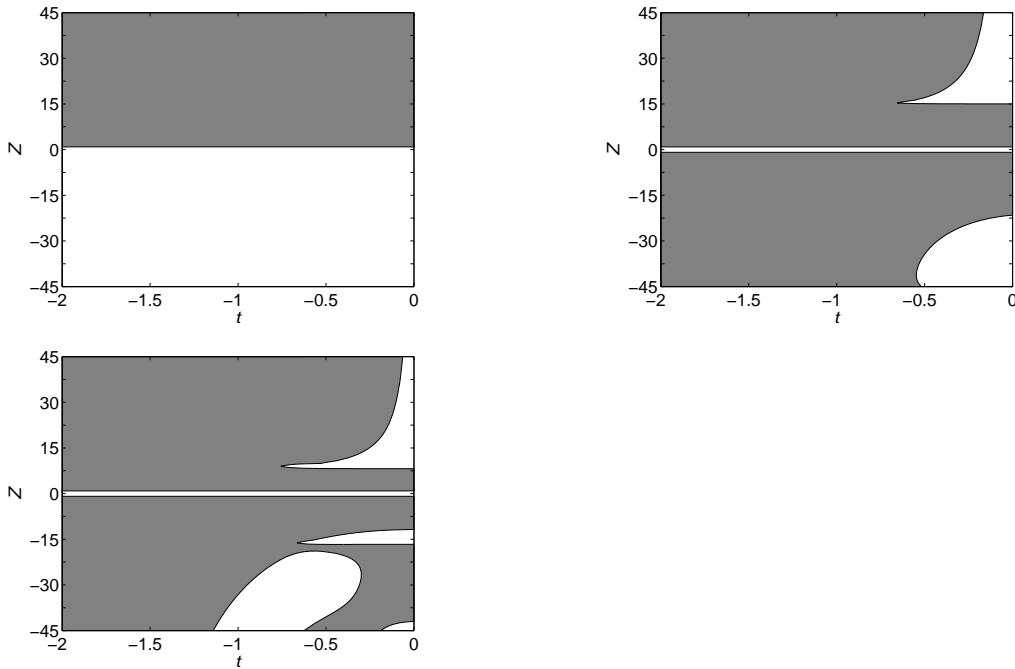


Fig. 3. Contour plots of the function $\varphi(t)$ for the central formula for the cases $P = 0$ (upper left), $P = 1$ (upper right) and $P = 2$ (below) in the case $M = 6$. The colors gray and white are used to distinguish positive and negative function values.

For the central formula, this means that we obtain an expression in the form

$$\text{lte} = h^M \frac{L_M^*(\mathbf{a})}{M!} D^{K+1} (D^2 - \mu^2)^{P+1} y(t_p) + \mathcal{O}(h^{M+2}).$$

E.g., in the case $M = 6$ this leads to the following results :

