Common Correlated Effects Estimation of Dynamic Panels with Cross-Sectional Dependence

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Abstract

We derive inconsistency expressions for dynamic panel data estimators under error cross-sectional dependence generated by an unobserved common factor in both the fixed effect and the incidental trends case. We show that for a temporally dependent factor, the standard within groups (WG) estimator is inconsistent even as both \( N \) and \( T \) tend to infinity. Next we investigate the properties of the common correlated effects pooled (CCEP) estimator of Pesaran [Econometrica, 2006] which eliminates the error cross-sectional dependence using cross-sectional averages of the data. In contrast to the static case, the CCEP estimator is only consistent when next to \( N \) also \( T \) tends to infinity. It is shown that for the most relevant parameter settings, the inconsistency of the CCEP estimator is larger than that of the infeasible WG estimator, which includes the common factors as regressors. Restricting the CCEP estimator results in a somewhat smaller inconsistency. The small sample properties of the various estimators are analysed using Monte Carlo experiments. The simulation results suggest that the CCEP estimator can be used to estimate dynamic panel data models provided \( T \) is not too small. The size of \( N \) is of less importance.

JEL Classification: C13, C15, C23

Keywords: Cross-Sectional Dependence; Dynamic Panel; Common Correlated Effects

1 Introduction

Over the last decades, estimation of dynamic panel data models has received a lot of attention. Nickell (1981) demonstrated that in dynamic panel data regressions the within groups (WG) estimator is inconsistent for fixed \( T \) and \( N \to \infty \). Given that the asymptotic bias may be quite sizable in many cases relevant to applied research, various alternative estimators have been suggested including general method of moments (GMM) estimators (Arellano and Bond, 1991; Blundell and Bond, 1998), bias-corrected WG estimators (Kiviet, 1995; Hahn and Kuersteiner, 2002; Bun and Carree, 2005; Everaert and Pozzi, 2007; Choi et al., 2010) and likelihood-based estimators (Lancaster, 2002; Moreira, 2009). However, new challenges arise when it comes

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to the estimation of dynamic panel data models. The recent panel data literature shifted its attention to the estimation of models with error cross-sectional dependence. A particular form that has become popular is a common factor error structure with a fixed number of unobserved common factors and individual-specific factor loadings (see e.g. Coakley et al., 2002; Phillips and Sul, 2003; Bai and Ng, 2004; Pesaran, 2006).

The most obvious implication of error cross-sectional dependence is that standard panel data estimators are inefficient and estimated standard errors are biased and inconsistent. Phillips and Sul (2003) for instance show that if there is high cross-sectional correlation there may not be much to gain from pooling the data. However, error cross-sectional dependence can also induce a bias and even result in inconsistent estimates. In general, inconsistency arises as an omitted variables bias when the observed explanatory variables are correlated with the unobserved common factors (see e.g. Pesaran, 2006). More specifically, Phillips and Sul (2007) show that in dynamic panel data models with fixed $T$ and $N \to \infty$, the unobserved common factors induce an additional (random) inconsistency term for the WG estimator even under the assumption of temporally independent factors such that these are not correlated with the lagged dependent variable. This bias disappears as $T \to \infty$. Sarafidis and Robertson (2009) show that also dynamic panel data IV and GMM estimators (in levels or first-differences) are inconsistent for fixed $T$ and $N \to \infty$ as the moment conditions used by these estimators are invalid under error cross-sectional dependence. Choi et al. (2010) suggest that their common recursive mean adjustment to reduce the dynamic panel data bias of the WG estimator can also be used under error cross-sectional dependence, but an explicit bias formula is not provided.

In this paper we further analyze the impact of error cross-sectional dependence in linear dynamic panels. Explicit inconsistency formulas are derived for both the naive within groups (W Gn) estimator, which ignores error cross-sectional dependence, and the common correlated effects pooled (CCEP) estimator of Pesaran (2006), which is explicitly designed to deal with unobserved common factors in the error term. We study both the standard fixed effects model and the incidental trends model. We first extend the work of Phillips and Sul (2007) by relaxing the assumption of a temporally independent common factor. In line with their results we find that for fixed $T$ and $N \to \infty$ the inconsistency of the W Gn estimator is a combination of the nonrandom dynamic panel data bias and a random component induced by the common factor in the error term. Importantly, the latter component of the inconsistency becomes nonrandom but does not disappear when next to $N \to \infty$ we also let $T \to \infty$ since the temporal dependence in the unobserved common factor implies that the error term is correlated with the lagged dependent variable even for both $N$ and $T \to \infty$. This finding should warn against the use of the W Gn estimator in cross-sectionally dependent dynamic panels even when $T$ is large. Second, we extend the work of Pesaran (2006) by analyzing the asymptotic behavior of the CCEP estimator in a dynamic panel data setting. The basic idea of CCEP estimation is to deal with error cross-sectional dependence by filtering out the unobserved common factors using the cross-section averages of both the dependent and the explanatory variables. We show that contrary to the static model, the CCEP estimator is no longer consistent for $N \to \infty$ and fixed $T$. Similar to the results for the WG estimator, the inconsistency is a combination of the standard nonrandom dynamic panel data bias and a random component which is now induced by orthogonalising on the cross-sectional averages. The main difference with the W Gn estimator is that both components of the inconsistency disappear as we also let $T \to \infty$. As a benchmark, we also derive the inconsistency of the infeasible within groups (W Gi) estimator,
which includes the unobserved factor as an explanatory variable. This WGi estimator also has a random inconsistency for fixed $T$ and $N \to \infty$ which disappears as $T \to \infty$. However, for the cases most relevant to applied research, the inconsistency of the CCEP is bigger than that of the WGi estimator. One possible reason for this is that the CCEP estimator as suggested by Pesaran (2006) ignores the restrictions on the individual-specific factor loadings as implied by the derivation of the cross-sectional averages augmented specification of the model. Imposing these restrictions, the inconsistency of the restricted CCEP estimator is closer to that of the WGi estimator.

We next analyse the small sample properties of the WG and CCEP estimators using a small-scale Monte Carlo experiment. First, the results illustrate that the standard WG estimator, ignoring error cross-sectional dependence, has a persistent (as $N$ and $T$ growing larger) bias for a temporally dependent factor. Second, the results show that both the unrestricted and the restricted CCEP estimators have a higher bias than the infeasible WG estimator for small values of $T$ but the restricted CCEP estimator outperforms the unrestricted CCEP estimator and is not much worse than the infeasible WG estimator for moderate $T$. In line with the results in Pesaran (2006), the small sample properties of the CCEP estimators are not very sensitive to the size of $N$. Overall, the results suggest that the CCEP estimator is quite useful for estimating cross-sectional dependent dynamic panel data models provided $T$ is not too small.

The remainder of this paper is organized as follows. Section 2 sets out the basic model and its assumptions. Section 3 derives explicit inconsistency expressions as $N \to \infty$ and large $T$ expansions thereof for the WGi, the WGi and the unrestricted and restricted CCEP estimators in a dynamic model with error cross-sectional dependence. Section 4 adds exogenous explanatory variables. Section 5 reports the results of a small-scale Monte Carlo experiment. Section 6 concludes.

2 Model and assumptions

Consider the following first-order autoregressive panel data model

\[(\text{Fixed effects model})\]
\[y_{it} = \alpha_i + \rho y_{i,t-1} + \nu_{it},\]  \hspace{1cm} (1a)

\[(\text{Incidental trends model})\]
\[y_{it} = \alpha_i + b_i t + \rho y_{i,t-1} + \nu_{it}.\] \hspace{1cm} (1b)

where $y_{it}$ is the observation of the dependent variable for cross-sectional unit $i$ ($i = 1, \ldots, N$) at time $t$ ($t = 1, \ldots, T$) and the individual effects $\alpha_i$ and $b_i$ are fixed parameters. We restrict $|\rho| < 1$ such that $y_{it}$ is stationary. For notational convenience we assume $y_{i0}$ is observed. We further assume:

**Assumption A1. (Cross-section dependence)** The error term $\nu_{it}$ has a single-factor structure

\[\nu_{it} = \gamma_i F_t + \varepsilon_{it},\] \hspace{1cm} (2)

\[F_t = \theta F_{t-1} + \mu_t, \quad |\theta| < 1,\] \hspace{1cm} (3)

where $F_t$ is an individual-invariant time-specific unobserved effect with $\mu_t \sim i.i.d. (0, \sigma^2_{\mu})$. The individual-specific factor loadings $\gamma_i$ satisfy $\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \gamma_i^2 = m^2_\gamma$ being finite.
The restriction of a single-factor structure is for notational convenience only. The inconsistencies for \( N \to \infty \) of the estimators presented below can straightforwardly be extended to the multi-factor case but the large \( T \) approximations of the inconsistencies would be more complicated as these would depend on the joint DGP of the factors.

**Assumption A2.** (Error condition) \( \varepsilon_{it} \sim i.i.d. (0,\sigma^2_{\varepsilon_i}) \) across \( i \) and \( t \) and independent of \( \alpha_j, y_{j0} \) and \( F_s \) \( \forall i, j, t, s \) and
\[
\text{plim} \frac{1}{N} \sum_{i=1}^{N} \sigma^2_{\varepsilon_i} = \sigma^2_{\varepsilon}.
\]

This assumption allows for cross-sectional heteroscedasticity under a mild regularity condition as in Phillips and Sul (2007).\(^1\)

The model in equations (1a)-(1b) and (2)-(3) can be written in component form as
\[
y_{it} = y^+_i + \gamma_i F^+_t,
\]
with
\[
F^+_t = (1 - \rho L)^{-1} F_t = (\rho + \theta) F^+_{t-1} - \rho \theta F^+_{t-2} + \mu_t.
\]

(Fixed effects model) \( y^+_it = \alpha + \rho y^+_{i,t-1} + \varepsilon_{it} \),

(Incidental trends model) \( y^+_it = \alpha + b_t + \rho y^+_{i,t-1} + \varepsilon_{it} \).

This component form is convenient for the proofs presented below as the components \( y^+_it \) and \( F^+_t \) are independent.

**Assumption A3.** (Initial conditions) Initial conditions are in the infinite past, such that the initial observations \( y^+_{i0} \) and \( F_0 \) satisfy
\[
y^+_{i0} = \frac{\alpha_i}{1 - \alpha} + \sum_{j=0}^{\infty} \rho^j \varepsilon_{i,-j},
\]
\[
F_0 = \sum_{j=0}^{\infty} \theta^j \mu_{i,-j}.
\]

The assumptions on the initial observations will be used in deriving large \( T \) approximations of the inconsistency expressions for \( N \to \infty \) of the estimators presented below, but the (in)consistency of the estimators does not rely on the specification of the initial conditions.

Moreover, for the presentation of the estimators below, stacking the model in (1a)-(1b) and (2) for each

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\(^1\)As in Phillips and Sul (2007) this cross-sectional heteroscedasticity does not affect the asymptotic results.
\( \text{i yields} \)

\begin{align*}
\text{(Fixed effects model) } & \quad y_i = \alpha_i + \rho y_{i,-1} + \gamma_i F + \varepsilon_i, \quad (9a) \\
\text{(Incidental trends model) } & \quad y_i = \alpha_i + b_i \tau + \rho y_{i,-1} + \gamma_i F + \varepsilon_i, \quad (9b)
\end{align*}

where \( y_i = (y_{i1}, \ldots, y_{iT})' \), \( y_{i,-1} = (y_{i0}, \ldots, y_{i,T-1})' \), \( F = (F_1, \ldots, F_T)' \), \( \varepsilon_i = (\varepsilon_{i1}, \ldots, \varepsilon_{iT})' \), \( \tau \) is a \((T \times 1)\) vector of ones and \( \tau = (1, \ldots, T)' \).

The affix notation on \( y \) will be used to denote the individual mean of \( y_{it} \), \( \bar{y}_{it} \) will be used to denote that \( y_{it} \) has been demeaned while \( \check{y}_{it} \) signifies that \( y_{it} \) has been demeaned and detrended. Similar notation is used for all other series.

