Gauss, Ramanujan and Hypergeometric Series Revisited[1]

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Abstract: (in English)
Srinivasa Ramanujan has been compared with Euler, Gauss and Jacobi for *natural genius*. In this article, after brief biographical notes on these mathematicians, a study is made of the contributions of Gauss and Ramanujan to Hypergeometric series.

Abstract: (in Dutch)

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Karl Friedrich Gauss (1777 – 1855) [1, 4]:

Gauss, the German mathematician, hailed as the *'Prince of Mathematicians', 'Titan of Science' [5], 'World's greatest mathematician'* and *'Prolific as Euler'*, has been acknowledged as one of the three leading mathematicians of all time, the others being Archimedes and Sir Issac Newton. His outstanding work includes the discovery of the method of least squares, the discovery of the non-Euclidian geometry and important contributions to the theory of numbers.

Gauss recalled at a later age that he could *count before he could talk*. He surprised his teacher J.G. Büttner at the age of ten by summing 1 to 100, mentally and when asked, explained his method of addition:

$$\left.\begin{array}{rcrcccrcrc} 1 & + & 2 & + & \cdots & + & 49 & + & 50 & + \\ 100 & + & 99 & + & \cdots & + & 52 & + & 51 & + \\ \hline = 101 & + & 101 & + & \cdots & + & 101 & + & 101 & \end{array}\right\} = 101 \times 50 = 5050,$$

in the process discovering the summation formula: $\sum_{i=1}^{n} i = n(n+1)/2$, when the teacher expected him to be occupied with the task the hard way, like other students of his age! This incident made the teacher get a special arithmetic text book for Gauss from Hamburg in recognition of his 'genius'. Büttner also requested Gauss' father to relieve the boy from doing his family business (spinning flax) and helped him to be admitted to a Gymnasium.

In 1791, the Duke of Brunswick, Carl Wilhelm Ferdinand, was willing to be a patron for *the continued training of such a gifted person*, starting his patronage with an annual stipend of ten tallers.

In 1794, Gauss discovered the method of Least squares. In 1796, Gauss discovered how to construct a regular polygon of 17 sides (heptadecagon)using only a compass and a straight edge, which is related to the problem of finding the roots of the equation: $x^{17} - 1 = 0$, geometrically on a unit circle. In fact, he proved that a regular polygon with n sides is constructible if and only if n is a distinct prime number of the form $p_k = 2^{2^k} + 1$. Thus, when $k = 0, 1, 2, 3$, the corresponding numbers $p_k = 3, 5, 17, 257$ are prime umbers and so regular polygons with these number of sides are constructible. This result happens to be the first item in a mathematical diary he kept until 1814. Gauss's diary – like the notebooks of Ramanujan – makes it possible to verify the priority of Gauss in many of the discoveries he did not publish. This little booklet of only 19 pages, considered as one of the most precious documents in the history of science, was unnoticed till 1898, and though not lost, was found among family papers with one of Gauss's grandsons. It contains 146 concise statements of the results of his researches, between 1796 to 1814. Among the unpublished discoveries of his are: non-Euclidian geometry and non-commutative algebras.

In 1799, Gauss completed his doctoral dissertation, under the nominal supervision of Johann Friedrich Pfaff, for which on July 16, 1799, he was awarded the *Doctor Philosophiae* degree of the University of Helmstedt, even without a conventional viva-voce examination. The Duke funded the publication of this dissertation [4]. Gauss' doctoral thesis contained the first proof of the fundamental theorem of algebra, which states that every algebraic equation has a root of the form $a + i\ b$ where a and b are real numbers and i is $\sqrt{-1}$. In general, numbers of the form $a + i\ b$ are called as *complex numbers*, and, in particular, when a and b are integers, the numbers $a + i\ b$ are called as *Gaussian numbers*. Complex numbers are represented in a plane called the Gaussian plane.