(i) $P = -1$

$$\text{lte} = \frac{h^6}{6} y^{(6)}(t_p) + \mathcal{O}(h^8)$$

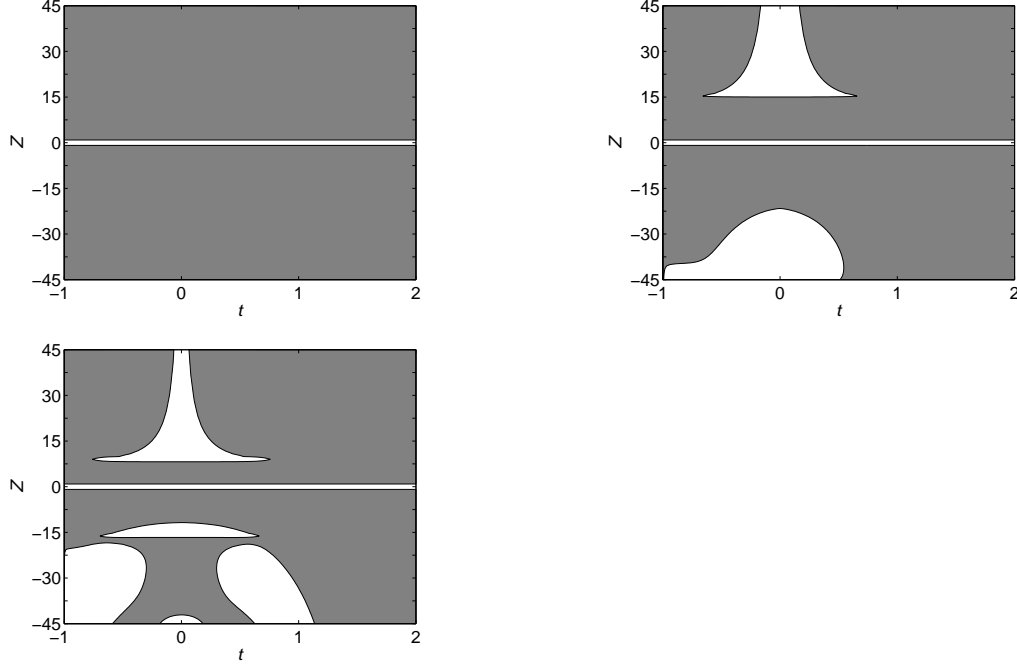


Fig. 4. Contour plots of the function $\varphi(t)$ for the begin formula for the cases $P = 0$ (upper left), $P = 1$ (upper right) and $P = 2$ (below) in the case $M = 6$. The colors gray and white are used to distinguish positive and negative function values.

(ii) $P = 0$

$$\text{lte} = \frac{h^6}{6} \left(y^{(6)}(t_p) - \mu^2 y^{(4)}(t_p) \right) + \mathcal{O}(h^8)$$

(iii) $P = 1$

$$\text{lte} = \frac{h^6}{6} \left(y^{(6)}(t_p) - 2\mu^2 y^{(4)}(t_p) + \mu^4 y^{(2)}(t_p) \right) + \mathcal{O}(h^8)$$

(iv) $P = 2$

$$\text{lte} = \frac{h^6}{6} \left(y^{(6)}(t_p) - 3\mu^2 y^{(4)}(t_p) + 3\mu^4 y^{(2)}(t_p) - \mu^6 y(t_p) \right) + \mathcal{O}(h^8).$$

For the begin formula, one also finds

(i) $P = -1$

$$\text{lte} = \frac{59h^6}{360} y^{(6)}(t_p) + \frac{h^7}{360} y^{(7)}(t_p) + \mathcal{O}(h^8)$$

(ii) $P = 0$

$$\text{lte} = \frac{59h^6}{360} \left(y^{(6)}(t_p) - \mu^2 y^{(4)}(t_p) \right) + \frac{h^7}{360} \left(y^{(7)}(t_p) - \mu^2 y^{(5)}(t_p) \right) + \mathcal{O}(h^8)$$

(iii) $P = 1$

$$\begin{aligned} \text{lte} &= \frac{59 h^6}{360} \left(y^{(6)}(t_p) - 2 \mu^2 y^{(4)}(t_p) + \mu^4 y^{(2)}(t_p) \right) \\ &+ \frac{h^7}{360} \left(y^{(7)}(t_p) - 2 \mu^2 y^{(5)}(t_p) + \mu^4 y^{(3)}(t_p) \right) + \mathcal{O}(h^8) \end{aligned}$$

(iv) $P = 2$

$$\begin{aligned} \text{lte} &= \frac{59 h^6}{360} \left(y^{(6)}(t_p) - 3 \mu^2 y^{(4)}(t_p) + 3 \mu^4 y^{(2)}(t_p) - \mu^6 y(t_p) \right) \\ &+ \frac{h^7}{360} \left(y^{(7)}(t_p) - 3 \mu^2 y^{(5)}(t_p) + 3 \mu^4 y^{(3)}(t_p) - \mu^6 y^{(1)}(t_p) \right) + \mathcal{O}(h^8). \end{aligned}$$

4 Parameter selection

We now come to the problem of attributing a value to the parameter μ for the EF methods, i.e. the cases with $P > -1$. The determination of the parameter is an essential part in the EF framework. Most papers on the subject only deal with the case $P = 0$. Here, we will use an algorithm valid for any $P \geq 0$. It is an adaptation of an algorithm which was originally presented in [22] and [24] for solving second order boundary value problems. A short description of this algorithm can also be found in [23].

The algorithm is based on the expression for the lte. The idea is to look for a value μ_j of μ that annihilates its leading term at the point t_j , i.e.

$$D^{(K+1)} (D^2 - \mu_j^2)^{(P+1)} y(t_j) = 0 \quad j = 1, \dots, N. \quad (4.10)$$

For $P = 1$ for instance, this means

$$y^{(K+5)}(t_j) - 2 y^{(K+3)}(t_j) \mu_j^2 + y^{(K+1)}(t_j) \mu_j^4 = 0 \quad j = 1, \dots, N. \quad (4.11)$$

In order to obtain values for the $y^{(i)}$ -values which appear in this expression, we can differentiate the differential equation and re-express higher-order derivatives in terms of y , y' , y'' and y''' . These derivatives can be approximated by means of (sufficiently accurate) finite difference formulas. This finally leads to expressions that only contain y -values. To obtain a first approximation for these y -values, we can apply the classical method.