3 Estimators

In this section we analyse the asymptotic properties of various estimators for \( \rho \) in equations (1a) and (1b) under assumptions A1-A3. We are mainly interested in the properties of the CCEP estimator, which was suggested by Pesaran (2006) for a static model with cross-sectional dependence. We consider both an unrestricted and a restricted version of the CCEP estimator. We also include two benchmark estimators. First, the WGN estimator is included to illustrate the extent of the bias that can occur if error cross-section dependence is ignored. Second, the WGi estimator is included to provide an upper bound to the performance of the CCEP estimators. Each of these four estimators for \( \rho \) in the models in (1a) and (1b) is given by

\[
\hat{\rho} = \frac{\sum_{i=1}^{N} \hat{y}_{i,-1}' y_i M y_i}{\sum_{i=1}^{N} \hat{y}_{i,-1}' M y_i y_{i,-1}} = \rho + \frac{\sum_{i=1}^{N} \hat{y}_{i,-1}' M (\gamma_i F + \varepsilon_i)}{\sum_{i=1}^{N} \hat{y}_{i,-1}' M y_{i,-1}}, \quad (10)
\]

but differ in their choice of \( M \).

3.1 Naive within groups

Fixed effects model

The WGN estimator for \( \rho \) in (1a) is obtained by choosing \( M \) in (10) to be the projection matrix \( M_i = I_T - \bar{u}' / T \) that demean the data. Nickell (1981) showed that for a model with no cross-sectional dependence, the within groups estimator for \( \rho \) yields inconsistent results for \( N \to \infty \) and \( T \) fixed but this inconsistency disappears as \( T \to \infty \). Phillips and Sul (2007) show that the consistency of the WGN estimator as \( N \) and \( T \to \infty \) continues to hold in dynamic panel data models with cross-sectional dependence induced by a temporarily independent factor. The next proposition looks into the asymptotic properties of the WGN estimator in the case of a temporarily dependent factor. All proofs are in Appendix A.

\[ \text{Note that identification of } \rho \text{ requires } \sum_{i=1}^{N} \hat{y}_{i,-1}' M y_{i,-1} \text{ to be nonsingular.} \]
Proposition 1.a. In model (1a) under assumptions A1-A3, the W Gn estimator is inconsistent as \( N \to \infty \)

\[
\lim_{N \to \infty} (\hat{\rho}_{\text{WGN}} - \rho) = -\frac{A(\rho, T) + m_{N,T}^2 \sum_{t=1}^{T} \tilde{F}_t}{B(\rho, T) + m_{N,T}^2 \sum_{t=1}^{T} \tilde{F}_t^2},
\]  

(11)

where \( A(\rho, T) = \frac{1}{1-\rho} \left( 1 - \frac{1}{1-\rho^T} \right) \) and \( B(\rho, T) = \frac{T}{1-\rho^T} \left( 1 - \frac{1}{1-\rho^T} + 2\rho \frac{1-\rho^T}{(1-\rho^T)^2} \right) \).

For large \( T \), the inconsistency in (11) has the following expansion

\[
\lim_{N \to \infty} (\hat{\rho}_{\text{WGN}} - \rho) = -\frac{1 + \rho}{T} + \eta \left( E(g_{FT}) + \frac{1 + \rho}{T} \right) + O_p \left( \frac{1}{\sqrt{T}} \right),
\]

(12)

with \( g_{FT} = \sum_{t=1}^{T} \tilde{F}_t \sum_{t=1}^{T} \tilde{F}_t^2 \) and \( \eta = \frac{m_{N,T}^2 \sigma^2 \gamma}{\sigma^2} \).

For \( \theta = 0 \), the large \( T \) expansion in (12) reduces to

\[
\lim_{N \to \infty} (\hat{\rho}_{\text{WGN}} - \rho) = -\frac{1 + \rho}{T} - \frac{2\rho}{T} + O_p \left( \frac{1}{\sqrt{T}} \right),
\]

(13)

while for \( \theta \neq 0 \) we have

\[
\lim_{(N,T) \to \infty} (\hat{\rho}_{\text{WGN}} - \rho) = \frac{(1 - \rho^2) \theta}{(1 + \theta \rho) + (1 - \theta \rho) (1 - \theta^2)} \frac{\sigma^2}{m_{N,T} \sigma^2 \rho}.
\]

(14)

Proposition 1.a is similar to the results in Phillips and Sul (2007), the only difference being that the data generating process of \( F_t \) is different. Equation (11) shows that the inconsistency of \( \hat{\rho}_{\text{WGN}} \) for \( N \to \infty \) and fixed \( T \) has two sources. The first is the standard Nickell dynamic panel data bias, which depends on the persistence \( \rho \) in \( y_{it} \) and on the time dimension \( T \). This can be seen by setting the error cross-sectional dependence to zero (\( m_{N,T}^2 = 0 \)) such that equation (11) reduces to the standard Nickell bias formula \(-A(\rho, T)/B(\rho, T)\) which has a large \( T \) expansion \(- (1 + \rho)/T\). The second source stems from the error cross-sectional dependence. It is apparent from equation (11) that this part of the inconsistency is random for fixed \( T \) as \( 1/T \sum_{t=1}^{T} \tilde{F}_t \) and \( 1/T \sum_{t=1}^{T} \tilde{F}_t^2 \) depend on the particular realisation for \( F_t \). The large \( T \) expansion in equation (12) shows that consistency of the WGN estimator for both \( N \) and \( T \to \infty \) depends on the asymptotic behavior of \( E(g_{FT}) \), which in turn depends on the temporary dependence in \( F_t \).

Equation (13) reproduces the approximate inconsistency expression of Phillips and Sul (2007) which shows that for a temporally independent factor \( (\theta = 0) \) the inconsistency of the WGN estimator disappears as \( T \to \infty \). The most important novel conclusion from Proposition 1 is that inertia in \( F_t \) \( (\theta \neq 0) \) implies that the WGN estimator is inconsistent as both \( N, T \to \infty \) since \( E(g_{FT}) \) does not converge to zero. Essentially, this is an omitted variable bias as \( \theta \neq 0 \) implies \( E(y_{i,t-1} F_t) \neq 0 \) such that omitting \( F_t \) from the regression results in an inconsistent estimator for \( \rho \) even if both \( N \) and \( T \) become infinitely large. Note that given the restrictions that \( |\rho| < 1 \) and \( |\theta| < 1 \), the sign of the inconsistency depends on the sign of \( \theta \). These results are visualized in panel (a) of Figure 1 which plots the inconsistency for \( \theta = 0 \) and \( \theta = 0.9 \) calculated from equation (11) for \( N \to \infty \) and various values of \( T \) together with the large \( T \) expansion from equation (13) for \( \theta = 0 \) and the inconsistency for \( N, T \to \infty \) from (14) for \( \theta = 0.9 \). The 5th and 95th percentiles...
show the randomness in the inconsistency for $N \to \infty$ and $T$ fixed. The graph also shows that the large $T$ approximation for $\theta = 0$ is fairly accurate.

**Figure 1:** Inconsistency of the WGN estimator for $\rho = 0.5$

(a) Fixed effects model

(b) Incidental trends model

Note: $\sigma^2 = 1 - \rho^2$, $\sigma^2_\rho = 1 - \theta^2$ and $\gamma_i \sim i.i.d. U[\gamma_L, \gamma_U]$ such that $\gamma^2 = 1.09$.

**Incidental trends model**

The WGN estimator for $\rho$ in (1b) is obtained by choosing $M$ in (10) to be the projection matrix $M_T = I_T - \tau_i (\tau_i')^{-1} \tau_i'$, with $\tau_i = (\iota, \tau)$, that demeans and detrends the data.

**Proposition 1.b.** In model (1b) under assumptions A1-A3, the WGN estimator is inconsistent as $N \to \infty$

$$\lim_{N \to \infty} (\hat{\rho}_{\text{WGn}} - \rho) = -C(\rho, T) + \frac{m_2^2}{\sigma^2} \sum_{t=1}^T \tilde{F}_{t-1}^+ \tilde{F}_t, \tag{15}$$

where $C(\rho, T) = \frac{1}{1 - \frac{2}{1 - \rho}} \left(\frac{T}{1 - \rho} - \frac{1}{1 - \rho} C_1\right)$ and $D(\rho, T) = \frac{T - 2}{1 - \rho} \left(1 - \frac{1}{T} \frac{4 \rho}{1 - 2 \rho} D_1\right)$, with $C_1$ and $C_2$ being defined in equations (A-12)-(A-13).

For large $T$, the inconsistency in (15) has the following expansion

$$\lim_{N \to \infty} (\hat{\rho}_{\text{WGn}} - \rho) = -2 \frac{1 + \rho}{T} + \eta \left(E(h_{FT}) + 2 \frac{1 + \rho}{T} \right) + O_p \left(\frac{1}{\sqrt{T}}\right), \tag{16}$$

with $h_{FT} = \sum_{t=1}^T \tilde{F}_{t-1}^+ \tilde{F}_t / \sum_{t=1}^T (\tilde{F}_{t-1}^+)^2$ and $\eta$ as defined in Proposition 1. A such, for $\theta = 0$ the large $T$ expansion in (16) reduces to

$$\lim_{N \to \infty} (\hat{\rho}_{\text{WGn}} - \rho) = -2 \frac{1 + \rho}{T} + O_p \left(\frac{1}{\sqrt{T}}\right), \tag{17}$$

while for $\theta \neq 0$ the inconsistency as $N, T \to \infty$ is given by (14).

The results in Proposition 1.b show that the inconsistency for $N \to \infty$ and $T$ fixed is larger than in the
fixed effects model. For $\theta = 0$, the expansion in (17) reproduces the expression in Phillips and Sul (2007) which shows that the first term in the inconsistency expression, which is due to the elimination of the fixed effects and the incidental trends, is approximately twice as large as in the fixed effects model. For $\theta \neq 0$, the inconsistency for $N, T \to \infty$ is the same as in the fixed effects model. These results can also be seen from panel (b) in Figure 1.

3.2 Infeasible within groups

Fixed effects model

The WGi estimator, i.e. including $F_i$ as an observed regressor, for $\rho$ in (1a) is obtained by choosing $M$ in (10) to be the projection matrix $M_{F_i} = I_T - F_i (F_i' F_i)^{-1} F_i'$, with $F_i = (i, F)$.

**Proposition 2.a.** In model (1a) under assumptions A1-A3, the WGi estimator is inconsistent as $N \to \infty$

$$\text{plim}_{N \to \infty} (\hat{\rho}_{\text{WG}, \theta} - \rho) = \frac{-A(\rho, T) - \sum_{t=1}^{T-1} \rho^{t-1} \bar{g}_{F,t}}{B(\rho, T) - \frac{1}{1-\rho^2} \left( 1 + 2\rho \sum_{t=1}^{T-1} \rho^{t-1} \bar{g}_{F,t} \right) + T \frac{\sigma_{\varepsilon}^2}{\sigma_e^2} \bar{k}_F}, \quad (18)$$

where $\bar{g}_{F,t} = \sum_{s=t+1}^{T} \bar{k}_{s,s-t}$ and $\bar{k}_F = \frac{1}{T} \sum_{t=1}^{T} \left( \bar{F}_{t-1}^+ \right)^2 \left( 1 - \frac{\left( \frac{1}{T} \sum_{t=1}^{T} \bar{F}_{t-1}^+ \right)^2}{\frac{1}{T} \sum_{t=1}^{T} \bar{F}_{t-1}^2 + \sum_{t=1}^{T} \left( \bar{F}_{t-1}^+ \right)^2} \right)$, with $\bar{k}_{s,s-t}$ being the $(s, s-t)$th element in $\bar{F} \left( \bar{F}' \bar{F} \right)^{-1} \bar{F}'$.

For large $T$, the inconsistency in (18) has the following expansion

$$\text{plim}_{N \to \infty} (\hat{\rho}_{\text{WG}, \theta} - \rho) = -\frac{1}{T} \left( 1 + \rho + \frac{\theta (1 - \rho^2)}{(1 - \theta \rho)} \right) \left( 1 + \frac{m_e^2}{(1 - \theta \rho)^2} \frac{\sigma_{\varepsilon}^2}{\sigma_e^2} \right)^{-1} + o_p \left( \frac{1}{T} \right). \quad (19)$$

Proposition 2.a shows that the inconsistency of the WGi estimator for $N \to \infty$ also has two sources. The first is again the standard Nickell bias. The second source now stems from orthogonalizing on the observed factor $F_i$. For fixed $T$, this induces randomness in the inconsistency as the orthogonalisation depends on the particular realization of the process $F_i$. Moreover, equation (19) shows that temporal dependence in the common factor ($\theta \neq 0$) also induces a nonrandom inconsistency for $N \to \infty$, which disappears as $T \to \infty$. The intuition for this result is that for fixed $T$ the transformed error term $M_{F_i} \varepsilon_i$ is, next to being a function of the average error term $\tau_i$ due to the within transformation, now also a function of the entire series $F$ (as represented by $\bar{g}_{F,t}$) due to the orthogonalisation on $F_i$ which results in correlation with the explanatory variable $y_{i,t-1}$. The denominator of equation (19) further shows that the inconsistency is smaller when the error cross-sectional dependence is stronger as this implies more variability in the explanatory variable $y_{i,t-1}$, which is induced by $F_{t-1}$ and is not completely filtered out by including $F_i$ as a control variable in the regression. This additional variability is captured by the term $\bar{k}_F$ in the denominator of equation (18). The inconsistency for $N \to \infty$ calculated from equation (18) and its large $T$ approximation calculated from equation (19) are plotted in panel (a) of Figure 2. The graph shows that the large $T$ approximation is again fairly accurate.
Figure 2: Inconsistency for $N \to \infty$ of the WGi estimator for $\rho = 0.5$ and $\theta = 0.9$

(a) Fixed effects model
(b) Incidental trends model

Note: See Figure 1.