Gauss published his number theoretic work of 1798, in 1801. In this treatise: *Disquisitiones Arithmeticae*, Gauss created the modern rigorous approach to mathematics. It contains the new algebra of congruence. He introduced the $\equiv$ symbol for congruence and the notation $a \equiv b\ (mod\ m)$, which has since become a standard in the theory of numbers. The *law of quadratic reciprocity* discovered by him was called *the gem of arithmetic* by him: *if* p *and* q *are prime numbers, then either* $x^2 \equiv q\ (mod\ p)$ *and* $x^2 \equiv p\ (mod\ q)$ *are both solvable or both unsolvable, unless both* p *and* q *are of the form* $4n + 3$*, in which case one is solvable and the other is not.* Gauss's *Arithmetical Investigations* is divided into seven parts: Congruences in general, Congruences of the first degree, Residues of powers, Congruences of the second degree, Quadratic forms, Applications and Divisions of the circle. This work (judged extremely hard to read) has been disseminated by his successor at Göttingen, Peter Gustav Dirichlet (1805 – 59).

In 1801, Gauss acquired the sextant and worked out the theory of the motion of the moon and his own astronomical ephimeres - almanac listing the daily positions of the planets, especially in the Solar system. On Jan. 1, 1801, Joseph Piazzi discovered the asteroid Ceres (765 Km diameter) and after 41 days of observations, over a geocentric arc of 3 degrees, he lost track of Ceres. Gauss' renowned contemporaries, Lagrange, Lambert and Laplace abandoned the problem of locating its position. *An opinion had universally prevailed that a complete determination from observations embracing a short interval of time was impossible, an ill-founded opinion, for it is now clearly shown that the orbit of a heavenly body may be determined quite nearly from good observations embracing only a few days; and this without any hypothetical assumption*, wrote Gauss in *Theoria Motus*. Gauss used elliptic orbit theory and his method of Least Squares to predict the orbit of Ceres. Ceres was observed by de Zach in Gotha on Dec.7, 1801 and by Olbers on Jan. 1, 1802. In recognition of his growing reputation, Gauss was Elected Member, Academy of Science, St. Petersburg, Leningrad, on Jan. 31, 1802 and the Duke Ferdinand increased the financial support to Gauss from Jan. 1, 1803. Gauss eventually published his work: *Theoria Motus Corporum Coelestium in Sectionibus Conicis Solem*

Ambientium (1809) - and it became a Bible for astronomy and numerical computations. Gaussian law of normal distribution occurs in it.

Gauss calculated the orbits of the planetoids Pallas (discovered by Olbers in 1802), of Juno (discovered by Harding in 1804) and of Vesta (discovered by Olbers in 1807). Within 10 hours of the discovery of Vesta by Olbers, in 1807, Gauss astonished the astronomers by calculating its orbit. While Gauss could calculate the orbits in a single hour, Euler required 3 days and Euler eventually became blind in one eye. *I should also have gone blind if I had calculated in that fashion for 3 days!*, commented Gauss, who worked out the optical theory and invented, in 1821, the heliotrope an instrument for distance measurements using the reflected rays of the Sun. After a week with Gauss, Jacobi wrote to his brother: *Mathematics would be in a very different position if practical astronomy had not directed this colossal genius from his glorious career.*This in spite of the fact that whenever Jacobi went to Gauss *each time Gauss pulled 30-year-old manuscripts out of his desk and showed Jacobi what Jacobi had just shown him* [11], much to Jacobis chagrin!

A new observatory was planned in Göttingen, in 1804. At the end of 1804, Gauss was engaged to Johanna Osthoff, the daughter of a tannery owner in Braunschweig, whom he was meeting for a year. Gauss married Johanna on Oct. 9, 1805 and lived happily in Braunschweig with increased financial support from his patron the Duke Ferdinand. Gauss met his benefactor for the last time in May 1806, for the Duke was seriously injured and defeated while leading the Prussian and Saxon troops against the French led by Emperor Napoleon at the battle of Auerstädt and eventually died at Altona, on Nov. 10, 1806. Gauss, very loyal to the Duke and to Germany, was greatly affected by this tragedy.

Gauss named his first son Joseph (born on Aug. 21, 1806), after the Italian astronomer Piazzi, the discoverer of Ceres. It was only on July 25, 1807 that Gauss was named Professor of Astronomy as well as Director of Observatory and in Nov. 1807, Gauss moved to Göttingen. Gauss lived in the Sternwarte (Observatory) of Göttingen from 1816 and worked untiringly till the very end of his life. In a letter to his Hungarian friend Wolfgang Bolyai, Gauss wrote: "*Astronomy and pure mathematics are the magnetic poles toward which the compass of my mind ever turns*". A memorial plaque with the inscription: *Erster electriches Telegraph, Gauss Weber, Ostern 1833* adorns this building to this day.