The equation (4.10) is of degree $P + 1$ in μ^2 . This means that for $P = 0$, a

unique value for μ_j (the sign does not matter in this discussion) is obtained. For $P \geq 1$ in the other hand, $P + 1$ choices can be made in each point t_j . In deciding which value to choose, two observations are of importance.

- (i) When the solution is of the form $y(t) = t^p e^{\alpha t}$ with $p \in \mathbb{N}$, then one can show that for $P \geq p$, $\mu_j = \alpha$ will be a (constant) solution of (4.10). To be correct : $\mu_j^2 = \alpha^2$ will be a solution of multiplicity $P - p + 1$. With this choice we will in principle obtain machine accuracy, since the solution $y(t)$ then falls within the fitting space.
- (ii) When the solution $y(t)$ is not of the form $t^p e^{\alpha t}$, then in each point t_j we can only try to determine a value for μ_j such that $y(t)$ is locally as good as possible approximated by a function within the fitting space. Sometimes however, the root(s) of (4.10) may become very large at certain points t_j (e.g. due to a denominator that become very small) and experiments have shown that for such large values of μ_j , the accuracy decreases. A good criterion is to attribute at each point of the interval the minimum value (in norm) of the suggested values for μ^2 .

These two observations lead to the following conclusions :

- (i) If possible, choose a rule for which there is a constant μ_j for all j .
- (ii) If μ_j cannot be held constant for all j , then try to find a rule for which μ_j can be held small. This means that a rule with $P = 0$ may be less suited in this case and a rule with $P \geq 1$ should be preferred.

5 Numerical examples

5.1 Problem 1

$$y^{(4)} - \frac{384 t^4}{(2 + t^2)^4} y = 24 \frac{2 - 11 t^2}{(2 + t^2)^4}$$

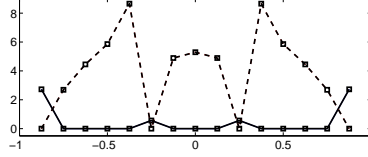


Fig. 5. Real (solid line) and imaginary (dashed line) values of μ_j for Problem 1 in case $M = 8$ and $P = 0$ for $h = 1/8$.

with boundary conditions

$$\begin{aligned} y(-1) &= \frac{1}{3}, & y(1) &= \frac{1}{3}, \\ y''(-1) &= \frac{2}{27}, & y''(1) &= \frac{2}{27}. \end{aligned}$$

The solution is given by $y(t) = \frac{1}{2 + t^2}$.

Since $y(t)$ does not belong to the fitting space of a EF-rule, the value of the parameter μ will not be constant over the interval of integration.

Suppose we first integrate this problem numerically with a classical rule of order 4, i.e. $M = 8$ and suppose we next want to improve the accuracy of the solution by computing the EF solution in the $P = 0$ case. Then the determination of μ_j starts from the expression

$$y^{(8)}(t_j) - y^{(6)}(t_j) \mu_j^2 = 0.$$

After reexpressing the higher order derivatives in terms of y, y', y'' and y''' and approximating the derivatives by means of finite difference schemes of order $\mathcal{O}(h^4)$, we obtain a numerical solution for μ_j , as depicted in Figure 5. The obtained value for μ_j is not constant over the interval and becomes quite large (in modulus) at certain points. Therefore we also consider the case $P = 1$, for which we start from

$$y^{(8)}(t_j) - 2y^{(6)}(t_j) \mu_j^2 + y^{(4)}(t_j) \mu_j^4 = 0.$$

This then leads to the values of two roots $\mu_{1,j}$ and $\mu_{2,j}$, depicted at the top of Figure 6. Again, we notice that at certain points each one of the two possible values $\mu_{1,j}$ and $\mu_{2,j}$ becomes (too) large, but when we define μ_j in each point as the root with the smallest modulus, we obtain smaller, acceptable values.

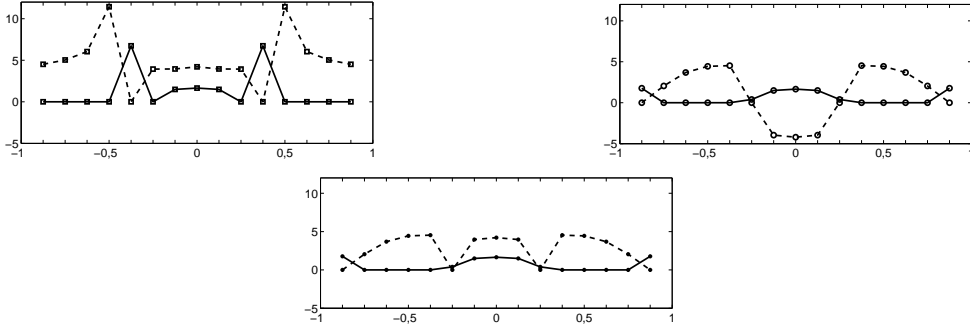


Fig. 6. Real (solid line) and imaginary (dashed line) values of $\mu_{1,j}$ (top, left) and $\mu_{2,j}$ (top, right) and μ_j with smallest modulus (bottom) for Problem 1 in case $M = 8$ and $P = 1$ for $h = 1/8$.