Incidental trends model

The WGi estimator for $\rho$ in (1b) is obtained by choosing $M$ in (10) to be the projection matrix $M_F = I_T - F_\tau (F_\tau' F_\tau)^{-1} F_\tau'$, with $F_\tau = (\iota, \tau, F)$.

Proposition 2.b. In model (1b) under assumptions A1-A3, the WGi estimator is inconsistent as $N \to \infty$

$$\lim_{N \to \infty} (\hat{\rho}_{WGi} - \rho) = \frac{-C(\rho, T) - \sum_{t=1}^{T-1} \rho^{t-1} \bar{y}_{F,t}}{D(\rho, T)} - \frac{1}{1-\rho} \left( 1 + 2\rho \sum_{t=1}^{T-1} \rho^{t-1} \bar{y}_{F,t} \right) + T \frac{m^2}{\sigma^2} \tilde{k}_F,$$

where $\bar{y}_{F,t} = \sum_{s=t+1}^T \bar{\kappa}_{s,s-t}$ and $\tilde{k}_F = \frac{1}{T} \sum_{t=1}^{T} \left( \tilde{F}_{t-1} \right)^2 \left( 1 - \frac{\sum_{j=1}^{T} \tilde{F}_{j} \tilde{F}_{j-1}}{\sum_{j=1}^{T} (\tilde{F}_{j-1})^2} \right)$, with $\bar{\kappa}_{s,s-t}$ being the $(s,s-t)$th element in $\tilde{F} \left( \tilde{F}' \right)^{-1} \tilde{F}'$.

For large $T$, the inconsistency in (20) has the following expansion

$$\lim_{N \to \infty} (\hat{\rho}_{WGi} - \rho) = -\frac{1}{T} \left( 2 (1 + \rho) + \theta \left( 1 - \rho^2 \right) \left( 1 + \frac{m^2}{\sigma^2} \frac{\sigma^2}{\sigma^2} \right)^{-1} + o_p \left( \frac{1}{T} \right).$$

The results in Proposition 2.b show that the inconsistency for $N \to \infty$ and $T$ fixed is larger than in the fixed effects model, i.e. the first term in the large $T$ expansion of the inconsistency is again twice as large as in the fixed effects model. These results can also be seen from panel (b) in Figure 2.

3.3 CCEP

The CCEP estimator suggested by Pesaran (2006) eliminates the unobserved common factors by including cross-section averages of the dependent and the explanatory variables. Taking cross-sectional averages of the
fixed effects model in (1a) gives

\[ y_t = \alpha + \rho y_{t-1} + \gamma F_t + \varepsilon_t, \]  

(22)

with \( \bar{y}_t \) denoting the cross-sectional mean of \( y_{it} \) and similarly for the other variables. In line with Pesaran (2007) and Phillips and Sul (2007) we make the full rank assumption \( \gamma \neq 0 \) (for fixed \( N \) and \( N \to \infty \)) such that equation (22) can be solved for \( F_t \) as

\[ F_t = \frac{1}{\gamma} (\bar{y}_t - \bar{\alpha} - \rho \bar{y}_{t-1} - \bar{\varepsilon}_t). \]  

(23)

Note that

\[ \text{plim}_{N \to \infty} \varepsilon_t = 0, \]  

(24)

\[ \text{plim}_{N \to \infty} \bar{y}_t = \lim_{N \to \infty} (\bar{y}_t^+ + \gamma F_t^+) = (1 - \rho)^{-1} \alpha + \gamma F_t^+, \]  

(25)

with \( \alpha = \text{plim} \bar{\alpha} \) and \( \gamma = \text{plim} \bar{\gamma} \neq 0 \).

Inserting (23) in (1a) yields the following augmented form

\[ y_{it} = \alpha_i + \rho y_{i,t-1} + \frac{\gamma_i}{\gamma} (\bar{y}_t - \bar{\alpha} - \rho \bar{y}_{t-1} - \bar{\varepsilon}_t) + \varepsilon_{it}, \]  

(26)

with \( \gamma_1 = \gamma_i / \bar{\gamma}, \gamma_2 = \rho \gamma_1, \alpha^*_i = \alpha_i - \gamma_1 \bar{\alpha} \) and \( \varepsilon^*_{it} = \varepsilon_{it} - \gamma_1 \bar{\varepsilon}_t \).

Similarly, for the incidentals trends model in (1b) we obtain

\[ F_t = \frac{1}{\gamma} (\bar{y}_t - \bar{\alpha} - \bar{b}t - \rho \bar{y}_{t-1} - \bar{\varepsilon}_t), \]  

(27)

\[ y_{it} = \alpha^*_i + b^*_i t + \rho y_{i,t-1} + \gamma_1 \bar{y}_t + \gamma_2 \bar{y}_{t-1} + \varepsilon^*_{it}, \]  

(28)

with \( b^*_i = b_i - \gamma_1 \bar{b} \).

The augmented forms in (26) and (28) are in terms of observable variables and can therefore be estimated with least squares. This is what Pesaran (2006) refers to as CCEP estimators. We consider two versions. The first, CCEP\( \text{Pu} \) ignores the restrictions on \( \gamma_1, \gamma_2, \alpha^*_i \) and \( b^*_i \) as implied by (26) or (28), while the second CCEP\( \text{Pr} \) takes these restrictions into account. Note that the augmented forms are easily generalized to a multi-factor model (see e.g. equation (37) in Phillips and Sul, 2007). The advantage of the CCEP\( \text{Pu} \) over the CCEP\( \text{Pr} \) approach is that it does not require a decision on the number of common factors. The number of factors can however be estimated using the panel information criteria suggested by Bai and Ng (2002).
3.3.1 Unrestricted CCEP

Fixed effects model

The unrestricted CCEP estimator for \( \rho \) in (26) is obtained by choosing \( M \) in (10) to be the projection matrix \( M_{G_x} = I_T - G_x (G_x'G_x)^{-1} G_x' \), with \( G_x = (\bar{y}, \bar{y}_{-1}) \) and \( G = (\bar{y}, \bar{y}_{-1})' \) and \( \bar{y}_{-1} = (\bar{y}_0, \ldots, \bar{y}_{T-1})' \).

Theorem 1.a. In model (1a) under assumptions A1-A3, the CCEPu estimator is inconsistent as \( N \to \infty \)

\[
\text{plim}_{N \to \infty} (\hat{\rho}_{\text{CCEPu}} - \rho) = \frac{-A(\rho, T) - \sum_{t=1}^{T-1} \rho^{t-1} \tilde{g}_{F,t}^+}{B(\rho, T) - \frac{2}{1-\rho^2} \left( 1 + \rho \sum_{t=1}^{T-1} \rho^{t-1} \tilde{g}_{F,t}^+ \right)},
\]

where \( \tilde{g}_{F,t}^+ = \sum_{s=t+1}^{T} \tilde{\kappa}_{s,s-t}^+ \) with \( \tilde{\kappa}_{s,s-t}^+ \) being the \((s, s-t)\)th element in \( \tilde{H} \left( \tilde{H}' \tilde{H} \right)^{-1} \tilde{H}' \), with \( \tilde{H} = \left( \tilde{F}^+, \tilde{F}^+_{-1} \right) \).

The inconsistency in (29) has the following large \( T \) expansion

\[
\text{plim}_{N \to \infty} (\hat{\rho}_{\text{CCEPu}} - \rho) = -\frac{1}{T} \left( 1 + 2\rho + \frac{\theta(1-\rho^2)}{1-\theta \rho} \right) + o_p \left( \frac{1}{T} \right).
\]

The implication of Theorem 1.a is that the CCEPu estimator is consistent for both \( N \) and \( T \to \infty \) but has a different inconsistency expression compared to the WGi estimator for \( N \to \infty \) and \( T \) fixed. The intuition for this is that the error term \( \varepsilon_{it} \) is now orthogonalised on a constant, \( \bar{y}_i \) and \( \bar{y}_{i-1} \), with, as can be seen from (25), the latter two converging to \( F_{i-1}^+ \) and \( F_{i-1}^+ \) respectively as \( N \to \infty \). For fixed \( T \), this implies two differences compared to the WGi estimator. First, orthogonalising on \( F_{i-1}^+ \) results in an extra correlation between the orthogonalised error term and the explanatory variable \( y_{i,t-1} \) as the latter is by construction a function of \( F_{i-1}^+ \). As such, comparing (19) and (30), the numerator of the latter contains an extra term in \( \rho \). Second, the extra variability in the explanatory variable \( y_{i,t-1} \) induced by \( F_{i-1} \) is now completely filtered out by orthogonalising on \( F_{i-1}^+ \). As such, stronger error cross-sectional dependence raises the denominator in (19) but doesn’t affect (30). Further comparing (19) and (30), it is clear that both inconsistencies need not have the same direction and that the absolute value of the inconsistency of the CCEPu estimator is not necessarily bigger than that of the WGi estimator. However, for the majority of values for \( \rho \) and \( \theta \), the absolute value of the inconsistency is larger for the CCEPu estimator. For the most relevant case of both \( \rho > 0 \) and \( \theta > 0 \), the inconsistency of both the WGi and CCEPu estimator is negative.\(^3\)

However, the inconsistency of the latter is bigger compared to the former. Thus, in these cases, approximating the unobserved \( F_i \) using cross-sectional averages of the observed data results in a larger inconsistency for \( N \to \infty \) and \( T \) fixed compared to the benchmark infeasible estimator with \( F_i \) observed. These results are shown in panel (a) of Figure 3, which plots the inconsistency calculated from equation (29) for \( N \to \infty \) and various values of \( T \) together with the large \( T \) expansion from equation (30).

Incidental trends model

The unrestricted CCEP estimator for \( \rho \) in (28) is obtained by choosing \( M \) in (10) to be the projection matrix \( M_{G_x} = I_T - G_x (G_x'G_x)^{-1} G_x' \), with \( G_x = (\bar{y}, \bar{y}_{-1}) \) and \( G = (\bar{y}, \bar{y}_{-1})' \) as defined above.

\(^3\)One example where the approximate inconsistency of the CCEPu estimator is smaller than that of the WGi estimator is the parameter setting used in Figures 1-4 but with \( \rho = 0.5 \) being replaced by e.g. \( \rho = -0.5 \).
Theorem 1.b. In model (1b) under assumptions A1-A3, the CCEPu estimator is inconsistent as $N \to \infty$

$$\text{plim}_{N \to \infty} (\hat{\rho}_{CCEPu} - \rho) = \frac{-C(\rho, T) - \sum_{t=1}^{T-1} \rho^{t-1} \tilde{y}_{F,t}^+}{D(\rho, T) - \frac{2}{1-\rho^2} \left(1 + \rho \sum_{t=1}^{T-1} \rho^{t-1} \tilde{y}_{F,t}^+ \right)},$$

where $\tilde{y}_{F,t}^+ = \sum_{s=t+1}^T \tilde{k}_{s,s-t}^+$ with $\tilde{k}_{s,s-t}^+$ being the $(s, s-t)$ th element in $\tilde{H} \left(\tilde{H}' \tilde{H}\right)^{-1} \tilde{H}'$, with $\tilde{H} = \left(\tilde{F}^+, \tilde{F}_{-1}^+\right)$. The inconsistency in (31) has the following large $T$ expansion

$$\text{plim}_{N \to \infty} (\hat{\rho}_{CCEPu} - \rho) = -\frac{1}{T} \left(2 + 3 \rho + \frac{\theta(1 - \rho^3)}{1 - \theta \rho} \right) + o_p \left(\frac{1}{T} \right).$$

The results in Theorem 1.b again show that the inconsistency for $N \to \infty$ and $T$ fixed is larger than in the fixed effects model, i.e. a factor $-\frac{1+\rho}{T}$ is added to the approximate inconsistency expression. These results are visualized in panel (b) of Figure 3.