Gauss became the father of a daughter (Feb. 29, 1808) and a son (Sep. 10, 1809) but a month after the birth of the third child, his wife died (Oct. 11, 1809). Five month's later the third child also passed away. Gauss proposed on March 27, 1810 to his wife's closest friend, Minna Waldeck and married her on Aug. 4, 1810. Two sons and a daughter were born out of this wedlock (in

1811, 1813 and 1816).

In 1811, Gauss extended the calculus to functions of a complex variable and discovered a fundamental theorem of integration: if a function $f(z)$ is analytic at all points on and within a closed curve C in the Gaussian plane, then the integral of $f(z)$ along C is zero. Since he did not publish this theorem, it is known today, after its re-discoverer, as Cauchy's integral theorem.

In Jan. 1812, recognizing the need for convergence of infinite series, he published his comprehensive, important work on hypergeometric series, in his famous thesis: "*Disquisitiones generales circa seriem intinitam ...* Gauss considered this series as a function of four variables: $F(\alpha, \beta; \gamma; x)$, which in the present day notation is: ${}_2F_1(\alpha, \beta; \gamma; x)$. This work of Gauss is the subject matter of our study in this article.

Gauss wrote extensively on the theory of errors of measurement in his *Theoria Motus* and subsequent works that the normal frequency distribution which is the familiar bell-shaped curve is often referred to as the Gaussian distribution (which found a place on the face of the 10 DM currency note of Germany). Gauss is considered as the effective founder of differential geometry. He introduced the concept of the Gaussian or total curvature of a surface, as a measure of curvature (which remains unchanged under a continuous deformation of a flexible and inextensible surface).

Gauss developed the theory of elliptic functions, though he did not publish his results. The remarkable property of double periodicity of these functions was rediscovered by Niels Abel and Karl Jacobi (1827 – 29).

A unit of magnetism is named after Gauss. The eyepiece used for auto collimation in spectrometers and refractometers is designed by him. He invented the heliotrope, the biflar magnetometer and with Weber devised the mirror galvanometer. In 1834-35, Gauss and Weber invented an apparatus for transmission of messages. This telegraph system of Gauss and Weber was destroyed by lightning in 1845. There exists a Gauss-Weber, life-size, bronze monument in Göttingen where the period when Gauss was the Astronomer Royal marked a golden era of science in that country.

Three works of Gauss were translated into English books and these are: *Theory of the Motion of the Heavenly Bodies Moving about the Sun in Conic Sections* (Dover Publications, 1963), referred in the text above as *Theoria Motus*; *General Investigations of Curved Surfaces* (Raven Press, 1965) and *Arithmetical Investigations* (Yale University Press, 1966), referred to in the text above as *Disquisitiones Arithmeticae.*

The *Collected Works of Gauss* ("Carl Friedrich Gauss' Werke"), Volumes I to

XII, contain his 150 research publications, his journal (*Notizenjournal*), part of his extensive correspondence (about 7000 letters to and from Gauss exist) with F.W. Bessel, Wolfgang Bolyai, Alexander von Humboldt, M. Olbers and H.C. Schumacher and essays about his scientific contributions to various fields. This work was directed by the Royal Scientific Society of Göttingen with E. Schering, F. Klein, E. Wiechert and L. Schlesinger as the editors, with assistance from six others. The twelve volumes appeared in succession during the years 1863 – 1933 by the publishing houses Perthes (Gotha), Teubner (Leipzig) and Springer (Berlin). This is perhaps one of the most exhaustive studies undertaken on the works of any mathematician in the history of mathematics.

Apart from his family – two wives and 6 children – Gauss's varied interests were in history, languages, literature and finance. He knew many languages including Danish, English, French, German, Greek, Latin and Russian. He had a collection of about 6000 books. So shrewd was he in handling his finances that *he left an estate over a 100 times as great as his annual income during the last half of his life... Gauss wrote a great deal; but to publish every fundamental discovery he made in a form satisfactory to himself would have required several long lifetimes* [2]. Gauss brought his immense energies to bear on a gamut of problems in mathematics and physics, establishing so many astounding results, that he leaves one wondering whether he *belonged to a higher species*!