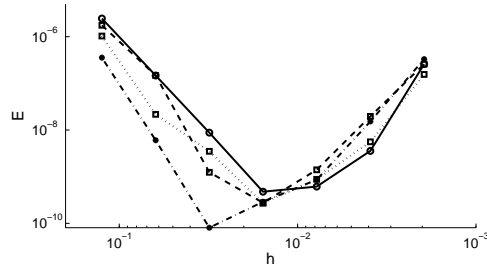


Fig. 7. the numerical solutions for various fixed meshsizes h obtained with the classical method (solid line), and the EF $P = 1$ method with either $\mu_{1,j}$ (dashed), $\mu_{2,j}$ (dotted) and μ_j with smallest modulus (dot-dashed)

The importance of a good choice for μ is shown in Figure 7 in which we depict the numerical solutions for various fixed meshsizes h obtained with the classical method (solid line), and the EF $P = 1$ method with either $\mu_{1,j}$ (dashed), $\mu_{2,j}$ (dotted) and μ_j with smallest modulus (dot-dashed).

In fact, the procedure thus followed turns the fourth order method into a sixth order method. This is made clear in Figure 8, where we show for each of the cases $M = 6$, $M = 8$ and $M = 10$ that the classical methods are of order 2, 4 and 6 respectively whilst their EF counterparts behave as methods of order 4, 6 and 8 resp. At least, this is what holds for sufficiently large values of h , i.e. when the system to solve is not too large.

As N grows, the five-diagonal coefficient matrix indeed becomes more and more ill-conditioned. In fact, the condition number of the matrix corresponding

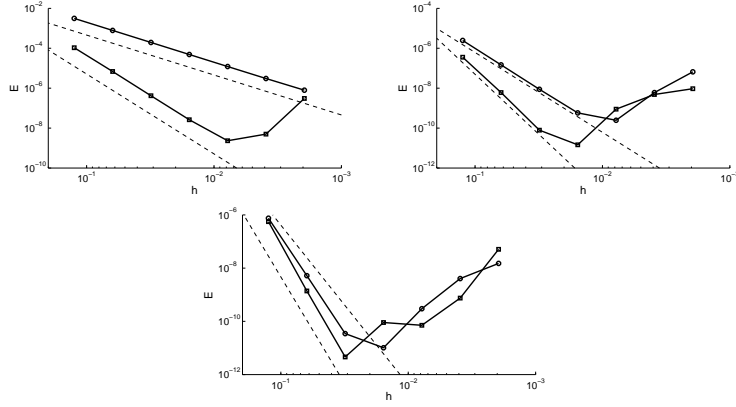


Fig. 8. The maximum error E as a function of the stepsize h for the cases $P = -1$ (circles) and $P = 1$ (squares) : on the left $M = 6$, in the middle the case $M = 8$ and on the right the case $M = 10$. The dashed lines indicate orders 2, 4, 6 and 8.

to the $P = -1$ case

$$A = \begin{pmatrix} 5 & -4 & 1 & & & \\ -4 & 6 & -4 & 1 & & \\ 1 & -4 & 6 & -4 & 1 & \\ & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & & 1 & -4 & 6 & -4 & 1 \\ & & & & 1 & -4 & 6 & -4 \\ & & & & & 1 & -4 & 5 \end{pmatrix}$$

grows like N^4 . We noticed that although it is possible to write $A = B^2$, whereby

$$B = \begin{pmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{pmatrix}$$

and whose condion number grows as N^2 , the numerical results do not improve. The same holds for the EF methods. Although is it still possible to write A as a product of two matrices, the numerical results do not improve.

This thus leaves us with the conclusion that our methods can only be applied for sufficiently large stepsizes. But even then, numerical results with 10 digits

of accuracy can still be obtained (for $M = 10$).

5.2 Problem 2

$$y^{(4)} - t = 4e^t$$

with boundary conditions

$$\begin{aligned} y(-1) &= -1/e, & y(1) &= e, \\ y''(-1) &= 1/e, & y''(1) &= 3e. \end{aligned}$$

The solution of this problem is given by $y(t) = e^t t$. In theory, this problem is solved up to machine accuracy by any EF-rule with $P \geq 1$ and $\mu_j = 1$. From the previous example, we know however that this will only be the case for sufficiently large values of h .