3.3.2 Restricted CCEP

Fixed effects model

The restricted CCEP estimator for $\rho$ in (26) can be obtained by minimizing the objective function

$$S_{NT}(\rho, F) = \frac{1}{NT} \sum_{i=1}^{N} (y_i - \rho y_{i-1})' M_{F_i} (y_i - \rho y_{i-1}).$$

Although $F$ is not observed when estimating $\rho$ and similarly, $\rho$ is not observed when estimating $F$, we can replace the unobserved quantities by initial estimates and iterate until convergence. The continuously-
updated estimator for \((\rho, F)\) is defined as

\[
(\hat{\rho}_{\text{CCEPr}}, \hat{F}) = \arg\min_{\rho,F} S_{NT}(\rho, F).
\]

More specifically, \(\hat{\rho}_{\text{CCEPr}}\) is the solution to equation (10) with the projection matrix \(M\) chosen to be \(M_{\hat{F}_i} = I_T - \hat{F}_i \left( \hat{F}_i' \hat{F}_i \right)^{-1} \hat{F}_i'\), where \(\hat{F}_i = (\iota, \hat{F})\), while \(\hat{F}\) is the solution to the following equation

\[
\hat{F} = \frac{1}{T} \left( \bar{y} - \hat{\rho}_{\text{CCEPr}} \bar{y}_t - 1 \right).
\]

Note that an estimate \(\hat{\alpha}\) for the constant \(\alpha\) in (23) has been dropped from (35) since this would be projected out anyway as \(\iota\) is included in \(\hat{F}_i\). Note that the restricted CCEP estimator bears some similarities with the continuously updated (Cup) estimator presented in Bai et al. (2009). The difference being that the CCEP estimates the unobserved components via the cross-sectional averages of both dependent and explanatory variables, whereas the Cup estimator uses a principal component approach.

**Theorem 2.a.** In model (1a) under assumptions A1-A3, the CCEPr estimator is inconsistent as \(N \to \infty\)

\[
\lim_{N \to \infty} (\hat{\rho}_{\text{CCEPr}} - \rho) = -A(\rho, T) - \frac{\sum_{t=1}^{T-1} \rho^t \tilde{g}_{F,t}}{B(\rho, T) - \frac{2}{1-\rho^2} \left( 1 + \rho \sum_{t=1}^{T-1} \rho^t \tilde{g}_{F,t} \right)}.
\]

where \(\tilde{g}_{F,t} = \sum_{s=t+1}^{T} \tilde{\tau}_{s,s-t}\) with \(\tilde{\tau}_{s,s-t}\) being the \((s,s-t)\)th element in \(\widetilde{F} \left( \widetilde{F}' \widetilde{F} \right)^{-1} \widetilde{F}'\).

The inconsistency in (36) has the following large \(T\) expansion

\[
\lim_{N \to \infty} (\hat{\rho}_{\text{CCEPr}} - \rho) = -\frac{1}{T} \left( 1 + \rho + \frac{\theta(1-\rho^2)}{1-\theta \rho} \right) + o_p \left( \frac{1}{T} \right).
\]

Comparing (37) and (19), the inconsistency of the CCEPr equals the inconsistency of the WGi multiplied by a factor \(\left( 1 + \frac{m^2}{(1-\theta \rho)^2} \frac{\sigma^2}{\sigma^2} \right) > 1\). This implies that the CCEPr inconsistency has the same direction and is bigger than that of the WGi estimator. The intuition for this is that for fixed \(T\), the deviation of \(\hat{F}_i\) from \(F_i\) is a function of \(F_{i-1}^+\), as can be seen from (A-36), which induces extra correlation between the transformed error term \(M_{\hat{F}_i} \varepsilon_{it}\) and \(y_{it-1}\). Further comparing (37), (30) and (19), the inconsistency of the CCEPr estimator is smaller than that of the CCEPu estimator and closer to that of the WGi estimator for the most relevant case of \(\rho > 0\) and \(\theta > 0\). The inconsistency for \(N \to \infty\) and its large \(T\) expansion are shown in panel (a) of Figure 4.

**Incidental trends model**

The restricted CCEP estimator for \(\rho\) in (28) is obtained by replacing \(M_{\hat{F}_i}\) in (33) by \(M_{\hat{F}_r}\) such that \(\left(\hat{\rho}_{\text{CCEPr}}, \hat{F} \right)\) is again the solution to equations (10)-(35) but with \(M_{\hat{F}_i}\) replaced by \(M_{\hat{F}_r} = I_T - \hat{F}_r \left( \hat{F}_r' \hat{F}_r \right)^{-1} \hat{F}_r'\) with \(\hat{F}_r = (\iota, \tau, \hat{F})\).
Theorem 2.b. In model (1b) under assumptions A1-A3, the CCEPr estimator is inconsistent as $N \to \infty$

$$\text{plim}_{N \to \infty} \left( \hat{\rho}_{\text{CCEPr}} - \rho \right) = \frac{-C(\rho, T) - \sum_{t=1}^{T-1} \rho^{t-1} \tilde{g}_{F,t}}{D(\rho, T) - \frac{2}{1-\rho^2} \left( 1 + \rho \sum_{t=1}^{T-1} \rho^{t-1} \tilde{g}_{F,t} \right)},$$

where $\tilde{g}_{F,t} = \sum_{s=t+1}^{T} \tilde{r}_{s,s-t}$ with $\tilde{r}_{s,s-t}$ being the $(s, s-t)$th element in $\tilde{F} \left( F \ F' \right)^{-1} F'$.

The inconsistency in (38) has the following large $T$ expansion

$$\text{plim}_{N \to \infty} \left( \hat{\rho}_{\text{CCEPr}} - \rho \right) = -\frac{1}{T} \left( 2(1+\rho) + \frac{\theta(1-\rho^2)}{1-\theta\rho} \right) + o_p \left( \frac{1}{T} \right).$$

The results in Theorem 2.b show that, in line with the results for the CCEPu estimator, a factor $-\frac{1+\rho}{T}$ is added to the approximate inconsistency expression compared to the fixed effects model. The inconsistency for $N \to \infty$ and its large $T$ expansion are shown in panel (b) of Figure 4.

4 Including exogenous variables

This section extends the model in (1a)-(1b) by including a vector of exogenous variables, $x_{it}$. Consider the following autoregressive model

(Fixed effects model) \quad y_{it} = \alpha_i + \rho y_{i,t-1} + x_{it}' \beta + \nu_{it}, \quad (40a)

(Incidental trends model) \quad y_{it} = \alpha_i + b_t + \rho y_{i,t-1} + x_{it}' \beta + \nu_{it}, \quad (40b)

with $x_{it} = (x_{it1}, \ldots, x_{itK})'$ a $(K \times 1)$ vector of explanatory variables which is assumed to be strictly exogenous with respect to the errors, i.e. $E(x_{it} \varepsilon_{is}) = 0 \ \forall i, t, s$, but allowed to be correlated with the individual effects.
and the common factor$^4$.

The estimator for $\rho$ is now given by

$$\hat{\rho} = \rho + \frac{\sum_{i=1}^{N} y_{i,-1} M_{x_i} M (\gamma_i F + \epsilon_i)}{\sum_{i=1}^{N} y_{i,-1} M_{x_i} M y_{i,-1}},$$

(41)

with the projection matrix $M$ for each of the four considered estimators as defined in Section 3 and $M_{x_i} = I_T - M x_i (x_i' M x_i)^{-1} x_i' M$ with $x_i = (x_{i1}, \ldots, x_{iT})'$. Equation (41) shows that adding an additional vector of exogenous explanatory variables has a double impact on $\hat{\rho}$. First, inconsistencies tend to be smaller in absolute value as the denominator of (41) is bigger than that of (10). This is due to the fact that adding explanatory variables increases the variance in $y_{i,t-1}$ and this extra variability is not fully projected out by orthogonalising on $x_{it}$. Second, the numerator of (41) differs from the one in (10) due to the extra orthogonalisation matrix $M_{x_i}$. However, for the WGi, CCEPu and CCEPr estimators, $\sum_{i=1}^{N} y_{i,-1} M_{x_i} M (\gamma_i F + \epsilon_i) \rightarrow \sum_{i=1}^{N} y_{i,-1} M \epsilon_i$ as $N \rightarrow \infty$ since the choice of $M$ for these estimators implies that $F_t$ is projected out and using that $E(x_{it} \epsilon_{is}) = 0$. As such, for $N \rightarrow \infty$ the numerator is the same as in the case of no exogenous variables. For the WGn estimator, $F_t$ is not projected out by $M$ such that here will be an extra term in the numerator of (41) which will depend on the correlation between $x_{it}$ and $F_t$.

Turning to the asymptotic behavior of $\hat{\beta}$, it is straightforward to show that letting $N \rightarrow \infty$

$$\text{plim}_{N \rightarrow \infty} (\hat{\beta} - \beta) = \frac{\text{plim}_{N \rightarrow \infty} \sum_{i=1}^{N} x_i' M \gamma_i F - \text{plim}_{N \rightarrow \infty} \sum_{i=1}^{N} x_i' M y_{i,-1} \text{plim}_{N \rightarrow \infty} (\hat{\rho} - \rho)}{\text{plim}_{N \rightarrow \infty} \sum_{i=1}^{N} x_i' M x_i},$$

(42)

where use is made of $E(x_{it} \epsilon_{is}) = 0$. Equation (42) shows that the inconsistency of $\hat{\beta}$ is a fraction of the inconsistency of $\hat{\rho}$, with this fraction depending on the relationship between the exogenous variables $x_{it}$ and $y_{i,t-1}$. For the WGi, CCEPu and CCEPr estimators, the first term in the numerator is zero since the choice of $M$ for these estimators implies that $F_t$ is projected out. For the WGn estimator this term is not zero such that this results in an additional inconsistency term when $x_{it}$ is correlated with $F_t$. This inconsistency does not disappear as $T \rightarrow \infty$.

5 Monte Carlo simulation

In this section we conduct a small-scale Monte Carlo experiment to investigate the small sample properties of the CCEPu and the CCEPr estimators under error cross-sectional dependence in both the fixed effects and the incidental trends case in comparison to the benchmark WGn and WGi estimators.

---

$^4$When the explanatory variables are correlated with the common factor $F_t$, the condition $\tau \neq 0$ can be generalized for the unrestricted CCEP estimator to a rank condition on the full matrix of all factor loadings as in Pesaran et al. (2013). For the restricted model $\tau \neq 0$ is still required, though.
### 5.1 Experimental design

The data generating process (DGP) we consider is given by

\[
y_{it} = \rho y_{i,t-1} + \beta x_{it} + \gamma_{it} F_t + \varepsilon_{it}, \quad \varepsilon_{it} \sim i.i.d. N \left( 0, (1 - \rho^2) \sigma^2_\varepsilon \right),
\]

\[
x_{it} = \phi_i F_t + \omega_{it}, \quad \omega_{it} \sim i.i.d. N \left( 0, \sigma^2_\omega \right),
\]

\[
F_t = \theta F_{t-1} + \mu_t, \quad \mu_t \sim i.i.d. N \left( 0, (1 - \theta^2) \sigma^2_\mu \right).
\]

We initialise \( y_{i,-49}, x_{i,-49} \) and \( F_{-49} \) at zero and discard the first 50 observations.

We compare the 4 alternative estimators over different values of the dynamic parameters, \( \rho \in \{0.5; 0.9\} \) and \( \theta \in \{0.5; 0.9\} \), for different samples sizes, \( T \in \{5; 10; 25; 50; 100\} \) and \( N \in \{20; 100\} \), with \( \sigma^2_\varepsilon = \sigma^2_\omega = \sigma^2_\mu = 1 \), \( \beta = 1 - \rho \), \( (\gamma_L, \gamma_U) = (0.5, 1.5) \) and \( (\phi_L, \phi_U) = (0.5, 1.5) \). Results for different parameter settings are available on request. The estimators include either fixed effects or both fixed effects and incidental trends.\(^5\)

All experiments are based on 5000 iterations. The estimators are compared in terms of (i) mean bias (bias), (ii) root mean squared error (rmse) and (iii) real size (size). The size is calculated for a two-sided hypothesis test at the 5% nominal level of significance using standard errors taken from the covariance matrix \( \Omega \) of the coefficients estimates \( (\hat{\rho}, \hat{\beta}) \) which, given that \( \varepsilon_{it} \) is a homoscedastic and serially uncorrelated error term, can be estimated in the standard way as

\[
\hat{\Omega} = \hat{\sigma}_\varepsilon^2 \left( \sum_{i=1}^{N} X_i X_i' \right)^{-1},
\]

where \( X_i = \frac{\partial M(\rho y_{i,t-1} + \beta x_i)}{\partial (\rho, \beta)} \) with the four alternative estimators differing in their choice of \( M \). Note that for the linear estimators WGN, WGI and CCEPu, \( X_i = MX_i \), with \( X_i = (y_{i,-1}, x_i) \), such that \( \hat{\Omega} = \hat{\sigma}_\varepsilon^2 \left( \sum_{i=1}^{N} X_i X_i' MX_i \right)^{-1} \). For the non-linear CCEPr estimator, the first derivatives in \( X_i \) are evaluated numerically.