Srinivasa Ramanujan (1887 – 1920) [8, 9]:

Ramanujan has been compared to the all time great mathematicians: Euler, Gauss and Jacobi, for his natural genius. Born in a middle class family, Ramanujan had only formal education up to the School final. He has been referred to as *intuition incarnate* [10] and was creative till the end of his short life.

Ramanujan was a bright student at School and impressed his classmates and teachers with his extraordinary intuition and astounding proficiency in arithmetic, algebra, geometry, number theory and trigonometry – several branches of Mathematics. In an arithmetic class on division, the teacher said that if three bananas were distributed to three boys, each would get a banana and the teacher went on to generalize this idea. Ramanujan is reported to have asked in the class: *Sir, if no banana is distributed to no student, will each student still get a banana?*, much to the teacher's surprise. The senior mathematics teacher of the School had such confidence in Ramanujan's ability that year after year, he entrusted Ramanujan with the task of preparing a conflict free time table for the school. Ramanujan won books as prizes for his proficiency in mathematics at school. Except for Loney's *Trigonometry*, Part II, the other books he received from the School were on English literature.

Ramanujan's chosen occupation was unguided research in Mathematics, inspired in his impressionable years by the thousands of formulae presented without proofs in G. S. Carr's: *A Synopsis of Elementary Results, a book on Pure Mathematics* (1886), which was borrowed and given to him (from the library of the Government Arts College, Kumbakonam, by two students who were boarding in his house), in 1903. Ramanujan passed the Matriculation Examination of the University of Madras as a student of the Kumbakonam Town High School, in 1904. *It was this book which awakened his genius. He set himself to establish the formulae therein. As he was without the aid of other books, each solution was a piece of research so far as he was concerned,* wrote his benefactor Ramachandra Rao [8].

Ramanujan joined the Government Arts College, Kumbakonam, in 1904 with a scholarship which he secured by passing a competitive examination in English and Mathematics. But he failed in his attempt to pass the Junior First Arts Class exams at the end of the first year in College and so was not promoted. He lost his scholarship. A year later he joined the Pachaiyappa's College in Madras, but failed to pass the First Degree in Arts examinations of the University of Madras, in 1907.

What Ramanujan did during those four years is not clear from any of the accounts of his life available today. He gave private coaching in mathematics and earned some money. He was in search of benefactors who could secure for him a job. To all those he considered as influential people, he showed his Notebooks, which contained results he had obtained in various branches of mathematics and tried to convince them of his extraordinary abilities to do Mathematics. In his search, he came across Mr. V. Ramaswamy Iyer, the Founder of the Indian Mathematical Society; Prof. C.L.T. Griffith, Professor of Mathematics at the Engineering College in Madras; Mr. B. Ramachandra Rao, then a Collector of Nellore, who was the first to offer him Rs. 20/- per month as a patron (for about a year in 1911 – 12); Mr.S. Narayana Iyer, Manager of the Madras Port Trust and through him Sir Francis Spring, Chairman of the Madras Port Trust and his teacher Prof. P.V. Seshu Aiyar, who moved to the Presidency College from the Government Arts College, Kumbakonam. These people – together with Prof. Richard Littlehailes (at the meteorological observatory in Madras), Prof. G.T. Walker, F.R.S., who visited the Port Trust in 1913 from Shimla, (where he was the Director General of Observatories) and Prof. E.H. Neville, visiting the University of Madras from Cambridge to deliver a series of lectures on Differential Geometry (to the Mathematics Honours students), in early 1914 – were the staunch supporters of Ramanujan, who left no stone unturned to secure support for Ramanujan's progress, from the concerned authorities, be it the Madras Port Trust or the University of Madras.

Ramanujan's earliest contributions were in the form of Questions or Answers to Questions in the Journal of the Indian Mathematical Society (JIMS), in

1911. In his brief illustrious career Ramanujan communicated 58 Questions or solutions to Questions section of the JIMS. His first full length paper was entitled: *Some properties of Bernoulli numbers*, also in the JIMS in 1911.

Despite the pecuniary circumstances, his mother's initiative resulted in Ramanujan getting married to nine year old Janaki, in 1909, at a place near Karur, and his father was not present at the five day wedding function [10]. The marriage was not consummated until after the bride came of age in 1912.