In practice, we do not know the μ_j ; we have to determine their values numerically. A natural question to ask is : what happens to the numerical solution in such a case? How accurate should the parameter μ be computed? In order to answer these questions we will consider the case $M = 6$ where $h = 1/16$. In this way we will not be disturbed by the ill-conditioning of the coefficient matrix.

The μ_j are obtained by annihilating the leading term of the lte, in which firstly the derivatives are reexpressed in terms of y, y', y'', y''' using the differential equation and secondly y', y'' and y''' are approximated in terms of the already computed solution $y^{(0)} = \{y_i | i = 1, \dots, N\}$ by finite difference schemes. Since the y_i are only $\mathcal{O}(h^2)$ accurate, it makes no sense to approximate the derivatives with finite difference schemes that are more accurate (than that). Using these values μ_j (let us denote these as $\mu^{(1)}$) in the EF method with $P = 1$, we then obtain an improved solution $y^{(1)}$, which is however far from accurate up to machine precision, as is shown in Figure 9. However, we can use this improved solution $y^{(1)}$ (which as we already know from the previous example is $\mathcal{O}(h^4)$ accurate) to obtain more accurate values for μ_j (let's denote these as $\mu^{(2)}$). In fact now we can approximate the derivatives by EF $\mathcal{O}(h^4)$ difference

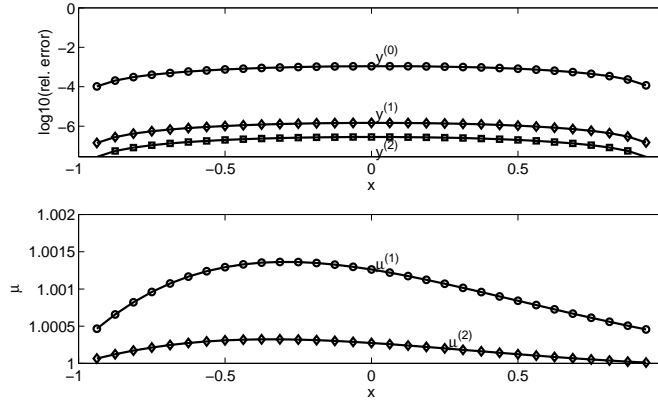


Fig. 9. The absolute error in $y^{(0)}$, $y^{(1)}$ and $y^{(2)}$ together with the computed values of $\mu^{(1)}$ and $\mu^{(2)}$ obtained by applying methods with $M = 6$ to Problem 2 with $h = 1/16$.

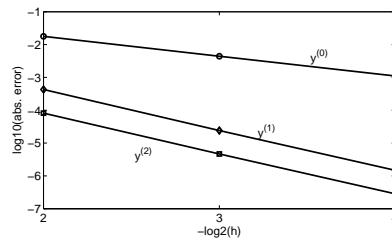


Fig. 10. The maximum absolute error in $y^{(0)}$, $y^{(1)}$ and $y^{(2)}$ obtained by applying methods with $M = 6$ to Problem 2 with $h = 1/4$, $h = 1/8$ and $h = 1/16$.

schemes of the type $P = 1$. One notices that the $\mu^{(2)}$ thus computed are quite accurate, leading to a solution $y^{(2)}$ which is again more accurate than $y^{(1)}$. One may try to proceed in this way to obtain a solution $y^{(3)}$, but experiments show that this is only possible if $\mu^{(3)}$ is computed by more advanced difference formulae.

In Figure 10 we show the accuracy of $y^{(0)}$, $y^{(1)}$ and $y^{(2)}$ thus obtained for $h = 1/4$, $h = 1/8$ and $h = 1/16$. Again we notice that $y^{(0)}$ confirms the classical method has order 2, whereas $y^{(1)}$ and $y^{(2)}$ indicate that the error for the EF method has increased to 4.

6 Conclusions

Fourth-order boundary value problems are solved by means of parameterized EF methods. The methods used are determined by imposing conditions

(related to combinations of polynomials, exponentials and/or trigonometric functions) onto a linear functional. The trigonometric/exponential part contains a parameter for which a suitable value can be found from the roots of the leading term of the local truncation error. If, for some level of tuning, a constant value is found for this parameter, then in principle a very accurate solution can be obtained. However, the methods strongly suffer from the fact that the system to be solved is ill-conditioned for small values of the mesh size. Therefore the methods should only be applied for moderate step sizes.

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