### 5.2 Simulation Results

The simulation results for the fixed effects and the incidental trends case can be found in Table 1 and 2 respectively. Before looking into the results in detail, two overall conclusions can be drawn. First, in line with the theoretical results the biases for small \( T \) are much bigger in the incidental trends case than in the fixed effects case. Second, comparing the results for \( N = 20 \) with those for \( N = 100 \) shows that the size of the cross-sectional dimension \( N \) does not have a considerable effect on the size of the biases for any of the considered estimators. This is in line with the results in Pesaran (2006).

With respect to estimating \( \rho \), the following conclusions stand out. First, the WGN estimator is biased for all combinations of \( N \) and \( T \), with the bias being negative for small \( T \) and positive for larger values of \( T \). This is in line with the theoretical results in Section 3.1 which show that the switch in sign is due to the fact that as \( T \) increases the Nickell part of the bias, which is negative, diminishes whereas the positive bias originating from the unobserved common component does not as \( \theta \neq 0 \) (see also Figure 1). In line with

\(^5\)Note that for simplicity the DGP for \( y_{it} \) does not include fixed effects or incidental trends. This is without loss of generality as they would be projected out anyway by fitting them in the regression model.
equation (14), the bias of the WGN estimator for larger values of $T$ is bigger for larger values of $\theta$ and smaller for larger values of $\rho$. Second, for small values of $T$ the WGI estimator is biased but as $T$ increases, this bias shrinks to zero. The absolute value of the bias of the WGI estimator is increasing in $\theta$ but, in contrast to the WG estimator in a model with no cross-sectional dependence, not increasing in $\rho$. This can also be seen from equation (19). Third, the CCEP estimators both have a considerably larger bias compared to the benchmark WGI estimator for small values of $T$ with CCEP$_r$ clearly outperforming CCEP$_u$. The absolute value of the bias of the CCEP estimators is increasing in both $\rho$ and $\theta$. The bias diminishes as $T$ increases and is more or less gone when $T = 50$ in the fixed effects case and when $T = 100$ in the incidental trends case. Note that for small values of $T$, both the WGN and WGI estimator have a considerably smaller rmse compared to the CCEP estimators but for larger values of $T$ the rmse of the CCEP estimators is more in line with that of the WGI estimator and is clearly smaller than that of the WGN estimator. Turning to the estimates for $\beta$, a relatively (to $\rho$) smaller bias is found for the WGI, CCEP$_u$ and CCEP$_r$ estimators with the bias disappearing as $T$ grows larger. For the WGN estimator, there is a considerable bias which does not decrease as $T$ grows larger. This is due to the fact that in the DGP $x_{it}$ is correlated with the common factor $F_t$ in $y_{it}$ but this common factor is omitted from the regression.

Looking at the size results, all estimators for $\rho$ are subject to large size distortions. This is not surprising, given the substantial downward biases for small $T$. For the WGI, CCEP$_u$ and CCEP$_r$ estimators the size improves along with the decrease in their bias as $T$ grows larger, though. As these estimators are also much less biased for $\beta$, size distortions for inference on $\beta$ are much more moderate and size is more or less correct from $T = 25$ onwards. The WGN estimator suffers from substantial size distortions for both $\rho$ and $\beta$ for all values of $T$.

In sum, the results show that ignoring cross-sectional dependence in a dynamic panel data model implies the standard WG estimator to break down. Although the WGN estimator tends to have a relatively small bias for $\rho$ when $T$ is small, in general it yields biased estimates for $\rho$ and $\beta$ and considerable size distortions with these biases and size distortions not disappearing as $T$ grows large. The CCEP estimators have a larger bias than the infeasible WG estimator for small $T$, with the CCEP$_r$ estimator being relatively less biased than the CCEP$_u$ estimator, but the biases disappear when $T$ grows larger. For moderate values of $T$, the CCEP estimators have more or less correct size for $\beta$ but remain oversized for $\rho$. 

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Note: No results are reported for the CCEPu estimator when $T = 5$ as in this case the number of unknown parameters exceeds the number of observations.
6 Concluding remarks

This paper examines the effects of error cross-sectional dependence, modelled as an unobserved common factor, on WG and CCEP estimators in a linear dynamic panel data model. In general, the asymptotic behavior as \( N \to \infty \) of each estimator stems from two sources: the well known Nickell dynamic panel data bias and a random inconsistency which depends on the particular realisation of the unobserved common factor. First, in line with Phillips and Sul (2007), we find that the naive WG estimator is inconsistent for \( N \to \infty \) and \( T \) fixed. For a temporally dependent factor, we further show that the inconsistency remains even for \( N, T \to \infty \). Second, contrary to the findings in Pesaran (2006) for a static model, the unrestricted CCEP estimator is inconsistent for \( N \to \infty \) and fixed \( T \). For a relevant range of parameter combinations, the inconsistency is larger compared to the infeasible WG estimator. Restricting the CCEP estimator by taking into account the restrictions on the individual-specific factor loadings as implied by the derivation of the specification of the model augmented with cross-sectional averages results in a somewhat smaller inconsistency. Letting \( N, T \to \infty \), both the unrestricted and the restricted CCEP estimators are consistent.

The main practical conclusions of this paper are that (i) in a dynamic panel data model with error cross-sectional dependence the naive WG estimator breaks down and (ii) while trying to account for this using the CCEP estimator one should not dismiss the bias for small to moderate values of \( T \). The latter implies that bias corrections for the CCEP estimator would be very useful in practice. Along the lines suggested by e.g. Hahn and Kuersteiner (2002), Bun and Carree (2005) and Phillips and Sul (2007), the asymptotic bias expressions presented in this paper can in principle be used for bias correction. For a temporally independent factor \((\theta = 0)\), this is easily implemented as the inconsistency of the CCEP estimators for fixed \( T \) is a function of \( \rho \) only such that it is possible to invert the bias formulas and express \( \rho \) as a function of \( \hat{\rho}_{\text{CCEP}} \) to construct bias-corrected CCEP estimators. For a temporally dependent factor \((\theta \neq 0)\), however, this is less straightforward as the bias expressions also depend on the persistence in the unobserved common factor. A feasible bias correction procedure requires estimating this persistence. As this is beyond the scope of the present paper, we leave this for future research.
References


Appendices

Appendix A  Proofs

Lemma A-1. Under Assumption A1 and A3 we have from (5)
\[
\lambda_0 = E (F_{i}^+) = (\rho + \theta) \lambda_1 - \rho \theta \lambda_2 + \sigma^2_{\mu},
\]
\[
\lambda_1 = E (F_{i}^+ F_{i-1}^-) = (\rho + \theta) \lambda_0 - \rho \theta \lambda_1,
\]
\[
\lambda_s = E (F_{i}^+ F_{i-s}^-) = (\rho + \theta) \lambda_{s-1} - \rho \theta \lambda_{s-2}, \quad \forall s \geq 2 \tag{A-1}
\]
which can be solved to obtain
\[
\lambda_0 = \frac{1 + \theta \rho}{(1 - \theta \rho)(1 - \theta - \rho + \theta \rho)(1 + \theta + \rho \theta \rho)} \sigma^2_{\mu}, \tag{A-2}
\]
\[
\lambda_1 = \frac{\theta + \rho}{(1 - \theta \rho)(1 - \theta^2)(1 - \rho^2)} \sigma^2_{\mu}. \tag{A-3}
\]
Next, using that \( F_i = F_i^+ - \rho F_{i-1}^- \) we have
\[
E (F_i F_{i-1}^-) = E (F_i^+ F_{i-1}^-) - \rho E (F_{i-1}^-)^2 = \lambda_1 - \rho \lambda_0 = \frac{\theta}{(1 - \theta \rho)(1 - \theta^2)} \sigma^2_\mu. \tag{A-4}
\]

Proof of Proposition 1.a. Suppose Assumptions A1-A3 hold, then:
\[
\text{plim}_{N \to \infty} (\hat{\rho}_{\omega, T} - \rho) = \text{plim}_{N \to \infty} \frac{(1 / NT) \sum_{i=1}^{N} y_{i,t-1} M_i (\gamma_i F + \varepsilon_i)}{(1 / NT) \sum_{i=1}^{N} y_{i,t-1} M_i y_{i,t-1}},
\]
\[
= \text{plim}_{N \to \infty} \frac{(1 / NT) \sum_{i=1}^{N} \sum_{t=1}^{T} (y_{i,t-1} + \gamma_i F_{i,t-1}^-) (\gamma_i \tilde{F}_1 + \tilde{\varepsilon}_i)}{(1 / NT) \sum_{i=1}^{N} \sum_{t=1}^{T} (y_{i,t-1} + \gamma_i \tilde{F}_1^-)^2} = \frac{-A(\rho, T) + \frac{m_2}{\sigma^2} \sum_{t=1}^{T} \tilde{F}_{i-1}^+ \tilde{F}_i}{B(\rho, T) + \frac{m_2}{\sigma^2} \sum_{t=1}^{T} \tilde{F}_{i-1}^+}, \tag{A-5}
\]
where from Nickell (1981) \( A(\rho, T) = \frac{1}{1 - \rho} \left( 1 - \frac{1 - \rho}{T} \right) \) and \( B(\rho, T) = \frac{T}{1 - \rho} \left( 1 - \frac{1 + \rho}{T} + \frac{2 \rho}{T^2} \right) \). As argued by Phillips and Sul (2007), using straightforward calculations these expressions continue to hold for the case of cross-sectional heteroscedasticity of the type given in A2.

Next, letting \( T \to \infty \) and using Lemma A-1, we have
\[
\frac{1}{T} \sum_{t=1}^{T} \left( \tilde{F}_{i-1}^+ \right)^2 = E (F_{i-1}^-)^2 + O_p \left( \frac{1}{\sqrt{T}} \right) = \frac{1 + \theta \rho}{(1 - \theta \rho)(1 - \theta^2)(1 - \rho^2)} \sigma^2_\mu + O_p \left( \frac{1}{\sqrt{T}} \right), \tag{A-6}
\]
\[
\frac{1}{T} \sum_{t=1}^{T} \tilde{F}_{i-1}^+ \tilde{F}_i = E (F_{i-1}^- F_i) + O_p \left( \frac{1}{\sqrt{T}} \right) = \frac{\theta}{(1 - \theta \rho)(1 - \theta^2)} \sigma^2_\mu + O_p \left( \frac{1}{\sqrt{T}} \right). \tag{A-7}
\]
Thus, the inconsistency in (11) has the following large T expansion

\[
\plim_{N \to \infty} (\hat{\rho}_{\text{w0n}} - \rho) = \frac{-A(\rho, T)}{T} \left( \frac{1}{T} \sum_{t=1}^{T} \left( \tilde{F}_{t-1}^+ \right)^2 \right)^{-1} + \frac{m_2^2}{\sigma_2^4} \sum_{t=1}^{T} \left( \tilde{F}_{t-1}^+ \right)^2 \sum_{t=1}^{T} \left( \tilde{F}_{t-1}^+ \right)^2 \sum_{t=1}^{T} \left( \tilde{F}_{t-1}^+ \right)^2 \sum_{t=1}^{T} \left( \tilde{F}_{t-1}^+ \right)^2,
\]

when \( \theta = 0 \), Tanaka (1983) shows that the unconditional mean of \( g_{FT} \) has a large T expansion given by

\[
E \left( g_{FT} \right) = -\frac{1+\rho}{T} + o\left( \frac{1}{T} \right) \quad \text{such that} \quad (A-8) \text{reduces to}
\]

\[
\plim_{N \to \infty} (\hat{\rho}_{\text{w0n}} - \rho) = -\frac{1+\rho}{T} - \frac{2\rho}{T} + O_p \left( \frac{1}{\sqrt{T}} \right).
\]