Through the help of his well wishers, Ramanujan first secured a job as a clerk in the Accountant General's office (from Jan. 12 to Feb. 21, 1912). A letter from Mr. Ramachandra Rao to Sir Francis Spring, and the help of Mr. Narayana Iyer, enabled Ramanujan to secure a post in the Accounts section (Class III, Grade IV) in the Madras Port Trust, on a salary of Rs. 30/- per month, from March 1, 1912. After this Ramanujan lived with Janaki and his mother in Madras till he left for England on March 17, 1914.

When Ramanujan approached Prof. Seshu Aiyar, at the Presidency College, Madras, with some results of his on Prime numbers, his attention was drawn to G.H. Hardy's Cambridge Tract on *Orders of Infinity.* The statement therein by Hardy that *no definite expression has yet been found for the number of prime numbers less than any given number*, made Ramanujan tell Seshu Aiyar that he had a formula which approximately gave the required result. Ramanujan was then directed by Seshu Aiyar to communicate the same to Prof. Hardy, F.R.S., then a Cayely Lecturer in Mathematics at the Trinity College, Cambridge. Ramanujan's first letter of Jan. 16, 1913, is perhaps one of the most historically important letters ever written by a mathematician, enclosing on some 9 sheets of paper scores of mathematical formulae without proofs and some of them Hardy later said he *had never seen anything in the least like them before* and which he said *defeated me completely* [9]. At the end of the day on which he received the letter, Prof. Hardy after a consultation about its contents with his colleague Prof. J.E. Littlewood came to the conclusion that the formulae *could be written down by a mathematician of the highest class.* Hardy soon set himself the task of getting Ramanujan to work even before he mailed the first encouraging reply to Ramanujan. In what today appears to be a fairy tale, Hardy succeeded in making Ramanujan to go to Cambridge, in about 15 months time!

Due to the orchestrated efforts of friends and benefactors of Ramanujan, for whom the support of Prof. Hardy acted as a catalyst, the British Government machinery was set in motion to award through the University of Madras, the first ever Research Scholarship, in April 1913 and later 250 pounds a year for five years plus 100 pounds to cover the passage and equipment, which enabled him to go to Cambridge. In a short span of five years, despite the war which started and ended during that period, Ramanujan's work on *Highly Composite*

Numbers was recognized for the award of the B.A. Degree by research of the Cambridge University (March 1916). For his work as a *Research student in Mathematics Distinguished as a pure mathematician particularly for his investigations in elliptic functions and the theory of numbers* he was elected a Fellow of the Royal Society of London, in Feb. 1918. Ramanujan was also elected to a Trinity College Fellowship, in October 1918 – which was a Prize Fellowship worth 250 pounds a year for six years with no duties or conditions!

During his five year stay at Cambridge, Ramanujan published 22 papers, five in collaboration with Hardy, including work on partitions, which resulted in an *astonishing theorem* for the calculating exactly the unrestricted partitions, $p(n)$, for any integer value of n, based on the *circle method*, a powerful tool in the analytic theory of numbers today.

It is unfortunate, that after two years of his stay in Cambridge, Ramanujan fell ill with suspected tuberculosis, which was considered then an infectious disease and had to spend more than two years in sanatoria in England. Ramanujan returned to India, in April 1919, emaciated from suspected tuberculosis but with a great reputation and he got the best medical attention possible. His wife, Janaki, joined him and nursed him during this period.

In his only letter to Hardy after his return, in Jan. 1920, Ramanujan wrote that he was working on what he christened as *mock theta functions* considered today as his legacy for posterity. It is surprising that after five years of having closely interacted and collaborated with Ramanujan, doing his best to get Ramanujan all the recognition that was possible in the academic world, Hardy did not reply to this last letter of Ramanujan. Ramanujan died on April 26, 1920, at the age of 32 years 4 months and 4 days. The news of Ramanujan's death, announced in a letter by Ramanujan's younger brother to Hardy, was a total surprise and shock to him!

Ramanujan's young widow spent the next eight years of her life with her brother in Bombay, during which period she learnt tailoring and English. She then returned to Madras and in two years started teaching tailoring and eked out a livelihood by stitching blouses for women and dresses for children. In 1950, one of her friends died suddenly, entrusting her 7 year-old son, W. Narayanan to her care. Mrs. Janakiammal lived up to a ripe old age of 94, fostering Narayanan through his schooling, marriage and took care of his family (of one son and two daughters). Recognition in the form of visits by reputed mathematicians, Professors Andrews, Askey, Berndt, Bollabos, making her recall the time when Ramanujan was alive and benefits in the form of endowments and honoraria started coming to her in a larger measure after the 75th Birthday and the birth centenary of Ramanujan, in Dec. 1987. She died on April 13, 1994 peacefully [12].

In 1923, Hardy spent some months studying one of the chapters of Ramanujan in his first Notebook. It is this which is the focus of our attention in this article. But before we start on the details about that work, some details about the Notebooks themselves are in order.

Ramanujan's Notebooks:

One hundred and fifteen years after the birth of Ramanujan, it is no exaggeration to say that the interest in Ramanujan is still alive and unabated in the world of mathematics. Ramanujan had noted down the results of his researches, without proofs, (as in the *Synopsis of Elementary mathematical Results*, by G.S.Carr), in three Notebooks, between the years 1903 - 1914, before he left for England. These were the Notebooks which he showed to his benefactors to convince them about his abilities as a Mathematician. The results in these Notebooks were organized by him. The first Notebook has 16 Chapters in 134 pages. The second is a revised, enlarged version of the first, containing 21 chapters in 252 pages. The third Notebook contains 33 pages of unorganized material. Ramanujan took these Notebooks with him to Cambridge. But, in one of his letters to a friend, he wrote that he had no time to look into them and most probably he did not put them to use during his five year stay abroad.

In March 1925, five years after Ramanujan's demise, Dr.S.R. Ranganathan, the then Librarian of Madras University, approached Prof Hardy and obtained the first Notebook of Ramanujan still with him. Hardy was only too glad to give the Notebook and he did so with the remark: *Ramanujan belongs to your country, the proper place for his Notebook is your own (Madras) University Library.* The second and third Notebooks of Ramanujan were donated to the Madras University Library after his death.

In 1957, at the instance of Prof. K. Chandrasekhar and Prof. Homi J. Bhabha, the Tata Institute of Fundamental Research, Bombay, brought out the facsimile edition of these Notebooks in two volumes, without any commentary. These were reprinted by Narosa Publishing House in 1987.

A resurgence in the work of Ramanujan occurred, in the Spring of 1976, when Prof. George E. Andrews, of the Pennsylvania State University, while going through the Estate of Prof. G.N. Watson (who died in 1965) with the Trinity College, made the historical discovery of 140 pages of Ramanujan's papers in a box, containing some 600 theorems mostly on mock theta functions. These were results which Ramanujan had noted down during the year when he was deathly ill, in India, after his return from Cambridge. To these papers, Prof. Andrews gave the romantic name, the 'Lost' Notebook of Ramanujan and wrote a series of articles [53, 56] in the American Mathematical Society. He set for himself the task of proving several of the theorems. The 'Lost' Notebook was published and released along with the reprints of the Notebooks of Ramanujan, by Narosa Publishing House on the occasion of the Ramanujan birth centenary celebrations, held in Madras, in Dec. 1987.

Of course, there have been many individual Mathematicians, all over the world, who worked on some aspects of the results in the Notebooks and several hundreds of research articles were published. A computer search by Bruce C. Berndt revealed that between 1973 – 84 over 200 research papers refer to Ramanujan in their titles or abstracts. But there was no concerted or dedicated effort, in the direction as suggested by Hardy, till the 1980s. Prof. Bruce C. Berndt, at the University of Illinois, Urbana-Champagne, took up the *heavy undertaking* or *formidable task* of editing the Notebooks of Ramanujan in right earnestness and set for himself the goal *to prove each of Ramanujan's theorems*, if they were new and for the known results refer to literature where proofs may be found. The details of the contents of the Notebooks and other Manuscripts is now recorded in the authentic work of Bruce C. Berndt [VIII].

The original Notebooks are now in the main Library of the Madras University (in Chepauk) and are not easily accessible. The pages are stuck in between translucent paper, a type of lamination of the pages. A few moth hole tracks are beginning to appear. The Notebooks should not be left in this condition and should be better preserved. They should be treated as the Mathematical heritage of India.