When \( \theta \neq 0 \), using (A-2) and (A-3) and letting \( T \to \infty \) we have from using (A-6)-(A-7) that \( E \left( g_{FT} \right) = \frac{(1-\rho^2)\theta}{1+\theta \rho} \) such that for \( (N,T)_{seq} \to \infty \) we have from (A-8)

\[
\plim_{(N,T)_{seq} \to \infty} (\hat{\rho}_{\text{w0n}} - \rho) = \frac{(1-\rho^2)\theta}{(1+\theta \rho) + (1-\theta \rho)(1-\theta^2) \frac{\sigma_2^2}{m_2^2 \sigma_2^2}}.
\]

Proof of Proposition 1.b. Suppose Assumptions A1-A3 hold, then using the same derivations as in the proof of Proposition 1.a and using the results from the proof of Proposition 4 in Phillips and Sul (2007), it is straightforward to show that

\[
\plim_{N \to \infty} (\hat{\rho}_{\text{w0n}} - \rho) = \plim_{N \to \infty} \left( \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} y_{it} M_T \left( \gamma_i F + \varepsilon_i \right) \right) = \frac{-C(\rho, T) + \frac{m_2^2}{\sigma_2^4} \sum_{t=1}^{T} \tilde{F}_{t-1}^+ \tilde{F}_{t-1}^+}{D(\rho, T) + \frac{m_2^2}{\sigma_2^4} \sum_{t=1}^{T} \left( \tilde{F}_{t-1}^+ \right)^2},
\]

where from Phillips and Sul (2007) we have that \( C(\rho, T) = \frac{T}{T-1} \frac{2}{T-1} \rho \left( T - 1 - \frac{2}{T-1} C_1 \right) \) and \( D(\rho, T) = \frac{T}{T-2} \frac{4\rho}{T-1} D_1 \), with

\[
C_1 = 1 - \frac{1}{T+1} \left( 1 + \frac{1-\rho^3}{T(1-\rho^3)} \right) + \left( \frac{1}{2} + \frac{1}{T+1} \left( 1 + \frac{1-\rho^3}{T(1-\rho^3)} \right) \rho^T \right),
\]

\[
D_1 = 1 - \frac{1}{T+1} \frac{2}{1+T} \left( 1 - \frac{1-\rho^3}{T(1-\rho^3)} \right) \rho^T + \left( \frac{3\rho}{1+T} + \frac{T+3}{2} \right) \rho^T.
\]

Defining \( h_{FT} = \sum_{t=1}^{T} \tilde{F}_{t-1}^+ \tilde{F}_{t-1}^+ \sum_{t=1}^{T} \left( \tilde{F}_{t-1}^+ \right)^2 \) , the results in equations (16) and (17) and the fact that for \( \theta \neq 0 \) the results in equation (14) for the fixed effects model continue to hold in the incidental trends case following directly from using that the results in (A-6)-(A-7) for \( \tilde{F}_{t-1}^+ \) and \( \tilde{F}_t \) continue to hold for \( \tilde{F}_{t-1}^+ \) and
\( \tilde{F}_t \) and noting that for \( \theta = 0 \) we have from Phillips and Sul (2007) that \( E (h_{FT}) = -\frac{2 \psi_{4n}}{T} + o \left( \frac{1}{T} \right) \).

**Proof of Proposition 2.a.** Choosing \( M \) in (10) to be the projection matrix \( M_{F_t} = I_T - F_t (F'_t F_t)^{-1} F'_t \), we have

\[
\hat{\rho}_{m0} = \frac{(1/NT) \sum_{i=1}^{N} y'_{i,t-1} M_{F_t} y_i}{(1/NT) \sum_{i=1}^{N} y'_{i,t-1} M_{F_t} y_i} = \rho + \frac{(1/NT) \sum_{i=1}^{N} y'_{i,t-1} M_{F_t} \epsilon_i}{(1/NT) \sum_{i=1}^{N} y'_{i,t-1} M_{F_t} y_i}.
\]

(A-14)

First, under Assumptions A1-A3 the probability limit as \( N \to \infty \) of the numerator of equation (A-14) is given by

\[
\text{plim}_{N \to \infty} \frac{1}{NT} \sum_{i=1}^{N} y'_{i,t-1} M_{F_t} \epsilon_i = \text{plim}_{N \to \infty} \frac{1}{NT} \sum_{i=1}^{N} y_{it} \left( \epsilon_{it} - \bar{\kappa}_{st} (\epsilon_{is} - \bar{\epsilon}_i) \right),
\]

\[
= \frac{1}{T} \sum_{t=1}^{T} \text{plim}_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} y_{it} - \frac{1}{T} \sum_{t=1}^{T} \left( 1 - \sum_{s=1}^{T} \bar{\kappa}_{st} \right) \text{plim}_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} y_{it} \bar{\epsilon}_i,
\]

(A-15)

where \( \bar{\kappa}_{st} = \tilde{F}_\tau \tilde{F}_t / \sum_{t=1}^{T} \tilde{F}_t^2 \). Using

\[
\text{plim}_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} y_{it} \epsilon_{it} = \text{plim}_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} E [y_{it} \epsilon_{it} - s] = \text{plim}_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \sigma_{\epsilon_i}^2 \rho^{s-1} = \sigma_{\epsilon_i}^2 \rho^{s-1}, \quad \forall s \geq 1,
\]

(A-16)

\[
= 0, \quad \forall s < 1,
\]

(A-17)

the first term of (A-15) drops and the last term rewrites to

\[
= -\sigma_{\epsilon_i}^2 \frac{1}{T} \left( \sum_{t=2}^{T} \bar{\kappa}_{t,t-1} + \rho \sum_{t=3}^{T} \bar{\kappa}_{t,t-2} + \rho^2 \sum_{t=4}^{T} \bar{\kappa}_{t,t-3} + \ldots + \rho^{T-2} \bar{\kappa}_{T,1} \right),
\]

\[
= -\sigma_{\epsilon_i}^2 \frac{1}{T} \rho^{t-1} \sum_{s=t+1}^{T} \bar{\kappa}_{s,s-t} = \sigma_{\epsilon_i}^2 \frac{1}{T} \rho^{t-1} \tilde{g}_{F,t},
\]

(A-18)

where \( \tilde{g}_{F,t} = \sum_{s=t+1}^{T} \bar{\kappa}_{s,s-t} = \sum_{t=1}^{T} \tilde{F}_t / \sum_{t=1}^{T} \tilde{F}_t^2 \).

The second term is given by

\[
= \frac{1}{T} \sum_{t=1}^{T} \left( 1 - \sum_{s=1}^{T} \bar{\kappa}_{st} \right) \text{plim}_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} y_{it} \bar{\epsilon}_i = \frac{1}{T^2} \sigma_{\epsilon_i}^2 \sum_{t=1}^{T} (1 - \rho^{t-1}) \left( 1 - \frac{\tilde{F}_t \sum_{s=1}^{T} \tilde{F}_s}{\sum_{s=1}^{T} \tilde{F}_s^2} \right)
\]

(A-19)
Using (A-16) - (A-19), equation (A-15) can be written as

\[
\lim_{N \to \infty} \frac{1}{NT} \sum_{i=1}^{N} y_{i,-1}^{t} M_{F_{i}} \varepsilon_{i} = -\frac{\sigma_{\varepsilon}^2}{T} A(\rho, T) - \frac{\sigma_{\varepsilon}^2}{T} \sum_{t=1}^{T-1} \rho^{t-1} g_{F_{i},t}.
\]

Second, the probability limit as \( N \to \infty \) of the denominator of equation (A-14) is given by

\[
\lim_{N \to \infty} \frac{1}{NT} \sum_{i=1}^{N} y_{i,-1}^{t} M_{F_{i}} y_{i,-1} = \lim_{N \to \infty} \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} y_{i,t-1} \left[ (y_{i,t-1} - \bar{y}_{i,-1}) - \sum_{s=1}^{T} \bar{\kappa}_{st} (y_{i,s-1} - \bar{y}_{i,-1}) \right],
\]

\[
= \lim_{N \to \infty} \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \bar{\kappa}_{st} y_{i,t-1} y_{i,-1},
\]

\[
= \frac{1}{T} \sum_{t=1}^{T} \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} [y_{i,t-1} - y_{i,t-1} \bar{y}_{i,-1}] - \frac{1}{T} \sum_{t=1}^{T} \sum_{s=1}^{T} \bar{\kappa}_{st} \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} y_{i,t-1} y_{i,s-1}
\]

\[
+ \frac{1}{T} \sum_{t=1}^{T} \sum_{s=1}^{T} \bar{\kappa}_{st} \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} y_{i,t-1} \bar{y}_{i,-1}. \tag{A-21}
\]

The first and second term can be written as

\[
\frac{1}{T} \sum_{t=1}^{T} \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} [y_{i,t-1}^{+} - y_{i,t-1} \bar{y}_{i,-1}] = \frac{1}{T} \sum_{t=1}^{T} \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} [y_{i,t-1}^{+} - y_{i,-1}^{+}]^{2} + \gamma_{i}^{2} \left[ F_{t-1}^{+} - F_{t-1}^{-} \right]^{2},
\]

\[
= \frac{\sigma_{\varepsilon}^2}{T} \frac{T - 1}{1 - \rho^2} \left[ 1 - \frac{1}{T-1} \frac{2 \rho}{1 - \rho} \left[ 1 - \frac{1}{T-1} \frac{1 - \rho^T}{1 - \rho} \right] \right] + m_{\varepsilon} \frac{1}{T} \sum_{t=1}^{T} \left[ F_{t-1}^{+} - F_{t-1}^{-} \right]^{2},
\]

\[
= \frac{\sigma_{\varepsilon}^2}{T} B(\rho, T) + m_{\varepsilon} \frac{1}{T} \sum_{t=1}^{T} \left( \bar{F}_{t-1}^{+} \right)^{2},
\]

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} y_{i,t-1} y_{i,s-1} = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} (y_{i,t-1}^{+} + \gamma_{i} F_{t-1}^{+}) (y_{i,s-1}^{+} + \gamma_{i} F_{s-1}^{+}),
\]

\[
= \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} y_{i,t-1}^{+} y_{i,s-1}^{+} + \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \gamma_{i}^{\frac{|t-s|}{T}} F_{t-1}^{+} F_{s-1}^{+} = \frac{\rho^{t-s}}{1 - \rho^2} \sigma_{\varepsilon}^2 + m_{\varepsilon} F_{t-1}^{+} F_{s-1}^{+},
\]

while the last term in (A-21) can be dropped since

\[
\frac{1}{T} \sum_{t=1}^{T} \sum_{s=1}^{T} \bar{\kappa}_{st} \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} y_{i,t-1} \bar{y}_{i,-1} = \frac{1}{T} \sum_{t=1}^{T} \left( \bar{F}_{t} \sum_{s=1}^{T} \bar{F}_{s} \right) \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} y_{i,t-1} \bar{y}_{i,-1} = 0.
\]

Thus, equation (A-21) can be written as

\[
\lim_{N \to \infty} \frac{1}{NT} \sum_{i=1}^{N} y_{i,-1}^{t} M_{F_{i}} y_{i,-1} = \frac{\sigma_{\varepsilon}^2}{T} B(\rho, T) + m_{\varepsilon} \frac{1}{T} \sum_{t=1}^{T} \left( \bar{F}_{t-1}^{+} \right)^{2} - \frac{1}{T} \sum_{t=1}^{T} \sum_{s=1}^{T} \bar{\kappa}_{st} \left( \frac{\rho^{t-s}}{1 - \rho^2} \sigma_{\varepsilon}^2 + m_{\varepsilon} F_{t-1}^{+} F_{s-1}^{+} \right),
\]
Proof of Proposition 2.b. where use is made of lemma A-1. Hence, Dividing (A-20) by (A-22) yields the result in equation (18).