Hypergeometric series:

Leonhard Euler [6] is perhaps the first to study the hypergeometric functions, in 1748. The modern framework for the hypergeometric series and the corresponding hypergeometric functions is however due to Carl Friedrich Gauss [7], in 1812, when he delivered his famous thesis: *Disquisitiones generales circa seriem infinitam*

$$1+\frac{\alpha\beta}{1\cdot\ \gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot\ 2\cdot\ \gamma(\gamma+1)}xx+\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1\cdot\ 2\cdot\ 3\cdot\ \gamma(\gamma+1)(\gamma+2)}x^3+etc.$$

The usage of the notation ${}_2F_1(\alpha,\beta;\gamma;1)$ for the hypergeometric function was introduced by E.W. Barnes [2], in 1907, who developed its contour integral representation for the generalized hypergeometric series ${}_AF_B(x)$ – a single infinite contour integral expressed as some infinite series of residues – different from the simple integral transforms introduced by Euler.

Ramanujan's first few results in his historic letter are on hypergeometric series, which according to Hardy were *much more intriguing, and it becomes obvious that Ramanujan must possess much more general theorems.* It is also surprising that Ramanujan published no papers on Hypergeometric series!

It is the considered opinion of Hardy and other experts that the second Notebook of Ramanujan is a revised enlarged version of the first. But as far as this chapter XII of the first Notebook is concerned, it becomes Chapter X in the second Notebook. Bruce Berndt has studied in detail the entries in this second Notebook, entry - by - entry and has provided the proofs for every one of them. This is a painstakingly careful work and done during the years 1975 to 1999. The following table provides a summary of the results noted by Ramanujan in the first and the second Notebooks on Hypergeometric series:

Chapter Number	Notebook	Entries	Corollaries	Examples
XII	Vol.1	47	26	2
X	Vol. 2	35	34	22

There are thus a total of 91 entries in Chapter X of Notebook 1 and 75 in Chapter XII of Notebook 2 of Ramanujan. Though Ramanujan mentioned a few of his results in his first letter to Hardy, it is surprising that Ramanujan did not publish any paper on Hypergeometric series. In both Chapter XII of the first notebook and Chapter X of the second notebook, Ramanujan starts with the most general theorem, called the Dougall-Ramanujan summation theorem for a ${}_7F_6$ and the simpler theorems are given in several entries as consequences of this main theorem in these chapters. Experts on Ramanujan's work opine that Ramanujan went from examples to the general case. In the chapters on hypergeometric series, however, Ramanujan has presented from the general, particular cases. There is one more chapter on hypergeometric series, in the notebooks (Chapter XIII in the first Notebook and Chapter XI in the second note book) on hypergeometric series. Besides there are a few scattered results in other places in his notebooks. For complete details we refer the interested reader to the five part exhaustive work of Berndt[3] on Ramanujan's Notebooks.

The details of the nature of the entries made by Ramanujan and how Hardy studied Chapter XII of the first notebook and Berndt edited Chapter X of the second notebook on hypergeometric series and presented them to the Mathematicians of the world is technical and hence presented elsewhere. Here we point out some examples of a different interpretation that can be made than that already made [3] and then draw relevant conclusions.

Ramanujan did not give chapter headings in his Notebooks, except for Chapter 1 of Notebook 1 [**?**] which alone has the title: *Magic Squares*. Though chapter 1 of the second Notebook is also on the same theme of Magic squares, he has omitted mentioning this heading.

Ramanujan has numbered the pages in his own handwriting in the Notebooks. The organized material alone is on sequentially numbered right hand side pages in Notebook 1 [**?**] while the left hand side pages are unnumbered and on some there are (unnumbered) jottings. In the case of the second Notebook, Ramanujan has numbered both the left and right hand side pages sequentially.

While in Notebook 1, chapter 1 is only 3 pages long and has 4 Entries, chapter 1 of Notebook 2 is 8 pages long and has Entries not sequentially numbered and there are several additional examples, as faithfully pointed out in [**?**, I]. Berndt [**?**] has commented on the *Location in Notebook 2 of the Material in the 16 chapters of Notebook 1*, in an unumbered chapter at the end of Part IV. Following the contents of this chapter, one can note that from chapter 2 onwards, there is no longer a one-to-one correspondence between the Entries in the different chapters of the two Notebooks. So much so, it is interesting to note that the Entry 1 of Chapter XII of the first Notebook 1 corresponds to the Entry 1 of Chapter X of the second Notebook and is pertaining to the Dougall-Ramanujan summationn theorem.

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