Next, letting $T \to \infty$

$$\tilde{g}_{F,t} = \frac{T^{-t} \left( E(F,F_{s-1}) + O_p \left( \frac{1}{\sqrt{T}} \right) \right) }{ E(F_s^2) + O_p \left( \frac{1}{\sqrt{T}} \right)} = \frac{T - t}{T} \theta^t + O_p \left( \frac{1}{\sqrt{T}} \right), \quad (A-23)$$

$$\tilde{k}_F = E \left( \left( F_{t-1}^+ \right)^2 \right) \left( 1 - \frac{E(F,F_{t-1}^+)^2}{E(F_s^2) E(F_{t-1}^+)^2} \right) + O_p \left( \frac{1}{\sqrt{T}} \right) = \frac{1}{(1-\theta \rho) (1-\rho^2) \sigma^2 + O_p \left( \frac{1}{\sqrt{T}} \right)}, \quad (A-24)$$

where use is made of lemma A-1. Hence,

$$\sum_{t=1}^{T-1} \rho^{t-1} g_{F,t} = \sum_{t=1}^{T-1} \rho^{t-1} \left( \frac{T - t}{T} \theta^t + O_p \left( \frac{1}{\sqrt{T}} \right) \right) = \theta \sum_{t=1}^{T-1} \theta^t - \frac{1}{\rho^2} \sum_{t=1}^{T-1} t (\rho t)^{t-1} + O_p \left( \frac{1}{\sqrt{T}} \right),$$

$$= \frac{\theta}{1-\theta \rho} \left[ 1 - \frac{1}{1-\theta} \theta^T \right] + O_p \left( \frac{1}{\sqrt{T}} \right) = \frac{\theta}{1-\theta} \rho^T + O_p \left( \frac{1}{\sqrt{T}} \right), \quad (A-25)$$

Thus, the inconsistency in (18) has the following large $T$ expansion

$$\lim_{N \to \infty} (\rho_{\text{WA}} - \rho) = \frac{1}{T} B(\rho, T) + \frac{1}{(1-\theta \rho)^2 (1-\rho^2)^2} m^2 \sigma^2 + o_p (1),$$

$$= - \frac{1}{T} \left( 1 + \rho + \frac{\theta (1-\rho^2)}{1-\theta \rho} \right) \left( 1 + \frac{m^2 \sigma^2}{(1-\theta \rho)^2 \sigma^2} \right)^{-1} + o_p \left( \frac{1}{T} \right). \quad (A-26)$$

**Proof of Proposition 2.b.** Under Assumptions A1-A3, it is straightforward to obtain the result in (20) using
the same derivations as in the proof of Proposition 2.a and using the results from the proof of Proposition 4 in Phillips and Sul (2007). The large T expansion in (21) then follows directly from the definitions of \( C(\rho, T) \) and \( D(\rho, T) \) and using that the results in (A-25) and (A-24) for \( \hat{y}_{F,t} \) and \( \hat{k}_F \) continue to hold for \( \tilde{y}_{F,t} \) and \( \tilde{k}_F \).

**Proof of Theorem 1.a.** Choosing \( M \) in (10) to be the projection matrix \( M_G = I_T - G_i(G'_iG_i)^{-1}G_i \), we have

\[
\hat{\rho}_{CCEM} = \frac{(1/NT) \sum_{i=1}^{N} y'_{i,-1}M_G y_i}{(1/NT) \sum_{i=1}^{N} y'_{i,-1}M_G y_i} = \rho + \frac{(1/NT) \sum_{i=1}^{N} y'_{i,-1}M_G \epsilon_i}{(1/NT) \sum_{i=1}^{N} y'_{i,-1}M_G y_i}. \tag{A-27}
\]

Using (24) and (25), the probability limit as \( N \to \infty \) of the numerator of equation (A-27) equals

\[
\lim_{N \to \infty} \frac{1}{NT} \sum_{i=1}^{N} y'_{i,-1}M_G \epsilon_i = \lim_{N \to \infty} \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} y_{i,t} (\epsilon_{it} - \bar{\epsilon}_t),
\]

with \( M_{H_i} = I_T - H_i(H'_iH_i)^{-1}H'_i \), \( H_i = (i, F^+, F'^+_i) \). Similar to the derivation of (A-20), (A-28) rewrites to

\[
\lim_{N \to \infty} \frac{1}{NT} \sum_{i=1}^{N} y'_{i,-1}M_{H_i} \epsilon_i = \lim_{N \to \infty} \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} y_{i,t} (\epsilon_{it} - \bar{\epsilon}_t) = -\frac{\sigma^2}{T} A(\rho, T) - \frac{\sigma^2}{T} \sum_{t=1}^{T} \rho^{-1} \tilde{g}^{-1}_{F,t}, \tag{A-29}
\]

where \( \tilde{g}^{-1}_{F,t} = \sum_{s=t+1}^{T} \tilde{r}_{s,s-t}^{++} \) and

\[
\tilde{r}_{st}^{++} = \frac{1}{\alpha_0} \left( \alpha_1 \hat{F}_t^{++} \hat{F}_s^{++} - \alpha_2 \hat{F}_t^{++} \hat{F}_{s-1}^{++} + \alpha_3 \hat{F}_{t-1}^{++} \hat{F}_s^{++} - \alpha_2 \hat{F}_{t-1}^{++} \hat{F}_{s-1}^{++} \right), \tag{A-30}
\]

with \( \alpha_0 = \alpha_1 \alpha_3 - \alpha_2^2, \alpha_1 = \sum_{t=1}^{T} \left( \hat{F}_t^{++} \right)^2, \alpha_2 = \sum_{t=1}^{T} \hat{F}_t^{++} \hat{F}_{t-1}^{++} \) and \( \alpha_3 = \sum_{t=1}^{T} \left( \hat{F}_t^{++} \right)^2 \).

Second, as (25) implies that for \( N \to \infty \) \( M_GF^+ = 0 \) such that \( M_Gy_{i,-1} = M_Gy_{i,-1}^+ \), the probability limit as \( N \to \infty \) of the denominator of equation (A-27) is given by

\[
\lim_{N \to \infty} \frac{1}{NT} \sum_{i=1}^{N} y'_{i,-1}M_G y_i = \lim_{N \to \infty} \frac{1}{NT} \sum_{i=1}^{N} y'_{i,-1}M_G y_i^+,
\]

and

\[
= \lim_{N \to \infty} \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} y'_{i,t-1} \left( y_{i,t-1}^+ - \overline{y}_{i,-1}^+ \right) - \sum_{s=1}^{T} \tilde{r}_{st}^{++} (y_{i,s-1}^+ - \overline{y}_{i,-1}) \right),
\]

\[
= \lim_{N \to \infty} \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \frac{1}{NT} \sum_{i=1}^{N} \sum_{s=1}^{T} \tilde{r}_{st}^{++} y_{i,t-1}^+ y_{i,s-1}^-, \sum_{t=1}^{T} \sum_{s=1}^{T} \tilde{r}_{st}^{++} y_{i,t-1}^+ y_{i,s-1}^-
\]

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\[ T \sum_{s=t+1}^{T} \bar{F}_{s+} \bar{F}_{s-t} = T - \frac{t}{T} E(\bar{F}_{s+} \bar{F}_{s-t}) + O_p \left( \frac{1}{\sqrt{T}} \right) = \frac{T - t}{T} \lambda_1 + O_p \left( \frac{1}{\sqrt{T}} \right), \]

such that

\[
\bar{g}_{F,t} = \frac{1}{\alpha_0} \left( \bar{\alpha}_1 \sum_{s=t+1}^{T} \bar{F}_{s+} \bar{F}_{s-t} - \bar{\alpha}_2 \sum_{s=t+1}^{T} \bar{F}_{s+} \bar{F}_{s-t-1} + \bar{\alpha}_3 \sum_{s=t+1}^{T} \bar{F}_{s-1} \bar{F}_{s-t} - \bar{\alpha}_2 \sum_{s=t+1}^{T} \bar{F}_{s-1} \bar{F}_{s-t} \right),
\]

with \( \omega_1 = \lambda_1/\lambda_0 \). Next

\[
- \frac{1}{T} \sum_{t=1}^{T-1} \rho^{-t} \bar{g}_{F,t} = - \frac{1}{T} \sum_{t=1}^{T-1} \rho^{-t} \left( \frac{T - t}{T} 2 \omega_1 - \omega_1 \omega_{t+1} - \omega_1 \omega_{t-1} \right) + O_p \left( \frac{1}{\sqrt{T}} \right),
\]

\[
= - \frac{(1 + \theta \rho)^2}{(1 - \theta^2)(1 - \rho^2)} \frac{1}{T} \sum_{t=1}^{T-1} \rho^{-t} \left( 1 - \frac{t}{T} \right) \left( (2 - (\theta + \rho) \omega_1) \omega_1 - (1 - \theta \rho) \omega_1 \omega_{t+1} \right) + O_p \left( \frac{1}{T} \right),
\]

\[
= - \frac{(1 + \theta \rho)}{T} \sum_{t=1}^{T-1} \rho^{-t} \left( 1 - \frac{t}{T} \right) \left( \frac{\omega_1 - \rho \omega_{t+1}}{1 - \rho^2} + \frac{\omega_1 - \theta \omega_{t+1}}{1 - \theta^2} \right) + O_p \left( \frac{1}{T} \right),
\]

\[
= - \frac{1}{T} \left( \frac{\theta}{1 - \theta \rho} + \frac{\rho}{1 - \rho^2} \right) + O_p \left( \frac{1}{T} \right), \tag{A-32}
\]

where use is made of \( \omega_t = (\theta + \rho) \omega_{t-1} - \theta \rho \omega_{t-2} \) \( \forall t \geq 2 \) and

\[
\sum_{t=2}^{T-1} \left( 1 - \frac{t}{T} \right) \rho^{-t} \omega_t = (\theta + \rho) \sum_{t=2}^{T-1} \left( 1 - \frac{t}{T} \right) \rho^{-t} \omega_{t-1} - \theta \rho \sum_{t=2}^{T-1} \left( 1 - \frac{t}{T} \right) \rho^{-t} \omega_{t-2},
\]

\[
= (\theta + \rho) \left( 1 - \frac{2}{T} \right) \rho \omega_1 + \rho \left( \sum_{t=2}^{T-1} \left( 1 - \frac{t}{T} \right) \rho^{-t} \omega_t - \frac{1}{T} \sum_{t=2}^{T-1} \rho^{-t} \omega_t \right).
\]
and such that

$$\sum_{t=1}^{T-1} \left(1 - \frac{t}{T}\right) \rho^{t-1} \omega_t = \omega_1 + \sum_{t=2}^{T-1} \left(1 - \frac{t}{T}\right) \rho^{t-1} \omega_t = \frac{\theta (1 - \rho^2) + \rho (1 - \theta^2 \rho^2)}{(1 - \rho^2)(1 - \theta^2 \rho^2)} + O\left(\frac{1}{T}\right),$$

and

$$\sum_{t=1}^{T-1} \left(1 - \frac{t}{T}\right) \rho^{t-1} \omega_{t-1} = 1 + \rho \sum_{t=1}^{T-1} \left(1 - \frac{t}{T}\right) \rho^{t-1} \omega_t - \frac{1}{T} \sum_{t=1}^{T} \rho^{t-1} \omega_{t-1},$$

$$= 1 + \rho \frac{\omega_1 - \theta^2 \rho^2}{(1 - \rho^2)(1 - \theta \rho)} + O\left(\frac{1}{T}\right) = \frac{(1 - \theta^2 \rho^2) + \rho (1 - \theta^2 \rho)}{(1 - \rho^2)(1 - \theta^2 \rho^2)} + O\left(\frac{1}{T}\right).$$

Using (A-32), as $T \to \infty$ (A-29) and (A-31) are given by

$$\begin{align*}
-\frac{\sigma_\varepsilon^2}{T} A(\rho, T) - \sigma_\varepsilon^2 \frac{1}{T} \sum_{t=1}^{T-1} \rho^{t-1} \eta_{F,t}^- &= -\frac{\sigma_\varepsilon^2}{T} \left(A(\rho, T) + \left(\frac{\theta}{1 - \theta \rho} + \frac{\rho}{1 - \rho^2}\right)\right) + o_p\left(\frac{1}{T}\right), \\
\frac{\sigma_\varepsilon^2}{1 - \rho^2} \left(1 - \frac{\rho^2}{T} B(\rho, T) - \frac{2}{T} \left(1 + \rho \sum_{t=1}^{T-1} \rho^{t-1} \gamma_{F,t}^+\right)\right) &= \frac{\sigma_\varepsilon^2}{T} B(\rho, T) + O\left(\frac{1}{T}\right).
\end{align*}
$$

Thus, the inconsistency in (29) has the following large $T$ expansion

$$\begin{align*}
\text{plim}_{N \to \infty} (\hat{\rho}_{\text{CCEP}} - \rho) &= -\frac{1}{T} \left(\frac{1}{1 - \rho} + \left(\frac{\theta}{1 - \theta \rho} + \frac{\rho}{1 - \rho^2}\right)\right) (1 - \rho^2) + o_p\left(\frac{1}{T}\right), \\
&= -\frac{1}{T} \left(1 + 2 \rho + \frac{\theta(1 - \rho^2)}{1 - \theta \rho}\right) + o_p\left(\frac{1}{T}\right).
\end{align*}
$$

**Proof of Theorem 1.b.** Under Assumptions A1-A3, it is straightforward to obtain the result in (31) using the same derivations as in the proof of Theorem 1.a and using the results from the proof of Proposition 4 in Phillips and Sul (2007). The large $T$ expansion in (32) then follows directly from the definitions of $C(\rho, T)$ and $D(\rho, T)$ and using that the result in (A-32) for $\bar{y}_{F,t}^+$ continues to hold for $\hat{y}_{F,t}^+$.

**Proof of Theorem 2.a.** First note that using (22) and (35)

$$\begin{align*}
\hat{F}_t &= \frac{1}{\gamma} \left(\eta_t - \bar{\alpha} - \hat{\rho}_{\text{CCEP}} \bar{y}_{t-1} \right) = \frac{1}{\gamma} \left((\rho - \hat{\rho}_{\text{CCEP}}) \bar{y}_{t-1} + (\bar{\alpha} - \bar{\gamma}) + \bar{\gamma} F_t + \varepsilon_t\right), \\
F_t &= \hat{F}_t + \frac{1}{\gamma} \left(\hat{\rho}_{\text{CCEP}} - \rho\right) \bar{y}_{t-1} + (\bar{\alpha} - \bar{\gamma}) - \varepsilon_t,
\end{align*}
$$
which for \( N \to \infty \) and using (24) and (25) reduces to

\[
F_i = \tilde{F}_i + \frac{1}{\bar{\gamma}} \left( \tilde{\rho}_{CCEP} - \rho \right) \left( (1 - \rho)^{-1} \tilde{\alpha} + \bar{\gamma} F_{t-1}^+ \right) + (\tilde{\alpha} - \alpha),
\]

\[
= \tilde{F}_i + (\tilde{\rho}_{CCEP} - \rho) F_{t-1}^+ + a^*,
\]

(A-36)

where \( a^* = \frac{1}{\bar{\gamma}} \left( \tilde{\rho}_{CCEP} - \rho \right) (1 - \rho)^{-1} \tilde{\alpha} + (\tilde{\alpha} - \alpha) \). Using (A-36) in (10) with \( M \) chosen to be \( M_{\tilde{F}_i} \)

\[
\lim_{N \to \infty} (\tilde{\rho}_{CCEP} - \rho) = \frac{\lim_{N \to \infty} (1 / NT) \sum_{i=1}^{N} y_{i,-1} M_{\tilde{F}_i} \left( \gamma_i \lim_{N \to \infty} (\tilde{\rho}_{CCEP} - \rho) F_{t-1}^+ + \varepsilon_i \right)}{\lim_{N \to \infty} (1 / NT) \sum_{i=1}^{N} y_{i,-1} M_{\tilde{F}_i} y_{i,-1}}
\]

(A-37)

Using (A-16)-(A-19), the numerator of (A-37) is given by

\[
\lim_{N \to \infty} \frac{1}{NT} \sum_{i=1}^{N} y_{i,-1} M_{\tilde{F}_i} \varepsilon_i = \frac{\lim_{N \to \infty} (1 / NT) \sum_{i=1}^{N} \sum_{t=1}^{T} y_{i,t-1} \left[ (\varepsilon_{it} - \bar{\varepsilon}_t) - \sum_{s=1}^{T} \bar{\tau}_{st} (\varepsilon_{is} - \bar{\varepsilon}_s) \right]}{\lim_{N \to \infty} (1 / NT) \sum_{i=1}^{N} y_{i,-1} \bar{\varepsilon}_i} - \frac{1}{T} \sum_{t=1}^{T} \sum_{s=1}^{T} \bar{\tau}_{st} \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} y_{i,t-1} \varepsilon_{is},
\]

(A-38)

where \( \bar{\varepsilon}_t = \sum_{s=t+1}^{T} \bar{\tau}_{s,s-t} \) with \( \bar{\tau}_{s,s-t} \) being the \((s, s-t)\)th element in \( \bar{\tau}_s \).

Similarly to the derivation of (A-22), the denominator of equation (A-37) is given by

\[
\lim_{N \to \infty} \frac{1}{NT} \sum_{i=1}^{N} y_{i,-1} M_{\tilde{F}_i} y_{i,-1}^+ = \lim_{N \to \infty} \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \left[ y_{i,t-1}^+ + \bar{\tau}_{st} y_{i,s-1}^+ - \sum_{s=1}^{T} \bar{\tau}_{st} \left( y_{i,s-1}^+ - y_{i,s}^+ \right) \right],
\]

\[
= \lim_{N \to \infty} \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \left( y_{i,t-1}^+ - y_{i,s}^+ \right)^2 - \lim_{N \to \infty} \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{s=1}^{T} \bar{\tau}_{st} y_{i,t-1}^+ y_{i,s-1}^+,
\]

\[
= \sigma_{\tilde{\varepsilon}}^2 \frac{1 - \rho^2}{1 - \rho^2} B(\rho, T) - \frac{2}{T} \left( 1 + \rho \sum_{t=1}^{T-1} \rho^{t-1} g_{\tilde{F},t} \right).
\]

(A-39)
Dividing (A-38) by (A-39) yields the result in equation (36).

Next, letting \((\rho^0, F^0)\) denote the true parameter \(\rho\) and the true factor \(F\) respectively such that, after centering, the objective function in (33) is given by

\[
S_{NT}(\rho, F) = \frac{1}{NT} \sum_{i=1}^{N} (\bar{y}_i - \rho \bar{y}_{i,-1})' M_{\bar{F}} (\bar{y}_i - \rho \bar{y}_{i,-1}) - \frac{1}{NT} \sum_{i=1}^{N} \bar{\varepsilon}_i' M_{\bar{F}_0} \bar{\varepsilon}_i,
\]

\[
= \frac{1}{NT} \sum_{i=1}^{N} \left( (\rho^0 - \rho) \bar{y}_{i,-1} + \gamma_i \bar{F}^0 + \bar{\varepsilon}_i \right)' M_{\bar{F}} \left( (\rho^0 - \rho) \bar{y}_{i,-1} + \gamma_i \bar{F}^0 + \bar{\varepsilon}_i \right) - \frac{1}{NT} \sum_{i=1}^{N} \bar{\varepsilon}_i' M_{\bar{F}_0} \bar{\varepsilon}_i,
\]

\[
= s_{NT}(\rho, F) + 2 \frac{(\rho^0 - \rho)}{NT} \sum_{i=1}^{N} \bar{y}_{i,-1}' M_{\bar{F}} \bar{y}_{i,-1} + 2 \frac{1}{NT} \sum_{i=1}^{N} \gamma_i \bar{F}' M_{\bar{F}} \bar{\varepsilon}_i + \frac{1}{NT} \sum_{i=1}^{N} \varepsilon_i' (M_{\bar{F}} - M_{\bar{F}_0}) \bar{\varepsilon}_i,
\]

where

\[
s_{NT}(\rho, F) = \frac{(\rho^0 - \rho)^2}{NT} \sum_{i=1}^{N} \bar{y}_{i,-1}' M_{\bar{F}} \bar{y}_{i,-1} + \frac{1}{NT} \sum_{i=1}^{N} \gamma_i^2 \bar{F}' M_{\bar{F}} \bar{F}^0 + 2 \frac{(\rho^0 - \rho)}{NT} \sum_{i=1}^{N} \bar{y}_{i,-1}' M_{\bar{F}} \bar{F}^0 \gamma_i.
\]

Using that for \(N \to \infty\), followed by an expansion as \(T \to \infty\)

\[
\frac{1}{NT} \sum_{i=1}^{N} \bar{y}_{i,-1}' M_{\bar{F}} \bar{\varepsilon}_i = \frac{1}{NT} \sum_{i=1}^{N} \bar{y}_{i,-1}' \bar{\varepsilon}_i - \frac{1}{N} \sum_{i=1}^{N} \bar{y}_{i,-1}' \left( \frac{\bar{F}' \bar{F}}{T} \right)^{-1} \frac{\bar{F}' \bar{\varepsilon}_i}{T} = o_p(1),
\]

\[
\frac{1}{NT} \sum_{i=1}^{N} \gamma_i \bar{F}' M_{\bar{F}} \bar{\varepsilon}_i = \frac{1}{NT} \sum_{i=1}^{N} \gamma_i \bar{F}' \bar{\varepsilon}_i - \frac{1}{N} \sum_{i=1}^{N} \gamma_i \bar{F}' \left( \frac{\bar{F}' \bar{F}}{T} \right)^{-1} \frac{\bar{F}' \bar{\varepsilon}_i}{T} = o_p(1),
\]

\[
\frac{1}{NT} \sum_{i=1}^{N} \varepsilon_i' (M_{\bar{F}} - M_{\bar{F}_0}) \bar{\varepsilon}_i = \frac{1}{N} \sum_{i=1}^{N} \varepsilon_i' \bar{F}' \left( \frac{\bar{F}' \bar{F}}{T} \right)^{-1} \frac{\bar{F}' \bar{\varepsilon}_i}{T} - \frac{1}{N} \sum_{i=1}^{N} \varepsilon_i' \left( \frac{\bar{F}' \bar{F}}{T} \right)^{-1} \
\frac{\bar{F}' \bar{\varepsilon}_i}{T} = o_p(1).
\]

we have

\[
S_{NT}(\rho, F) = s_{NT}(\rho, F) + o_p(1),
\]

uniformly over \(\rho\) and \(F\). First note that as \(M_{\bar{F}_0} \bar{F}^0 = 0\), \(s_{NT}(\rho^0, F^0) = 0\). Second, we show that for any \((\rho, F) \neq (\rho^0, F^0)\), \(s_{NT}(\rho, F) > 0\); thus \(s_{NT}(\rho^0, F^0)\) attains its unique minimum value at \((\rho, F) = (\rho^0, F^0)\).

Define

\[
A = \frac{1}{NT} \sum_{i=1}^{N} \bar{y}_{i,-1}' M_{\bar{F}} \bar{y}_{i,-1}; \quad B = \frac{1}{NT} \sum_{i=1}^{N} \gamma_i^2; \quad C = \frac{1}{NT} \sum_{i=1}^{N} \gamma_i M_{\bar{F}} \bar{y}_{i,-1}.
\]

Then

\[
s_{NT}(\rho, F) = (\rho^0 - \rho)^2 A + \bar{F}' M_{\bar{F}} M_{\bar{F}} \bar{F}^0 + 2 (\rho^0 - \rho) C' M_{\bar{F}} \bar{F}^0,
\]

\[
= (\rho^0 - \rho)^2 \left( A - C' B^{-1} C \right) + \left( \bar{F}' M_{\bar{F}} + (\rho^0 - \rho) C' B^{-1} \right) B' \left( M_{\bar{F}} \bar{F}^0 + B^{-1} C (\rho^0 - \rho) \right),
\]

\[
= (\rho^0 - \rho)^2 D \left( \bar{F} \right) + \theta' B \theta,
\]

\[
\geq 0,
\]

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since $D(F) = A - C'B^{-1}C$ and $B$ are both positive definite, where $\theta = M\tilde{F}\tilde{F}_0 + B^{-1}C(\rho^0 - \rho)$. Note that $s_{NT}(\rho, F) > 0$ if either $\rho \neq \rho^0$ or $F \neq F^0$. This implies that $\hat{\rho}_{CCEP_r}$ is consistent for $\rho$ for $N, T \to \infty$.

From (A-36) and the consistency of $\hat{\rho}_{CCEP_r}$ we have that for $N \to \infty$ followed by an expansion as $T \to \infty$

$$\hat{F}_t = F_t + o_p(1), \quad (A-41)$$

such that

$$\tilde{g}_{\hat{F},t} = \tilde{g}_{F,t} + o_p(1). \quad (A-42)$$

The large $T$ expansion of the $CCEP_r$ estimator in (37) then follows from substituting (A-42) in (36) and using similar derivations as to obtain (19).

Proof of Theorem 2.b. Under Assumptions A1-A3, it is straightforward to obtain the result in (38) using the same derivations as in the proof of Theorem 2.a and using the results from the proof of Proposition 4 in Phillips and Sul (2007). The large $T$ expansion in (39) then follows directly from the definitions of $C(\rho, T)$ and $D(\rho, T)$ and using that the result in (A-42) for $\tilde{g}_{\hat{F},t}$ continues to hold for $\tilde{g}_{\hat{F},t}$.